MULTI-OBJECT FILTERING FOR PAIRWISE MARKOV CHAINS

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ABSTRACT

The Probability Hypothesis Density (PHD) Filter is a recent solution to the multi-target filtering problem which consists in estimating an unknown number of targets and their states. The PHD filter equations are derived under the assumption that the dynamics of the targets and associated observations follow a Hidden Markov Chain (HMC) model. HMC models have been recently extended to Pairwise Markov Chains (PMC) models. In this paper, we focus on multi-target filtering when targets and associated measurements follow a PMC model, and we extend the classical PHD filter to such models. We also propose a Gaussian Mixture (GM) implementation of our PMC PHD filter for linear and Gaussian PMC. Our approach enables to extend multi-object filtering to more general tracking scenarios, and also enables to deduce an estimate of the measurement associated to each target.

1. INTRODUCTION

Single object target filtering has been studied for a long time now, beginning with the pionnering work of Kalman in the 1960s. Bayesian filtering has next been developed in complex non linear and / or non Gaussian stochastic models; available solutions include extended or unscented Kalman filter methods as well as sequential Monte Carlo (SMC) algorithms [1]. Most of these works assume that the state and the associated observation are an HMC, i.e. the state is a Markov chain (MC), and conditionally to a sequence of states, the observations are independent and the observation at time k depends only on the state at time k. Recent works have extended single object filtering from HMC to a more general class of models called PMC. In such models we only assume that the pair (hidden state and associated observation) follows a (vector) MC. As a result, the marginal hidden state process is not necessarily Markovian; and conditionally on a sequence of states, the observations are Markovian, but not necessarily independent. Kalman Filtering (KF) and Particle Filtering (PF) have been extended from HMC to PMC [2] [3] [4].

On the other hand, the problematic of multi-target filters regained interest thanks to a new approach based on Random Finite Sets (RFS) which are sets of random variables with random and time-varying cardinal [5] [6]. The interest of RFS based methods is that they avoid the use of a complex association mechanism which aims at optimizing the association between observations and targets. The PHD Filter is one such RFS based solution. It consists in propagating the PHD (or intensity), which is a positive density function operating in the single target state space domain and which enables to deduce the number of targets as well as the states of each target. PHD-based solutions involve the computation of complex integrals and thus the PHD cannot be computed exactly. In practice, two implementations of the PHD filter, with variants, are popular. On the one hand, the GM implementation [7] assumes that the PHD is a GM, but requires that each target (when it is detected) and associated measurement follow a linear and Gaussian model; on the other hand, the SMC implementation of the PHD [8] [9] consists in approximating the PHD by a set of weighted particles and does not require any assumption as regards the dynamics of the targets.

However, both the GM and SMC implementations are based on the assumption that the state of a target and its associated measurement follow an HMC model. In this paper, we thus focus on multi-object filtering in the case where each target and the associated measurement follow a PMC model, and we extend the classical PHD filter for tracking a random number of PMC. The practical interest of this extension is twofold. First, it enables to take into account more complex tracking situations. For instance, in a PMC model the pdf of a state $\mathbf{x}_k$ at time k given $\mathbf{x}_{k-1}$ can also depend on the associated measurement at time $k-1$, $\mathbf{y}_{k-1}$, and given $\mathbf{x}_k$ the associated measurement $\mathbf{y}_k$ can also depend on $\mathbf{x}_{k-1}$ and $\mathbf{y}_{k-1}$. The second advantage is that the PMC formulation of the PHD filter enables to deduce explicitly the measurements associated to each estimated state, even in the particular case where the PMC model reduces to an HMC one, which can be of interest in tracking applications. Finally, as in HMC models, the PMC-PHD filter cannot be computed exactly, so we propose a GM implementation of the PHD filter for linear and Gaussian PMC.

The paper is organized as follows. In Section 2 we first recall the multi-target filtering problem in terms of RFS. Section 3 is devoted to PMC models and to linear and Gaussian PMC models in which exact single-object filtering is feasible. In Section 4, we extend the PHD filter from HMC to PMC models, and we derive a GM implementation of the PHD filter for linear and Gaussian PMC. We perform simulations in section 5.

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2. MULTI-TARGET FILTERING

2.1. RFS description of the model

In a standard multi-object tracking problem both the number of targets and their states are unknown at a given time k. It is thus convenient to represent the set of targets by a RFS (i.e. a set of random vectors with random cardinal) $X_k = \{\mathbf{x}_{k,1}, \cdots, \mathbf{x}_{k,n}\}$, where n is an unknown integer and $\mathbf{x}_{k,i}$ is a vector of $\mathbb{R}^m$ for all k and i. The measurements at time k can also be represented by RFS $Z_k = \{\mathbf{z}_{k,1}, \cdots, \mathbf{z}_{k,q}\}$, where q is also an unknown (and random) integer and $\mathbf{z}_{k,i} \in \mathbb{R}^p$ for all k and i.

Let us next describe the link between RFS X_{k-1} and RFS X_k. Assume that the set of targets is represented by RFS X_{k-1} at time $k-1$. Then a target at time k is either: a target from RFS X_{k-1} which has survived; a new target which spawns from another target of the previous RFS X_{k-1}; or a new target. These constraints can also be formulated using RFS: let us denote by

- $\underset{\varsigma \in X_{k-1}}{\cup} S_{k|k-1}(\varsigma)$ the set of surviving targets between $k-1$ and k;
- $\underset{\varsigma \in X_{k-1}}{\cup} B_{k|k-1}(\varsigma)$ the set of targets spawned from set X_{k-1};
- Γ_k the set of birth targets.

