A MIXED GM/SMC IMPLEMENTATION OF THE PROBABILITY HYPOTHESIS DENSITY FILTER

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ABSTRACT

The Probability Hypothesis Density (PHD) filter is a recent solution for tracking an unknown number of targets in a multi-object environment. The PHD filter cannot be computed exactly, but popular implementations include Gaussian Mixture (GM) and Sequential Monte Carlo (SMC) based algorithms. GM implementations suffer from pruning and merging approximations, but enable to extract the states easily; on the other hand, SMC implementations are of interest if the discrete approximation is relevant, but are penalized by the difficulty to guide particles towards promising regions and to extract the states. In this paper, we propose a mixed GM/SMC implementation of the PHD filter which does not suffer from the above mentioned drawbacks. Due to the SMC part, our algorithm can be used in models where the GM implementation is unavailable; but it also benefits from the easy state extraction of GM techniques, without requiring pruning or merging approximations. Our algorithm is validated on simulations.

1. INTRODUCTION

Multi-target filtering consists in estimating the random states of an unknown number of targets from a set of observations which are either due to detected targets or are false alarms measurements. This problem has been studied for a long time now and has received much attention since the introduction of Random Finite Sets (RFS). RFS are sets of random variables with random and time-varying cardinal [1], and enable to avoid the use of an association mechanism between observations and targets, used in classical multi-target filters such as the Joint Probabilistic Data Association (JPDA) algorithm [2] or the Multiple Hypothesis Tracker (MHT) [3]. Direct implementations of RFS based solutions such as the Bayesian multi target filter (MTF) are generally not computable. Therefore Mahler proposed to propagate the so-called PHD or intensity, a positive density function which operates in the single target state space domain and which enables to deduce the number of targets as well as the states of each target.

The PHD propagation formula still involves the computation of complex integrals and thus the PHD cannot be computed exactly either. In practice, two implementations of the PHD filter, with variants, are popular. On the one hand, the GM implementation [4] assumes that the PHD is a GM, but requires that each target and associated measurement (when it is detected) follow a linear and Gaussian model; on the other hand, the SMC implementation of the PHD [5] consists in approximating the PHD by a set of weighted particles and does not require any assumption as regards the dynamics of the targets.

Even in the Gaussian and linear case, the PHD filter is not exactly computable. The GM implementation relies on approximations such as pruning and merging, and it can sometimes be necessary to limit the number of Gaussians of the mixture. This is due to the fact that the number of Gaussians grows exponentially with time. More precisely, if J_{k-1} is the number of Gaussians at time $k-1$ and $|Z_k|$ the number of measurements available at time k then even in the simple case where there is no birth and no spawning, the computation of the PHD a time k requires the computation of J_{k-1} prediction steps and $J_{k-1} \times |Z_k|$ update steps of the Kalman Filter (KF). On the other hand, the extraction of states is easy since it consists in looking for Gaussians with high weight.

By contrast, pruning and merging approximations are not necessary in SMC implementations. However, as in SMC methods for single object filtering, it may be difficult to guide particles into regions where targets are present and so the choice of the sampling importance distribution is critical. Consequently, a poor choice of the importance distribution can lead to unreliable estimates of the number of targets and states. Finally, it has been argued in [4] that contrary to the GM implementation, the extraction of states is a difficult task, since it requires clustering techniques which can be unreliable in some situations. However some practical heuristics have been proposed in [6] and improve classical clustering techniques.

In this paper, we combine the GM and SMC implementations of the PHD filter in order to take advantage of both implementations. More precisely, we propagate a Monte Carlo (MC) approximation of the PHD filter, which enables to avoid the use of pruning and merging approximations, and given an MC approximation of the PHD at time $k-1$, we derive an estimate of the number of targets and states at time k based on the procedure used in the GM implementation. Thus, we do not need to use clus-

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tering techniques to extract states, and we reduce the role of the importance distribution since our estimates do not rely on an MC approximation of the PHD at time k. We first focus on the Gaussian and linear model and we show that our approach enables to relax some assumptions used in the GM implementation; in particular, the probability of survival of targets does not need to be constant and the birth intensity of targets may not be a GM. Next we observe that our algorithm can be used in semi-linear Gaussian models, which is not the case for purely GM based implementations. Finally, our algorithm can be extended to non-linear Gaussian models up to some numerical approximations.

The paper is organized as follows. In §2 we recall the general PHD filter and its GM and SMC implementations. In §3 we derive our combined GM-SMC PHD filter for Gaussian and linear models, and then extend it to more general models. Finally in §4 compare the three implementations (SMC, GM and GM-SMC) via computer simulations.