Then given RFS X_{k-1}, RFS X_k is the union of three RFS:

$$X_k = \left[\underset{\varsigma \in X_{k-1}}{\cup} S_{k|k-1}(\varsigma) \right] \cup \left[\underset{\varsigma \in X_{k-1}}{\cup} B_{k|k-1}(\varsigma) \right] \cup \Gamma_k. \tag{1}$$

We assume in this paper without loss of generality that $B_{k|k-1}(\varsigma) = \emptyset$, i.e. there is not spawning.

We now specify physical constraints for the measurements available at time k and describe the relation between RFS Z_k and X_k. A measurement at time k is either: a measurement due to a detected target of RFS X_k; or a false alarm (or clutter) measurement. These constraints can again be formulated in terms of RFS: let us denote by

- K_k the set of clutter measurements;
- $\underset{\mathbf{x} \in X_k}{\cup} \vartheta(\mathbf{x})$ the set of measurements due to detected targets.

Then the measurements RFS Z_k, conditionally on a realization of the targets RFS X_k, can be described as the union of two RFS:

$$Z_k = K_k \cup \left[\underset{\mathbf{x} \in X_k}{\cup} \vartheta(\mathbf{x}) \right]. \tag{2}$$

Let us finally describe the dynamic behavior of a single surviving target between instants $k-1$ and k, and the relation between a measurement due to a detected target and the target itself. We assume here that the states and corresponding observations follow an HMC model, i.e. that the pdf of the couple ($\mathbf{x}_{0:k} \triangleq \{\mathbf{x}_0 \cdots \mathbf{x}_k\}, \mathbf{y}_{0:k} \triangleq \{\mathbf{y}_0 \cdots \mathbf{y}_k\}$) can be factorized as:

$$p(\mathbf{x}_{0:k}, \mathbf{y}_{0:k}) = p(\mathbf{x}_0) \prod_{i=1}^{k} f_{i|i-1}(\mathbf{x}_i|\mathbf{x}_{i-1}) \times \prod_{i=0}^{k} g_i(\mathbf{y}_i|\mathbf{x}_i). \tag{3}$$

2.2. The PHD filter

A PHD (or intensity) is a real and positive function $v(\mathbf{x})$ which has the following property: let $S \subset \mathbb{R}^m$ and X be an RFS of targets, then

$$\int_S v(\mathbf{x})\, \mathrm{d}\mathbf{x} = \mathrm{E}(|X \cap S|), \tag{4}$$

where $|X \cap S|$ is the cardinal of the set of targets which belong to region S. In other words, the integral of PHD $v(\mathbf{x})$ over region S is the expected number of objects in this region.

In this RFS framework, the multi-target filtering problem can be seen as the propagation of the intensity $v_k(\mathbf{x})$ of the targets at time k given all past measurements up to time k. The so-called PHD filter is a set of equations which propagate recursively the PHD. It can be derived under the following hypotheses [6]: targets evolve and generate measurements independently of one another; the clutter is independent of measurements due to detected targets; the clutter and targets follow a Poisson distribution.

Let us note $p_{s,k}(\mathbf{x})$ the probability that a target with state $\mathbf{x}$ at time $k-1$ still exists at time k; $p_{d,k}(\mathbf{x})$ the probability that a target with state $\mathbf{x}$ is detected at time k; κ_k the PHD of clutter RFS K_k at time k; γ_k the PHD of birth RFS Γ_k at time k. The propagation of the PHD $v_k(\mathbf{x})$ is the succession of a prediction step and of an updating step [6]:

$$v_{k|k-1}(\mathbf{x}) = \int p_{s,k}(\mathbf{x}_{k-1}) f_{k|k-1}(\mathbf{x}|\mathbf{x}_{k-1}) \times$$
$$v_{k-1}(\mathbf{x}_{k-1}) \mathrm{d}\mathbf{x}_{k-1} + \gamma_k(\mathbf{x}), \tag{5}$$
$$v_k(\mathbf{x}) = [1 - p_{d,k}(\mathbf{x})]\, v_{k|k-1}(\mathbf{x}) +$$
$$\sum_{\mathbf{z} \in Z_k} \frac{p_{d,k}(\mathbf{x}) g_k(\mathbf{z}|\mathbf{x}) v_{k|k-1}(\mathbf{x})}{\kappa_k(\mathbf{z}) + \int p_{d,k}(\mathbf{x}) g_k(\mathbf{z}|\mathbf{x}) v_{k|k-1}(\mathbf{x})\, \mathrm{d}\mathbf{x}}. \tag{6}$$

(5)-(6) cannot be computed exactly, except in the linear and Gaussian case where we assume that $v_{k|k}(\mathbf{x})$ is a GM. However, even in this case, the mixture grows exponentially so prunning and merging techniques are necessary. It leads to the GM implementation of the PHD filter which we briefly recall [7]. Let us assume that the probabilities of survival $p_{s,k}(\mathbf{x})$ and of detection $p_{d,k}(\mathbf{x})$ do not depend on $\mathbf{x}$ and that the HMC model is

$$\mathbf{x}_k = \mathbf{F}_k \mathbf{x}_{k-1} + \mathbf{u}_k, \tag{7}$$
$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k, \tag{8}$$

where $\mathbf{u}_1 \cdots, \mathbf{u}_k$ and $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are independent zero-mean Gaussian noise, with $\mathrm{E}(\mathbf{u}_k \mathbf{u}_k^T) = \mathbf{Q}_k^u$ and $\mathrm{E}(\mathbf{v}_k \mathbf{v}_k^T)$

$= \mathbf{R}_k^v$. Let us also assume that intensities $\gamma_k(\mathbf{x}_k)$ and $v_k(\mathbf{x}_k)$ are GM, i.e.

$$v_{k-1}(\mathbf{x}) = \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} \mathcal{N}(\mathbf{x}; \mathbf{m}_{k-1}^{(i)}; \mathbf{P}_{k-1}^{(i)}), \quad (9)$$

$$\gamma_{k-1}(\mathbf{x}) = \sum_{i=1}^{J_\gamma} w_{\gamma_{k-1}}^{(i)} \mathcal{N}(\mathbf{x}; \mathbf{m}_{\gamma_{k-1}}^{(i)}; \mathbf{P}_{\gamma_{k-1}}^{(i)}), \quad (10)$$

where $\mathcal{N}(\mathbf{x}; \mathbf{m}; \mathbf{P})$ is the Gaussian pdf with variable $\mathbf{x}$, mean $\mathbf{m}$ and covariance matrix $\mathbf{P}$. Since $p_{s,k}$ does not depend on $\mathbf{x}_{k-1}$ and $f_{k|k-1}(\mathbf{x}_k|\mathbf{x}_{k-1})$ is a Gaussian pdf, the mean of which is linear in $\mathbf{x}_{k-1}$, if we inject (9) and (10) in (5) we get a new GM. Further injecting this new mixture in (6) still provides a GM.

3. PMC MODELS

3.1. HMC vs PMC models

A pairwise process $\{\xi_k = (\mathbf{x}_k, \mathbf{y}_k)\}_{k \geq 0}$ is called a PMC if $\{\xi_k\}$ is an MC, i.e. the state *and* the observation jointly form an MC. So the pdf of the couple $(\mathbf{x}_{0:k}, \mathbf{y}_{0:k}) = \xi_{0:k}$ can be factorized as follow:

$$p(\xi_{0:k}) = p_0(\xi_0) \times p_{1|0}(\xi_1|\xi_0) \times \cdots \times p_{k|k-1}(\xi_k|\xi_{k-1}). \quad (11)$$

This class of models encompasses and extends the class of HMC models (3): note that in a PMC the transition pdf $p_{k|k-1}(\xi_k|\xi_{k-1})$ can be factorized as $p(\mathbf{x}_k\,|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \times p(\mathbf{y}_k|\mathbf{x}_k, \mathbf{x}_{k-1}, \mathbf{y}_{k-1})$, so (11) reduces to (3) if $p(\mathbf{x}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1})$ reduces to $f_{k|k-1}(\mathbf{x}_k|\mathbf{x}_{k-1})$ and $p(\mathbf{y}_k|\mathbf{x}_k, \mathbf{x}_{k-1}, \mathbf{y}_{k-1})$ to $g_k(\mathbf{y}_k|\, \mathbf{x}_k)$. However, in a PMC the states $\{\mathbf{x}_k\}$ are not necessarily an MC, and conditionally on $\mathbf{x}_{0:k}$, observations $\{\mathbf{y}_i\}_{i=0}^k$ are no longer necessarily independent (however they form an MC).

Sequential filtering relies on the following relation (here $\mathcal{N}$ calls for numerator):

$$p(\mathbf{x}_k|\mathbf{y}_{0:k}) = \frac{\int p(\mathbf{x}_{k-1}|\mathbf{y}_{0:k-1}) p_{k|k-1}(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \mathrm{d}\mathbf{x}_{k-1}}{\mathcal{N} \mathrm{d}\mathbf{x}_k}. \quad (12)$$

As in a classical HMC, this equation if often not computable and PF implementations have been proposed [3]. However, in a linear and Gaussian PMC (i.e. transitions $p_{k+1|k}$ are Gaussian with mean linear in $(\mathbf{x}_k, \mathbf{y}_k)$), exact filtering is possible as we now recall.

3.2. KF for Gaussian PMC

Let us now consider the following model:

$$\begin{bmatrix} \mathbf{x}_k \\ \mathbf{y}_k \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{F}_k^1 & \mathbf{F}_k^2 \\ \mathbf{H}_k^1 & \mathbf{H}_k^2 \end{bmatrix}}_{\mathbf{B}_k} \begin{bmatrix} \mathbf{x}_{k-1} \\ \mathbf{y}_{k-1} \end{bmatrix} + \mathbf{w}_k \quad (13)$$

where $\mathbf{w}_1 \cdots \mathbf{w}_k$ are independent, $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \Sigma_k)$ with $\Sigma_k = \begin{bmatrix} \mathbf{Q}_k & \mathbf{S}_k \\ \mathbf{S}_k^T & \mathbf{R}_k \end{bmatrix}$, and $\mathbf{x}_0 \sim \mathcal{N}(\mathbf{m}_0; \mathbf{P}_0)$. It remains possible in a Gaussian PMC to propagate exactly the filtering pdf $p(\mathbf{x}_k|\mathbf{y}_{0:k})$ [10, eqs. (13.56) and (13.57)]. In model (13) $p(\mathbf{x}_k|\mathbf{y}_{0:k-1})$ and $p(\mathbf{x}_k|\mathbf{y}_{0:k})$ are Gaussians. So let us set

$$p(\mathbf{x}_k|\mathbf{y}_{0:k-1}) = \mathcal{N}(\mathbf{x}_k; \mathbf{m}_{k|k-1}; \mathbf{P}_{k|k-1}), \quad (14)$$

$$p(\mathbf{x}_k|\mathbf{y}_{0:k}) = \mathcal{N}(\mathbf{x}_k; \mathbf{m}_{k|k}; \mathbf{P}_{k|k}). \quad (15)$$