2. CLASSICAL IMPLEMENTATIONS OF THE PHD FILTER

A PHD is a real and positive function which has the following property: let $S \subset \mathbb{R}^m$, then

$$\int_S v(\mathbf{x})\,\mathrm{d}\mathbf{x} = \mathrm{E}(|X \cap S|), \tag{1}$$

where $|X \cap S|$ is the cardinal of the set of the targets which belong to region S. In other words, the integral of PHD $v(\mathbf{x})$ over region S is the expected number of objects in this region.

In this framework, the multi-target filtering problem can be seen as the propagation of PHD $v_{k|k}$, i.e. the PHD of the set of targets at time k, given all observations up to time k. The so-called PHD filter, which is a set of equations which propagate recursively the PHD, can be derived under the following hypotheses [1]: targets evolve and generate measurements independently of one another; the clutter is independent of measurements due to detected targets and the clutter and the number of targets follow a Poisson distribution. If we note $f_{k|k-1}(\mathbf{x}_k|\mathbf{x}_{k-1})$ the transition pdf from state $\mathbf{x}_{k-1}$ to $\mathbf{x}_k$; $g_k(\mathbf{z}_k|\mathbf{x}_k)$ the likelihood of a measurement $\mathbf{z}_k$ with state $\mathbf{x}_k$; $p_{s,k}(\mathbf{x}_{k-1})$ the probability that a target with state parameters $\mathbf{x}_{k-1}$ at time $k-1$ still exists at time k; $p_{d,k}(\mathbf{x}_k)$ the probability that a target with state parameters $\mathbf{x}_k$ is detected at time k; κ_k the intensity of the clutter measurements at time k and γ_k the intensity of birth targets at time k, the propagation of PHD $v_{k|k}(\mathbf{x})$ is the succession of a prediction step and of an updating step (we assume for simplicity, but without loss of generality, that there is not spawning) [1]:

$$v_{k|k-1}(\mathbf{x}) = \int p_{s,k}(\mathbf{x}_{k-1}) f_{k|k-1}(\mathbf{x}|\mathbf{x}_{k-1}) \times$$
$$v_{k-1|k-1}(\mathbf{x}_{k-1})\mathrm{d}\mathbf{x}_{k-1} + \gamma_k(\mathbf{x}), \tag{2}$$
$$v_{k|k}(\mathbf{x}) = [1 - p_{d,k}(\mathbf{x})]\, v_{k|k-1}(\mathbf{x}) +$$
$$\sum_{\mathbf{z} \in Z_k} \frac{p_{d,k}(\mathbf{x}) g_k(\mathbf{z}|\mathbf{x}) v_{k|k-1}(\mathbf{x})}{\kappa_k(\mathbf{z}) + \int p_{d,k}(\mathbf{x}) g_k(\mathbf{z}|\mathbf{x}) v_{k|k-1}(\mathbf{x})\,\mathrm{d}\mathbf{x}}. \tag{3}$$

2.1. GM Implementation of the PHD filter [4]

Let us now assume that the model is linear and Gaussian, i.e that $f_{k|k-1}(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}(\mathbf{x}_k; \mathbf{F}_k\mathbf{x}_{k-1}; \mathbf{Q}_k)$ and $g_k(\mathbf{z}|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}; \mathbf{H}_k\mathbf{x}_k; \mathbf{R}_k)$, where $\mathcal{N}(\mathbf{x}; \mathbf{m}; \mathbf{P})$ is the Gaussian pdf with variable $\mathbf{x}$, mean $\mathbf{m}$ and covariance $\mathbf{P}$. We also assume that the probabilities of survival $p_{s,k}$ and of detection $p_{d,k}$ do not depend on the state, and that the PHD at time $k-1$ and the birth intensity are GM:

$$v_{k|k-1}(\mathbf{x}) = \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} \mathcal{N}(\mathbf{x}; \mathbf{m}_{k-1}^{(i)}; \mathbf{P}_{k-1}^{(i)}), \tag{4}$$
$$\gamma_k(\mathbf{x}) = \sum_{i=1}^{J_{\gamma_k}} w_{\gamma_k}^{(i)} \mathcal{N}(\mathbf{x}; \mathbf{m}_{\gamma_k}^{(i)}; \mathbf{P}_{\gamma_k}^{(i)}). \tag{5}$$

Plugging (4) and (5) in (2), we get a new GM of $J_{k|k-1} = J_{k-1} + J_{\gamma_k}$ components:

$$v_{k|k-1}(\mathbf{x}) = \sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} \mathcal{N}(\mathbf{x}; \mathbf{m}_{k|k-1}^{(i)}; \mathbf{P}_{k|k-1}^{(i)}). \tag{6}$$

The first J_{k-1} components are obtained with the prediction step of the KF and the first J_{k-1} weights are $w_{k|k-1}^{(i)} = p_{s,k} w_{k-1|k-1}^{(i)}$.