Then $p(\mathbf{x}_k|\mathbf{y}_{0:k-1})$ and $p(\mathbf{x}_k|\mathbf{y}_{0:k})$ can be computed from $p(\mathbf{x}_{k-1}|\mathbf{y}_{0:k-1})$ via the following equations:

$$\mathbf{m}_{k|k-1} = \mathbf{F}_k^1 \mathbf{m}_{k-1|k-1} + \mathbf{F}_k^2 \mathbf{y}_{k-1}, \quad (16)$$

$$\mathbf{P}_{k|k-1} = \mathbf{Q}_k + \mathbf{F}_k^1 \mathbf{P}_{k-1|k-1} \mathbf{F}_k^{1\,T}, \quad (17)$$

$$\mathbf{m}_{k|k} = \mathbf{m}_{k|k-1} + \mathbf{K}_k(\mathbf{y}_k - \tilde{\mathbf{y}}_k), \quad (18)$$

$$\mathbf{P}_{k|k} = \mathbf{P}_{k|k-1} - \mathbf{K}_k(\mathbf{S}_k^T + \mathbf{H}_k^1 \mathbf{P}_{k-1|k-1} \mathbf{F}_k^{1\,T}), \quad (19)$$

$$\tilde{\mathbf{y}}_k = \mathbf{H}_k^1 \mathbf{m}_{k-1|k-1} - \mathbf{H}_k^2 \mathbf{y}_{k-1}, \quad (20)$$

$$\mathbf{K}_k = (\mathbf{S}_k + \mathbf{F}_k^1 \mathbf{P}_{k-1|k-1} \mathbf{H}_k^{1\,T}) \mathbf{L}_k^{-1}, \quad (21)$$

$$\mathbf{L}_k = (\mathbf{H}_k^1 \mathbf{P}_{k-1|k-1} \mathbf{H}_k^1 + \mathbf{R}_k^T). \quad (22)$$

The classical linear and Gaussian model described in (7)-(8) is a particular case of this model, where we set $\mathbf{F}_k^1 = \mathbf{F}_k$, $\mathbf{H}_k^1 = \mathbf{H}_k \mathbf{F}_k$ and $\mathbf{F}_k^2 = \mathbf{H}_k^2 = \mathbf{0}$, $\mathbf{Q}_k = \mathbf{Q}_k^u$, $\mathbf{S}_k = \mathbf{Q}_k^u \mathbf{H}_k^T$ and $\mathbf{R}_k = \mathbf{R}_k^v + \mathbf{H}_k \mathbf{Q}_k^u \mathbf{H}_k^T$. It is easy to check that in this case, Eqs. (16)-(22) reduce to the classical KF equations.

4. PHD FILTER FOR PMC

We next extend the problematic of multi-target filtering to PMC models. To that end we need to modify the approach of Section 2. More precisely, contrary to the classical approach, here we characterize a target $(\mathbf{x}_k, \mathbf{y}_k)$ by its state $\mathbf{x}_k$ and associated observation $\mathbf{y}_k$. In this case, the RFS X_k at time k should be written as $X_k = \{[\mathbf{x}_{k,1}, \mathbf{y}_{k,1}]^T, \cdots, [\mathbf{x}_{k,n}, \mathbf{y}_{k,n}]^T\}$. We now look for estimating the number of couples and their extended state (so the states of targets and their associated measurements) at time k, from all the past measurements $Z_{0:k}$, contrary to the classical PHD filter where we only look for estimating the number of targets and their states at time k. We assume that the probabilities of survival and detection of the couple $(\mathbf{x}_k, \mathbf{y}_k)$ depend only on state $\mathbf{x}_k$. The transition of the couple $\xi_{k-1} = (\mathbf{x}_{k-1}, \mathbf{y}_{k-1})$, when it survives, is given by pdf $p_{k|k-1}(\xi_k|\xi_{k-1})$. Finally the likelihood of a measurement $\mathbf{z}$ with the couple is now $g_k(\mathbf{z}|\mathbf{x}_k, \mathbf{y}_k) = \delta_{\mathbf{z}}(\mathbf{y}_k)$: indeed, the measurement $\mathbf{z}$ associated to the couple $(\mathbf{x}_k, \mathbf{y}_k)$ can only be $\mathbf{y}_k$.

One can next adapt the derivation of the PHD filter in [6]. Assume that the PHD of RFS X_{k-1} is given by

$v_{k-1}(\mathbf{x}_{k-1},\mathbf{y}_{k-1})$. Then the PMC-PHD filter reads:

$$v_{k|k-1}(\mathbf{x},\mathbf{y}) = \int p_{s,k}(\mathbf{x}_{k-1}) p_{k|k-1}(\mathbf{x},\mathbf{y}|\mathbf{x}_{k-1},\mathbf{y}_{k-1}) \times v_{k-1}(\mathbf{x}_{k-1},\mathbf{y}_{k-1}) \mathrm{d}\mathbf{x}_{k-1}\mathrm{d}\mathbf{y}_{k-1} + \gamma_k(\mathbf{x},\mathbf{y}), \quad (23)$$

$$v_k(\mathbf{x},\mathbf{y}) = [1 - p_{d,k}(\mathbf{x})]\, v_{k|k-1}(\mathbf{x},\mathbf{y}) + \sum_{\mathbf{z}\in Z_k} \frac{p_{d,k}(\mathbf{x}) v_{k|k-1}(\mathbf{x},\mathbf{z})\delta_{\mathbf{z}}(\mathbf{y})}{\kappa_k(\mathbf{z}) + \int p_{d,k}(\mathbf{x}) v_{k|k-1}(\mathbf{x},\mathbf{z})\,\mathrm{d}\mathbf{x}}. \quad (24)$$

4.1. A GM implementation of the PMC-PHD Filter

As in the case of HMC models, the PMC based PHD filter (23)-(24) cannot be computed exactly in general, except in the following particular case. Let us now assume that the dynamics of each couple is described by (13), that the probabilities of survival and detection are constant, that the intensity of birth couples is

$$\gamma_k(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{J_{\gamma,k}} w_{\gamma,k}^{(i)} \mathcal{N}(\mathbf{x},\mathbf{y}; \mathbf{m}_{\gamma,k}^{(i)}, \mathbf{P}_{\gamma,k}^{(i)}) \quad (25)$$

and that the PHD at time $k-1$ is approximated by

$$v_{k-1}(\mathbf{x},\mathbf{y}) = v_{k-1}^1(\mathbf{x},\mathbf{y}) + v_{k-1}^2(\mathbf{x},\mathbf{y}), \quad (26)$$

where

$$v_{k-1}^1(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{J_{k-1}^1} w_{k-1}^{1,(i)} \mathcal{N}(\mathbf{x},\mathbf{y}; \mathbf{m}_{k-1}^{1,(i)}, \mathbf{P}_{k-1}^{1,(i)}), \quad (27)$$

$$v_{k-1}^2(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{J_{k-1}^2} w_{k-1}^{2,(i)} \mathcal{N}(\mathbf{x}; \mathbf{m}_{k-1}^{2,(i)}, \mathbf{P}_{k-1}^{2,(i)}) \delta_{\mathbf{z}^{(i)}}(\mathbf{y}), \quad (28)$$

where $\mathbf{z}^{(i)}$ belong to a given set $Z = \{\mathbf{z}^{(1)}, \cdots, \mathbf{z}^{(J_{k-1}^2)}\}$ with possible repetitions. We will explain the signification of v_{k-1}^1 and v_{k-1}^2 when we will deduce v_k^1 and v_k^2.

Combining (23) with (13) we see that $v_{k|k-1}$ is the following GM:

$$v_{k|k-1}(\mathbf{x},\mathbf{y}) = v_{k|k-1}^1(\mathbf{x},\mathbf{y}) + v_{k|k-1}^2(\mathbf{x},\mathbf{y}) + \gamma_k(\mathbf{x},\mathbf{y}), \quad (29)$$

where

$$v_{k|k-1}^1(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{J_{k-1}^1} w_{k|k-1}^{1,(i)} \mathcal{N}(\mathbf{x},\mathbf{y}; \mathbf{m}_{k|k-1}^{1,(i)}, \mathbf{P}_{k|k-1}^{1,(i)}), \quad (30)$$

$$v_{k|k-1}^2(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{J_{k-1}^2} w_{k|k-1}^{2,(i)} \mathcal{N}(\mathbf{x},\mathbf{y}; \mathbf{m}_{k|k-1}^{2,(i)}, \mathbf{P}_{k|k-1}^{2,(i)}), \quad (31)$$

where

$$w_{k|k-1}^{1,(i)} = p_{s,k} w_{k-1}^{1,(i)}, \quad (32)$$

$$w_{k|k-1}^{2,(i)} = p_{s,k} w_{k-1}^{2,(i)}, \quad (33)$$

$$\mathbf{m}_{k|k-1}^{1,(i)} = \mathbf{B}_k \mathbf{m}_{k-1}^{1,(i)}, \quad (34)$$

$$\mathbf{m}_{k|k-1}^{2,(i)} = \mathbf{B}_k \begin{bmatrix} \mathbf{m}_{k-1}^{2,(i)} \\ \mathbf{z}^{(i)} \end{bmatrix}, \quad (35)$$

$$\mathbf{P}_{k|k-1}^{1,(i)} = \Sigma_k + \mathbf{B}_k \mathbf{P}_{k-1}^{1,(i)} \mathbf{B}_k^T, \quad (36)$$

$$\mathbf{P}_{k|k-1}^{2,(i)} = \Sigma_k + \begin{bmatrix} \mathbf{F}_k^1 \\ \mathbf{H}_k^1 \end{bmatrix} \mathbf{P}_{k-1}^{2,(i)} [\mathbf{F}_k^{1\,T} \mathbf{H}_k^{1\,T}]. \quad (37)$$

Let us now implement (24). So we start from (29), which we rewrite as

$$v_{k|k-1}(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} \mathcal{N}(\mathbf{x},\mathbf{y}; \mathbf{m}_{k|k-1}^{(i)}, \mathbf{P}_{k|k-1}^{(i)}), \quad (38)$$

where

$$\mathbf{m}_{k|k-1}^{(i)} = \begin{bmatrix} \mathbf{m}_{k|k-1}^{\mathbf{x},(i)}, \\ \mathbf{m}_{k|k-1}^{\mathbf{y},(i)} \end{bmatrix}, \quad (39)$$

$$\mathbf{P}_{k|k-1}^{(i)} = \begin{bmatrix} \mathbf{P}_{k|k-1}^{\mathbf{x},(i)} & \mathbf{P}_{k|k-1}^{\mathbf{xy},(i)} \\ \mathbf{P}_{k|k-1}^{\mathbf{xy},(i)\,T} & \mathbf{P}_{k|k-1}^{\mathbf{y},(i)} \end{bmatrix}. \quad (40)$$