Next, $v_{k|k}(\mathbf{x})$ is the sum of the reweighted predicted GM $v_{k|k-1}(\mathbf{x})$, and of a new GM of $J_{k|k-1} \times |Z_k|$ components obtained by applying the update step of the KF on the predicted GM with each measurement $\mathbf{z} \in Z_k$, and by deriving appropriate weights, see [4] for further details.

In order to keep this GM tractable several approximations studied in [7] have to be done:

- a Gaussian with weight $w_{k|k}^{(i)}$ is under a given threshold T_p is deleted;
- given a threshold T_u, Gaussians which are too close are merged;
- if after these two steps, the number of components is above a given threshold J_{max}, only the J_{max} Gaussians with highest weights are kept.

Finally, an estimate of the number of targets is given by the sum of the weights of the mixture, and a Gaussian is considered as a target when its weight is above a given threshold, typically 0.5.

2.2. SMC Implementation of the PHD filter [5]

Let us now focus on the SMC implementation of the PHD filter. Assume that an approximation of $v_{k-1|k-1}$ is given by

$$\hat{v}_{k-1|k-1}(\mathbf{x}_{k-1}) = \sum_{i=1}^{L_{k-1}} w_{k-1}^{(i)} \delta_{\mathbf{x}_{k-1}^{(i)}}(\mathbf{x}_{k-1}). \quad (7)$$

Then by plugging this approximation in (2), we get an approximation of the predicted PHD:

$$\tilde{v}_{k|k-1}(\mathbf{x}_k) = \sum_{i=1}^{L_{k-1}} p_{s,k}(\mathbf{x}_{k-1}^{(i)}) w_{k-1}^{(i)} f_{k|k-1}(\mathbf{x}_k|\mathbf{x}_{k-1}^{(i)}) + \gamma_k(\mathbf{x}_k). \quad (8)$$

Next, let $q(\mathbf{x}_k|\mathbf{x}_{k-1})$ and $q'(\mathbf{x}_k)$ be two importance distributions. By sampling $\mathbf{x}_k^{(i)} \sim q(\mathbf{x}_k|\mathbf{x}_{k-1}^{(i)})$ for $1 \leq i \leq L_{k-1}$ and $\mathbf{x}_k^{(i)} \sim q'(\mathbf{x}_k)$ for $L_{k-1}+1 \leq i \leq L_{k-1} + L_{\gamma_k} = L'_k$, we get a discrete approximation $\hat{v}_{k|k-1}(\mathbf{x}_k) = \sum_{i=1}^{L'_k} w_{k|k-1}^{(i)} \delta_{\mathbf{x}_k^{(i)}}(\mathbf{x}_k)$ of $\tilde{v}_{k|k-1}$ where

$$w_{k|k-1}^{(i)} = w_{k-1}^{(i)} \times \frac{p_{s,k}(\mathbf{x}_{k-1}^{(i)}) f_{k|k-1}(\mathbf{x}_k^{(i)}|\mathbf{x}_{k-1}^{(i)})}{q(\mathbf{x}_k^{(i)}|\mathbf{x}_{k-1}^{(i)})}, 1 \leq i \leq L_{k-1}, \quad (9)$$

$$w_{k|k-1}^{(i)} = \frac{\gamma_k(\mathbf{x}_k^{(i)})}{L_{\gamma_k} \times q'(\mathbf{x}_k^{(i)})}, L_{k-1}+1 \leq i \leq L'_k. \quad (10)$$

Finally, by plugging $\hat{v}_{k|k-1}$ in (3), we get a discrete approximation $\hat{v}_{k|k}$ of the updated PHD $v_{k|k}$ given by

$$\hat{v}_{k|k}(\mathbf{x}_k) = \sum_{i=1}^{L'_k} w_k^{(i)} \delta_{\mathbf{x}_k^{(i)}}(\mathbf{x}_k), \quad (11)$$

where

$$w_k^{(i)} = (1 - p_{d,k}(\mathbf{x}_k^{(i)})) w_{k|k-1}^{(i)} + \sum_{\mathbf{z} \in Z_k} \frac{w_{k|k-1}^{(i)} p_{d,k}(\mathbf{x}_k^{(i)}) g_k(\mathbf{z}|\mathbf{x}_k^{(i)})}{\kappa(\mathbf{z}) + \sum_{i=1}^{L_{k|k-1}} w_{k|k-1}^{(i)} p_{d,k}(\mathbf{x}_k^{(i)}) g_k(\mathbf{z}|\mathbf{x}_k^{(i)})}. \quad (12)$$

One can possibly turn the set $\{w_k^{(i)}, \mathbf{x}_k^{(i)}\}_{i=1}^{L'_k}$ into a set of L_k particles by using a resampling step.

An estimate of the number of targets is then given by $\sum_{i=1}^{N} w_k^{(i)}$ but it is difficult to extract states. Clustering techniques are required to find regions where particles are concentrated. Another approach developed in [6] consists in looking measurements $\mathbf{z}$ such that

$$\sum_{i=1}^{L'_k} \frac{w_{k|k-1}^{(i)} p_{d,k}(\mathbf{x}_k^{(i)}) g_k(\mathbf{z}|\mathbf{x}_k^{(i)})}{\kappa(\mathbf{z}) + \sum_{i=1}^{L_{k|k-1}} w_{k|k-1}^{(i)} p_{d,k}(\mathbf{x}_k^{(i)}) g_k(\mathbf{z}|\mathbf{x}_k^{(i)})} \quad (13)$$

is above a given threshold. In this case, an estimate of the state associated to the measurement $\mathbf{z}$ is given by

$$\sum_{i=1}^{L'_k} \frac{w_{k|k-1}^{(i)} p_{d,k}(\mathbf{x}_k^{(i)}) g_k(\mathbf{z}|\mathbf{x}_k^{(i)})}{\kappa(\mathbf{z}) + \sum_{i=1}^{L_{k|k-1}} w_{k|k-1}^{(i)} p_{d,k}(\mathbf{x}_k^{(i)}) g_k(\mathbf{z}|\mathbf{x}_k^{(i)})} \mathbf{x}_k^{(i)}. \quad (14)$$

3. A MIXED GM-SMC IMPLEMENTATION OF THE PHD FILTER

We now develop an alternative approach to the GM and SMC implementations. We first focus on Gaussian and linear models (see §2.1). As in the GM implementation we assume that the probability of detection $p_{d,k}(\mathbf{x}_k) = p_{d,k}$ is constant and that γ_k is a GM with N_{γ_k} components; however, we do not need to assume that $p_{s,k}(\mathbf{x}_{k-1})$ is constant. As in SMC implementations, we assume that an approximation of $v_{k-1|k-1}$ is given by (7).

3.1. Prediction Step

Plugging the approximation $\hat{v}_{k-1|k-1}$ in (2), we get the approximation (8), which actually is a GM of $L_{k|k-1} = L_{k-1} + J_{\gamma_k}$ components:

$$\tilde{v}_{k|k-1}(\mathbf{x}) = \sum_{i=1}^{L_{k|k-1}} w_{k|k-1}^{(i)} \mathcal{N}(\mathbf{x}; \mathbf{m}_{k|k-1}^{(i)}; \mathbf{P}_{k|k-1}^{(i)}), \quad (15)$$

in which $w_{k|k-1}^{(i)} = p_{s,k}(\mathbf{x}_{k-1}^{(i)}) w_{k-1|k-1}^{(i)}$, $\mathbf{m}_{k|k-1}^{(i)} = \mathbf{F}_k \times \mathbf{x}_{k-1}^{(i)}$, $\mathbf{P}_{k|k-1}^{(i)} = \mathbf{Q}_k$ for $1 \leq i \leq L_{k-1}$, and other components are given by the GM γ_k.

3.2. Update Step

Next, as in the GM implementation, we derive an approximation of $v_{k|k}(\mathbf{x})$ without sampling new particles. Let

$$B(\mathbf{z}) = \kappa(\mathbf{z}) + \int p_{d,k} g_k(\mathbf{z}|\mathbf{x}) \tilde{v}_{k|k-1}(\mathbf{x}) \mathrm{d}\mathbf{x} \quad (16)$$

$$= \kappa(\mathbf{z}) + \sum_{i=1}^{L_{k|k-1}} p_{d,k} w_{k|k-1}^{(i)} q_k^{(i)}(\mathbf{z}), \quad (17)$$

where

$$q_k^{(i)}(\mathbf{z}) = \mathcal{N}(\mathbf{z}; \mathbf{H}_k \mathbf{m}_{k|k-1}^{(i)}; \mathbf{R}_k + \mathbf{H}_k \mathbf{Q}_k \mathbf{H}_k^T). \quad (18)$$

Then by plugging the approximation (15) of $v_{k|k-1}(\mathbf{x})$ in (3) and using classical results on Gaussian variables, we get a GM approximation of $v_{k|k}(\mathbf{x})$ given by

$$\tilde{v}_{k|k}(\mathbf{x}) = (1 - p_{d,k}) \tilde{v}_{k|k-1}(\mathbf{x}) + \sum_{\mathbf{z} \in Z_k} \frac{\sum_{i=1}^{L_{k|k-1}} p_{d,k} w_{k|k-1}^{(i)} q^{(i)}(\mathbf{z}) \psi_k^{(i)}(\mathbf{x}, \mathbf{z})}{B(\mathbf{z})}, \quad (19)$$