Then

$$v_k(\mathbf{x},\mathbf{y}) = v_k^1(\mathbf{x},\mathbf{y}) + v_k^2(\mathbf{x},\mathbf{y}), \quad (41)$$

where

$$v_k^1(\mathbf{x},\mathbf{y}) = (1 - p_{d,k}) v_{k|k-1}(\mathbf{x},\mathbf{y}), \quad (42)$$

$$v_k^2(\mathbf{x},\mathbf{y}) = \sum_{\mathbf{z}\in Z_k} v_{d,k}^2(\mathbf{x},\mathbf{z}) \delta_{\mathbf{z}}(\mathbf{y}), \quad (43)$$

where

$$v_{d,k}^2(\mathbf{x},\mathbf{z}) = \sum_{i=1}^{J_{k|k-1}} w_k^{2,(i)}(\mathbf{z}) \mathcal{N}(\mathbf{x}; \mathbf{m}_k^{2,(i)}(\mathbf{z}); \mathbf{P}_k^{2,(i)}), \quad (44)$$

and

$$w_k^{2,(i)}(\mathbf{z}) = \frac{p_{d,k} w_{k|k-1}^{(i)} q_k^{(i)}(\mathbf{z})}{\kappa_k(\mathbf{z}) + p_{d,k} \sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} q_k^{(i)}(\mathbf{z})}, \quad (45)$$

$$q_k^{(i)}(\mathbf{z}) = \mathcal{N}(\mathbf{z}; \mathbf{m}_{k|k-1}^{\mathbf{y},(i)}; \mathbf{P}_{k|k-1}^{\mathbf{y},(i)}), \quad (46)$$

$$\mathbf{m}_k^{2,(i)} = \mathbf{m}_{k|k-1}^{\mathbf{x},(i)} + \mathbf{K}_k^{(i)}(\mathbf{z} - \mathbf{m}_{k|k-1}^{\mathbf{y},(i)}), \quad (47)$$

$$\mathbf{P}_k^{2,(i)} = \mathbf{P}_{k|k-1}^{\mathbf{x},(i)} - \mathbf{K}_k^{(i)} \mathbf{P}_{k|k-1}^{\mathbf{xy},(i)\,T}, \quad (48)$$

$$\mathbf{K}_k^{(i)} = \mathbf{P}_{k|k-1}^{\mathbf{xy},(i)} (\mathbf{P}_{k|k-1}^{\mathbf{y},(i)})^{-1}. \quad (49)$$

Finally, PHD $v_k(\mathbf{x},\mathbf{y})$ has the same form of $v_{k-1}(\mathbf{x},\mathbf{y})$, see (26)-(28). Term $v_k^2(\mathbf{x},\mathbf{y})$ is due to detected targets and so if $\mathbf{y}$ does not belong to the set Z_k, then $v^2_k(\mathbf{x},\mathbf{y}) = 0$. The reason is that if a target is detected, then the associated measurement is necessarily in Z_k. PHD $v^1_k(\mathbf{x},\mathbf{y})$ is due to non-detected targets but it remains possible to have an estimate of states and associated measurements even if they are not detected.

Remark 1 Let us remark that the PHD at time $k-1$ could be rewritten as $v_{k-1}(\mathbf{x},\mathbf{y}) = p^1_{k-1}(\mathbf{y}|\mathbf{x})\tilde{v}^1_{k-1}(\mathbf{x}) + p^2_{k-1}(\mathbf{y}|\mathbf{x})\ \tilde{v}^2_{k-1}(\mathbf{x})$ where $\tilde{v}^j_{k-1}(\mathbf{x}) = \int v^j_{k-1}(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}$ and $p^j_{k-1}(\mathbf{y}|\mathbf{x}) = v^j_{k-1}(\mathbf{x},\mathbf{y}) / \int v^j_{k-1}(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}$ for $j=1$ or $j=2$. It is easy to show that $\tilde{v}^1_{k-1}(\mathbf{x})$ and $\tilde{v}^2_{k-1}(\mathbf{x})$ are GM as a function of $\mathbf{x}$, $\int p^1(\mathbf{y}|\mathbf{x})\mathrm{d}\mathbf{y} = \int p^2(\mathbf{y}|\mathbf{x})\mathrm{d}\mathbf{y} = 1$ and so $\int \tilde{v}^1_{k-1}(\mathbf{x}) + \tilde{v}^1_{k-1}(\mathbf{x})\ \mathrm{d}\mathbf{x} = \int v_{k-1}(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$. If we consider the linear and Gaussian HMC, i.e. $\mathbf{F}^2_k = \mathbf{H}^2_k = 0, \mathbf{F}^1_k = \mathbf{F}_k, \mathbf{H}^1_k = \mathbf{H}_k\mathbf{F}^1_k, \mathbf{R}_k = \mathbf{R}^v_k + \mathbf{H}_k\mathbf{Q}^u_k\mathbf{H}^T_k$ and if we take for birth targets $\mathbf{m}^{(i)}_{\gamma_k} = \begin{bmatrix} \mathbf{m}^{1,(i)}_{\gamma_k} \\ \mathbf{H}_k\mathbf{m}^{1,(i)}_{\gamma_k} \end{bmatrix}$ and $\mathbf{P}^{(i)}_{\gamma_k} = \begin{bmatrix} \mathbf{P}^{1,(i)}_{\gamma_k} & \mathbf{P}^{1,(i)}_{\gamma_k}\mathbf{H}^T_k \\ \mathbf{H}_k\mathbf{P}^{1,(i)}_{\gamma_k} & \mathbf{R}_k + \mathbf{H}_k\mathbf{P}^{1,(i)}_{\gamma_k}\mathbf{H}^T_k \end{bmatrix}$, then one can check that $\tilde{v}_k(\mathbf{x}) = \int v_k(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}$ is propagated by using the GM implementation of the PHD filter [7]. In other words, the Gaussian PMC-PHD filter reduces to the GM-PHD filter when the PMC reduces to an HMC, except that in our formulation we explicitly deduce information on the measurements associated to targets. This is a main difference with the GM implementation of the PHD filter for HMC models, which only enables to estimate the number of targets and the states. So, the PMC-PHD filter enables the use of more general models than the HMC-PHD filter and is not penalized when the dynamical model reduces to an HMC.