where

$$\psi_k^{(i)}(\mathbf{x}, \mathbf{z}) = \mathcal{N}(\mathbf{x}; \mathbf{m}_k^{(i)}(\mathbf{z}); \mathbf{P}_k^{(i)}), \quad (20)$$

$$\mathbf{m}_k^{(i)}(\mathbf{z}) = \mathbf{m}_{k|k-1}^{(i)} + \mathbf{K}_k^{(i)}(\mathbf{z} - \mathbf{H}_k \mathbf{m}_{k|k-1}^{(i)}), \quad (21)$$

$$\mathbf{P}_k^{(i)} = (\mathbf{I} - \mathbf{K}_k^{(i)} \mathbf{H}_k) \mathbf{P}_{k|k-1}^{(i)}, \quad (22)$$

$$\mathbf{K}_k^{(i)} = \mathbf{P}_{k|k-1}^{(i)} \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_{k|k-1}^{(i)} \mathbf{H}_k^T + \mathbf{R}_k)^{-1}. \quad (23)$$

3.3. Estimating the number of targets and extracting the states

An estimate of the number of targets is given by $\hat{N}_k = \int \tilde{v}_{k|k}(\mathbf{x})\mathrm{d}\mathbf{x}$ which is computable here:

$$\hat{N}_k = (1 - p_{d,k}) \sum_{i=1}^{L_{k|k-1}} w_{k|k-1}^{(i)} + \sum_{\mathbf{z}\in Z_k} \frac{\sum_{i=1}^{L_{k|k-1}} p_{d,k} w_{k|k-1}^{(i)} q^{(i)}(\mathbf{z})}{B(\mathbf{z})}. \quad (24)$$

Two options are available to extract states:

- The first one consists in using the procedure used in the GM implementation, i.e. in first merging close Gaussians, then in looking for Gaussians with high weights.
- Otherwise we first look for measurements $\mathbf{z}$ such that $\frac{\sum_{i=1}^{L_{k|k-1}} p_{d,k} w_{k|k-1}^{(i)} q_k^{(i)}(\mathbf{z})}{B(\mathbf{z})}$ is above a given threshold. Then we deduce an estimate of the state associated to this measurement given by $\frac{1}{B(\mathbf{z})} \sum_{i=1}^{L_{k|k-1}} p_{d,k} w_{k|k-1}^{(i)} q_k^{(i)}(\mathbf{z}) \mathbf{m}_k^{(i)}(\mathbf{z})$. This method has the advantage to involve the computation of $\mathbf{m}_k^{(i)}(\mathbf{z})$ only for such measurements $\mathbf{z}$.

Note that by contrast with the SMC implementation of the PHD filter, our estimates of the number of targets and of the states do not depend on a weighted set of samples $\{w_k^{(i)}, \mathbf{x}_k^{(i)}\}_{i=1}^{L_k}$.

3.4. SMC approximation of the PHD at time k

We finally derive a discrete approximation $\hat{v}_{k|k}(\mathbf{x}) = \sum_{i=1}^{L_k} w_k^{(i)} \delta_{\mathbf{x}_k^{(i)}}(\mathbf{x})$ of $v_{k|k}$ (remember that our approach requires an MC approximation of the PHD at each time step), by using either the method described in §2.2 or other SMC methods; for example, the Auxiliary PHD filter [8] uses the expression of $\tilde{v}_{k|k}(\mathbf{x})$ to sample particles.

3.5. Discussion and extensions

The previous paragraph was devoted to Gaussian and linear models. Note however that contrary to the GM implementation, the probability of survival $p_{s,k}(\mathbf{x}_{k-1})$ can indeed depend on $\mathbf{x}_{k-1}$. In addition, our approach can be used if one can compute $q_k^{(i)}(\mathbf{z}) = \int g_k(\mathbf{z}|\mathbf{x}) f_{k|k-1}(\mathbf{x}|\mathbf{x}_{k-1}^{(i)})\mathrm{d}\mathbf{x}$ and so $\psi_k^{(i)}(\mathbf{x},\mathbf{z}) = g_k(\mathbf{z}|\mathbf{x})\, f_{k|k-1}(\mathbf{x}|\mathbf{x}_{k-1}^{(i)}) \times 1/q_k^{(i)}(\mathbf{z})$ (see (18) and (20)).

3.5.1. Gaussian but semi-linear models

Note that, these terms are also computable in a wider class of models. Assume for instance that $f_{k|k-1}(\mathbf{x}|\mathbf{x}_{k-1})$ and $g_{k|k}(\mathbf{z}|\mathbf{x})$ are Gaussians, that the mean of $g_{k|k}$ is linear in $\mathbf{x}$ but that the mean of $f_{k|k-1}$ is nonlinear in $\mathbf{x}_{k-1}$. Then the GM implementation is not computable because it is not possible to compute (2), while the GM-SMC approach remains computable for such models since $q_k^{(i)}(\mathbf{z})$ and $\psi_k^{(i)}(\mathbf{x},\mathbf{z})$ are indeed computable.