4.2. Extraction of states and measurements associated

As in the classical implementation of the PHD filter, extraction of states consists in looking for local maxima of v_k. A solution is to consider couples such that weights $w^{\cdot,(i)}_k$ in $v^1_k(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{J^1_k} w^{1,(i)}_k\ \mathcal{N}(\mathbf{x},\mathbf{y};\mathbf{m}^{1,(i)}_k,\mathbf{P}^{1,(i)}_k)$ and $v^2_k(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{J^2_k} w^{2,(i)}_k\ \mathcal{N}(\mathbf{x};\mathbf{m}^{2,(i)}_k,\mathbf{P}^{2,(i)}_k)\delta_{\mathbf{z}^{(i)}}(\mathbf{y})$ are above a given threshold, typically 0.5. In this case, an estimate of a state and associated observation is $[\mathbf{m}^{1,(i)}_{k-1}]$ or $[\mathbf{m}^{2,(i)}_{k-1}, \mathbf{z}^{(i)}]$.

However, this GM implementation is not directly computable since the mixture grows exponentially. Pruning techniques which consists in deleting components with weight under a given threshold are applicable here. In addition, one should merge close components. However, (41), (42) and (43) suggest that we should consider two merging steps, one for the PHD associated to non-detected targets v^1_k and one for the PHD associated to detected targets v^2_k because their forms are different. For the merging step of PHD v^1_k, one can use the traditional procedure described in [7] in augmented dimension. For the second one, one can also use the same procedure to decide when we merge two Gaussians (i) and (j), provided the measurement associated to the two Gaussians is the same.

5. SIMULATIONS

We track and observe Cartesian coordinates of targets, $\mathbf{x}_k = [p_x, \dot{p}_x, p_y, \dot{p}_y]^T_k$. Let us consider PMC model (13) where we set

$$\mathbf{F}^1_k = \begin{bmatrix} 1/2 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathbf{F}^2_k = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \end{bmatrix},$$

$$\mathbf{H}^1_k = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{H}_k} \times \mathbf{F}^1_k, \mathbf{H}^2_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \times \mathbf{F}^2_k,$$

$$\mathbf{Q}_k = \sigma^2_v \begin{bmatrix} \frac{T^3}{3} & \frac{T^2}{2} & 0 & 0 \\ \frac{T^2}{2} & T & 0 & 0 \\ 0 & 0 & \frac{T^3}{3} & \frac{T^2}{2} \\ 0 & 0 & \frac{T^2}{2} & T \end{bmatrix}, \mathbf{S}_k = \mathbf{H}_k\mathbf{Q}_k \text{ and}$$

$\mathbf{R}_k = \underbrace{\mathrm{diag}[\sigma^2_x, \sigma^2_y]}_{\mathbf{R}^v_k} + \mathbf{H}_k\mathbf{Q}_k\mathbf{H}^T_k$. It is possible to compute $f_{k|k-1}(\mathbf{x}_k|\mathbf{x}_{k-1}) = \mathcal{N}(\mathbf{x}_k;\mathbf{F}_k\mathbf{x}_{k-1},\mathbf{Q}_k)$ and $g_k(\mathbf{y}_k|\mathbf{x}_k) = \mathcal{N}(\mathbf{y}_k;\mathbf{H}_k\mathbf{x}_k;\mathbf{R}^u_k)$ where $\mathbf{F}_k = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix}$. In this experiment, we set $T = 1s$, $\sigma_v = 1m^2/sec^3$, $\sigma_x = \sigma_y = 10m$. We generate a mean of 20 false alarms measurements, and the probability of survival and of detection are $p_s = 0.98$ and $p_d = 0.95$. The scenario is displayed in Fig. 1 (only coordinate p_x is displayed).

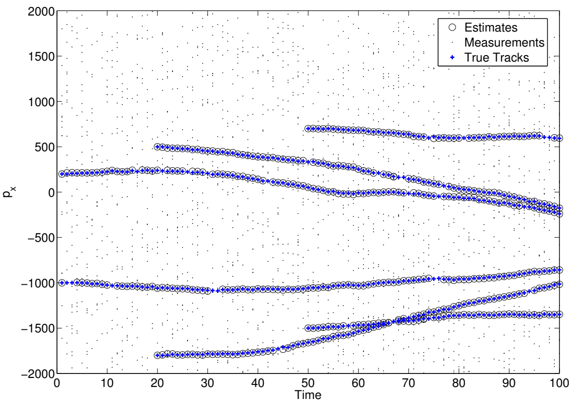

Fig. 1. Scenario with 6 targets - True tracks, estimates of the PMC PHD filter and measurements

We compare an HMC and a PMC based PHD filters. The first filter uses pdfs $f_{k|k-1}(\mathbf{x}_k|\mathbf{x}_{k-1})$ and $g_{k|k-1}(\mathbf{y}_k|\mathbf{x}_k)$ computed above while the second one uses pdf $p_{k|k-1}(\mathbf{x}_k,\mathbf{y}_k|\mathbf{x}_{k-1},\mathbf{y}_{k-1})$. The signals are generated from the PMC model, so it is expected that the PMC-PHD filter performs better. However, in this PMC model, $f_{k|k-1}$ and $g_{k|k-1}$ are computable, so it is interesting to consider the contribution of the PMC-model to the PHD filter. The pruning and merging thresholds are respectively $T_p = 10^{-5}$ and $U = 4m$ for both algorithms; for the HMC-PHD filter, we keep at most $J_{max} = 100$ Gaussians and for the PMC PHD filter we keep $J^1_{max} = 50$ Gaussians for the PHD of non-detected targets v^1_k and $J^2_{max} = 50$ Gaussians for the PHD of detected targets v^2_k. We consider that a Gaussian is a target when its weight is above 0.5.