3.5.2. About alternate birth intensities

Up to now the birth intensity γ_k was assumed to be a GM. However, one can adapt our approach to any birth intensity. If $\int g_k(\mathbf{z}|\mathbf{x})\gamma_k(\mathbf{x})\mathrm{d}\mathbf{x}$ is not computable, then one can derive an MC approximation $\hat{\gamma}_k$ of γ_k, using the procedure described in §2.2. An approximation of the predicted PHD is now given by the sum of a GM and of the MC approximation $\hat{\gamma}_k$ of γ_k:

$$\tilde{v}_{k|k-1} = \sum_{i=1}^{L_{k-1}} w_{k|k-1}^{(i)} \mathcal{N}(\mathbf{x}; \underbrace{\mathbf{F}_k \mathbf{x}_{k-1}^{(i)}}_{\mathbf{m}_{k|k-1}^{(i)}}; \underbrace{\mathbf{Q}_k}_{\mathbf{P}_{k|k-1}^{(i)}}) + \hat{\gamma}_k, \quad (25)$$

where weights $w_{k|k-1}^{(i)} = p_{s,k}(\mathbf{x}_{k-1}^{(i)}) w_{k-1|k-1}^{(i)}$ and $\hat{\gamma}_k = \sum_{i=1}^{L_{\gamma_k}} w_{\gamma_k}^{(i)} \delta_{\mathbf{x}_{\gamma_k}^{(i)}}$. The term $B(\mathbf{z})$ in (16) should be modified and is computed from the the MC approximation of $\int g_k(\mathbf{z}|\mathbf{x}) \times \gamma_k(\mathbf{x})\mathrm{d}\mathbf{x}$. We have now

$$B(\mathbf{z}) = \kappa(\mathbf{z}) + \sum_{i=1}^{L_{k-1|k-1}} p_{d,k} w_{k|k-1}^{(i)} q_k^{(i)}(\mathbf{z}) + \sum_{i=1}^{L_{\gamma_k}} w_{\gamma_k}^{(i)} g_k(\mathbf{z}|\mathbf{x}_{\gamma_k}^{(i)}). \quad (26)$$

Finally, an approximation of $v_{k|k}$ is of the following form:

$$\tilde{v}_{k|k}(\mathbf{x}) = \tilde{v}_{k|k}^1(\mathbf{x}) + \tilde{v}_{k|k}^2(\mathbf{x}), \quad (27)$$

where

$$\tilde{v}_{k|k}^1(\mathbf{x}) = (1 - p_{d,k}) \sum_{i=1}^{L_{k-1|k-1}} w_{k|k-1}^{(i)} \mathcal{N}(\mathbf{x}; \mathbf{m}_{k|k-1}^{(i)}; \mathbf{P}_{k|k-1}^{(i)}) + \sum_{\mathbf{z}\in Z_k} \frac{p_{d,k} \sum_{i=1}^{L_{k-1|k-1}} w_{k|k-1}^{(i)} q_k^{(i)}(\mathbf{z}) \psi_k^{(i)}(\mathbf{x},\mathbf{z})}{B(\mathbf{z})}, \quad (28)$$

where $q_k^{(i)}(\mathbf{z})$ and $\psi_k^{(i)}(\mathbf{x},\mathbf{z})$ are respectively defined in (18) and (20), and

$$\tilde{v}_{k|k}^2(\mathbf{x}) = (1 - p_{d,k}) \sum_{i=1}^{L_{\gamma_k}} w_{\gamma_k}^{(i)} \delta_{\mathbf{x}_{\gamma_k}^{(i)}} + \sum_{\mathbf{z}\in Z_k} \sum_{i=1}^{L_{\gamma_k}} w_{\gamma_k}^{(i)} \frac{p_{d,k} g_k(\mathbf{z}|\mathbf{x}_{\gamma_k}^{(i)}) \delta_{\mathbf{x}_{\gamma_k}^{(i)}}}{B(\mathbf{z})}. \quad (29)$$

The term $\tilde{v}_{k|k}^1$ is due to persistent targets while $\tilde{v}_{k|k}^2$ is due to birth targets. In this case, one can use the procedure used in the GM implementation to extract states due to persistent targets and that used in the SMC implementation to extract states due to birth targets. However, if we want to avoid the use of an SMC extraction procedure, one can only extract states due to persistent targets; the birth targets at time k become persistent target at time $k+1$ and so their extraction will become easy at the next iteration.