To compare both algorithms, we use the Optimal Subpattern Assignment (OSPA) metric which enables to compare multi-target filtering algorithms [11]. Let $X = \{x_1, \cdots, x_m\}$ and $Y = \{y_1, \cdots, y_k\}$ be two finite sets. X represents the estimated finite set of the targets and Y represents the true finite set of the targets. For $1 \leq p < +\infty$ and $c > 0$, we denote $d^{(c)}(x, y) = min(c, ||x - y||)$ ($||.||$ is the euclidean norm) and Π_k the set of permutations on $\{1, 2, ..., n\}$. The OSPA metric is defined by :

$$\overline{d}^c_p(X, Y) \triangleq \left(\frac{1}{n}\left(\underset{\pi\in\Pi_k}{min}\sum_{i=1}^{m} d^{(c)}(x_i, y_{\pi(i)})^p + c^p(n-m)\right)\right)^{\frac{1}{p}} \quad (50)$$

if $m \leq n$ and $\overline{d}^c_p(X, Y) \triangleq \overline{d}^c_p(Y, X)$ if $m > n$. We use $p = 1$ and $c = 20$ in our simulations. The OSPA distances averaged over 100 Monte-Carlo runs for both algorithms are displayed in Fig. 2. We observe that the PMC-PHD filter outperforms the HMC one. Here $p(\mathbf{x}_k, \mathbf{y}_k|\mathbf{x}_{k-1}, \mathbf{y}_{k-1}) \neq f_{k|k-1}(\mathbf{x}_k|\mathbf{x}_{k-1})g_k(\mathbf{y}_k|\mathbf{x}_k)$; so the PMC-PHD filter does not reduce to the HMC-PHD filter which is no longer optimal for the considered data. In conclusion, using the couple of pdfs $f_{k|k-1}(\mathbf{x}_k|\mathbf{x}_{k-1})$ and $g_k(\mathbf{y}_k|\mathbf{x}_k)$ may be insufficient to model a tracking issue: states at time k depend on states at time $k - 1$ but may also depend on measurements at time $k - 1$. In this this experiment, introducing the pdf of the pair improves the performances of the PHD filter by considering such dependences.

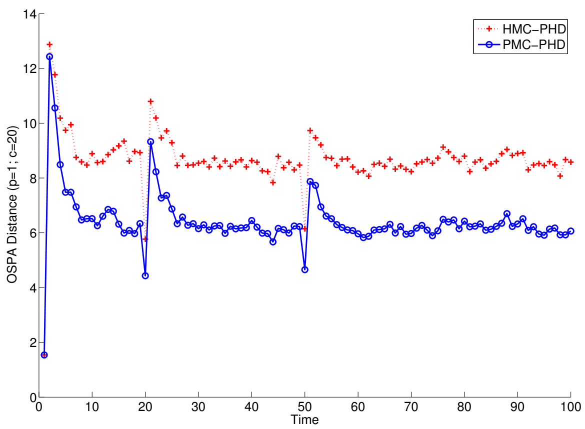

Fig. 2. OSPA Distance

6. CONCLUSION

In this paper, we introduced PMC models in a multi-target filtering context. We adapted the PHD filter to PMC models. The advantages of the PMC based PHD filter are twofold. PMC models enable to take into account more general pdf transitions than those used in HMC models; moreover such an approach enables to deduce also the measurement associated to each target. Our approach was validated by simulations.

7. REFERENCES

[1] B. Ristic, S. Arulampalam, and N. Gordon, *Beyond the Kalman Filter: Particle Filters for Tracking Applications*. Artech House, 2004.

[2] W. Pieczynski and F. Desbouvries, "Kalman filtering using pairwise Gaussian models," in *Proceedings of the International Conference on Acoustics, Speech and Signal Processing (ICASSP 03)*, Hong-Kong, 2003.

[3] F. Desbouvries and W. Pieczynski, "Particle filtering in pairwise and triplet Markov chains," in *Proc. IEEE - EURASIP Workshop on Nonlinear Signal and Image Processing*, Grado-Gorizia, Italy, June 8-11 2003.

[4] W. Pieczynski, "Pairwise Markov chains," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 25, no. 5, pp. 634–39, May 2003.

[5] J. Goutsias, R. P. Mahler, and H. T. Nguyen, *Random Sets Theory and Applications*. Springer- Verlag New York, 1997.

[6] R. Mahler, "Multitarget Bayes Filtering via First-Order Multitarget Moments," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 39, no. 4, October 2003.

[7] B.-N. Vo and W. Ma, "The Gaussian Mixture Probability Hypothesis Density Filter," *IEEE Tr. Sig. Pr.*, vol. 54, pp. 4091–4104, November 2006.

[8] B.-N. Vo, S. Singh, and A. Doucet, "Sequential Monte Carlo Methods for Multi-Target Filtering with Random Finite Sets," *IEEE Trans. Aerospace and Electronic Systems*, vol. 41, pp. 1224–1245, 2005.

[9] T. Zajic and R. Mahler, "A particle-systems implementation of the PHD multitarget tracking filter," in *Proceedings of SPIE*, vol. 5096, 2003.

[10] R. S. Lipster and A. N. Shiryaev, *Statistics of Random Processes, Vol. 2 : Applications*. Berlin: Springer Verlag, 2001, ch. 13 : "Conditionally Gaussian Sequences : Filtering and Related Problems".

[11] D. Schuhmacher, B.-T. Vo, and B.-N. Vo, "A consistent metric for performance evaluation of multi-object filters," *IEEE Transactions on Signal Processing*, vol. 56, no. 8, pp. 3447–3457, August 2008.