3.5.3. Non linear Gaussian models

Finally, there are some models for which the expressions of $q_k^{(i)}(\mathbf{z})$ and $\psi_k^{(i)}(\mathbf{x},\mathbf{z})$ are not computable. Well known techniques such as linearization and Unscented Transformation can be used to approximate $q_k^{(i)}(\mathbf{z})$ and $\psi_k^{(i)}(\mathbf{x},\mathbf{z})$ [9]. When they are used sequentially, they lead to the Extended Kalman or Unscented Kalman implementations of the PHD filter [4]. Of course, in an oriented GM-SMC implementation, they are just used locally to estimate the number of targets and their state, but not to compute and propagate a GM approximation of the PHD.

4. SIMULATIONS

4.1. The Optimal Subpattern Assignment (OSPA) metric

The OSPA metric enables to compare multi-target filtering algorithms [10]. Let $X = \{x_1, ..., x_m\}$ and $Y = \{y_1, ..., y_k\}$ be two finite sets. X represents the estimated finite set of the targets and Y represents the true finite set of the targets. For $1 \leq p < +\infty$ and $c > 0$, we denote $d^{(c)}(x,y) = min(c, ||x-y||)$ ($||.||$ is the euclidean norm) and Π_k the set of permutations on $\{1,2,...,n\}$. The OSPA metric is defined by :

$$\overline{d}_p^c(X,Y) \triangleq \left(\frac{1}{n}\left(\min_{\pi\in\Pi_k}\sum_{i=1}^{m} d^{(c)}(x_i, y_{\pi(i)})^p + c^p(n-m)\right)\right)^{\frac{1}{p}} \quad (30)$$

if $m \leq n$ and $\overline{d}_p^c(X,Y) \triangleq \overline{d}_p^c(Y,X)$ if $m > n$. We use $p = 2$ and $c = 100$ in our simulations.

4.2. Linear and Gaussian Model

We track the position and velocity of targets, so $\mathbf{x}_k = [p_x, \dot{p}_x, p_y, \dot{p}_y]_k^T$. We take $f_{k|k-1}(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}(\mathbf{x}_k, \mathbf{F}\mathbf{x}_{k-1}, \mathbf{Q})$ and $g_k(\mathbf{z}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{z}_k, \mathbf{H}\mathbf{x}_k, \mathbf{R})$ where

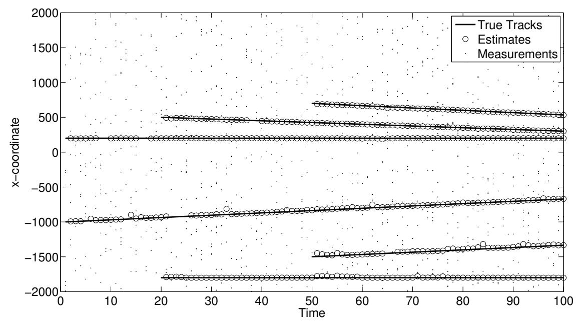
$$\mathbf{F} = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\mathbf{Q} = \sigma_v^2 \begin{bmatrix} \frac{T^3}{3} & \frac{T^2}{2} & 0 & 0 \\ \frac{T^2}{2} & T & 0 & 0 \\ 0 & 0 & \frac{T^3}{3} & \frac{T^2}{2} \\ 0 & 0 & \frac{T^2}{2} & T \end{bmatrix} \text{ and } \mathbf{R} = \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}.$$

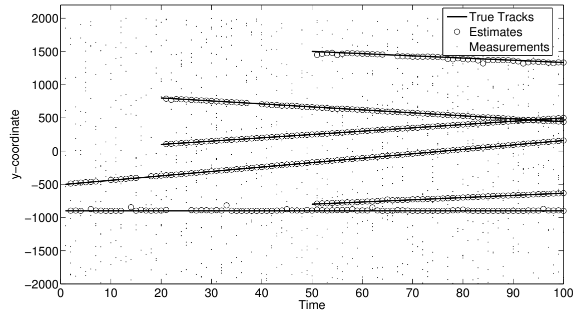
Other parameters are $T = 2s$, $\sigma_v = 3m^2/sec^3$ but $\sigma_x = \sigma_y = 0.3m$, which means that likelihood $g_k(\mathbf{z}|\mathbf{x}_k)$ is sharp. This is challenging for the SMC-PHD implementation above when the sampling importance distribution is the transition pdf $f_{k|k-1}$ $(\mathbf{x}_k|\mathbf{x}_{k-1})$: it does not take into account the available measurements $\mathbf{z} \in Z_k$ and so few particles are in regions where terms $g_k(\mathbf{z}|\mathbf{x}_k)$ are large, even if the measurement $\mathbf{z}$ is due to a target. So few particles are in regions where targets are present, and the estimates of the number of targets and of their states are not accurate.

We compare the SMC and GM-SMC implementations of the PHD filter, and both algorithms use the transition density to sample particles (remember that in our combined approach, one should propagate the discrete approximation, even if it not used for computing an estimator of the number of targets, see §3.4). Particles are initialized around the measurements [6], we use $N_b = 20$ particles per newborn targets and $N = 200$ particles per persistent targets in both algorithms. The probability of detection is $p_{d,k} = 0.95$ for $1 \leq k \leq 100$, the probability of survival is $p_{s,k} = 0.98$ for $1 \leq k \leq 100$, and we generate 10 false alarm measurements (in mean).

We also implement the GM implementation, with $T_p = 10^{-5}$ for the pruning threshold, $T_m = 4m$ for the merging threshold and we keep at most $N_{\max} = 100$ Gaussians. A scenario with 6 targets which appear at $k = 0$, $k = 20$ and $k = 50$ is presented in Fig. 1.



(a)



(b)

Fig. 1. Scenario with 6 targets - True tracks, estimates and measurements

The OSPA distance and the estimated number of targets are displayed in Fig. 2 and Fig. 3. The combined approach outperforms the SMC one and copes with the issue of guiding particles in promising regions. Indeed, even if we use the transition density for getting a discrete approximation of $v_{k-1|k-1}$, we get a correct estimator of the number of targets when we use the GM-SMC algorithm by contrast with the SMC one where the new set $\{w_k^{(i)}, \mathbf{x}_k^{(i)}\}_{i=1}^{L_k}$ is used to deduce a discrete approximation of $v_{k|k}$ and then an estimator of the number of targets. It means that for a same MC approximation of the

PHD at time $k-1$, which actually is not accurate approximation since MC samples are not in regions where targets are present (see Fig. 3), the GM-SMC approach succeeds in estimating the true number of targets and their states.

On the other hand, the GM-SMC approach also outperforms the GM implementation in terms of OSPA distance, above all when time increases and so when the approximations of the GM implementation are severe since the true PHD is a GM with more than N_{max} components. Finally, the number of targets is well estimated by the GM and GM-SMC algorithms, but the GM-SMC estimator is more accurate, see Fig. 4.

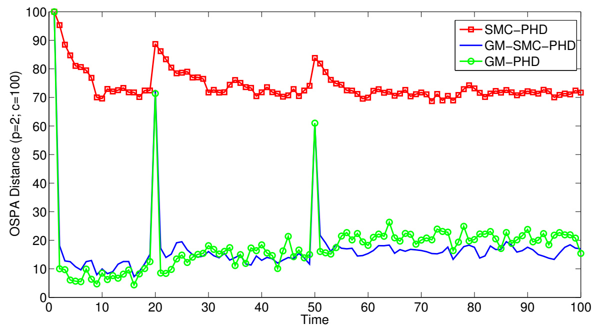


Fig. 2. OSPA distance

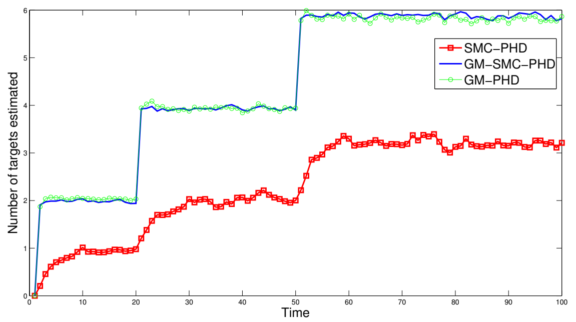


Fig. 3. Estimator of the number of targets

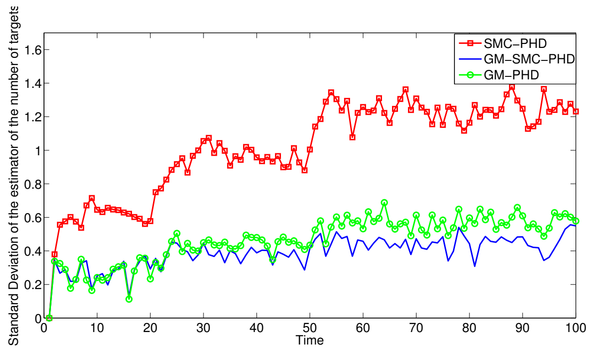


Fig. 4. Standard Deviation of the estimator of the number of targets

5. CONCLUSION

We proposed a mixed GM-SMC implementation of the PHD filter. Starting from an MC approximation of the PHD at time $k-1$, we showed that one can estimate the number of targets and the states by using the extraction procedure of the GM implementation. Estimates at time k do not rely on the MC approximation of the PHD at time k and so are less responsive to the critical choice of the importance distribution. More importantly, the approach remains valid when we relax assumptions of the GM implementation and in some non-linear Gaussian models. Approximations techniques have been proposed when the approach is not directly computable. Simulations confirmed that our GM-SMC method outperforms both classical GM and SMC implementations.

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