the numbers $0.4 + j0.4$, $0.4 - j0.4$, $-0.4 + j0.4$, $-0.4 - j0.4$ in Table I reflect our belief that input points should correspond to one of four catalog $\pm 1 \pm j$. Fig. 1 shows the learning characteristics of the fuzzy adaptive filters under SNR = 15 dB. In Fig. 1, we see that learning speed can be greatly improved by incorporating these fuzzy rules. The computation complexity of the complex LMS fuzzy filter is shown in Table II.

We compared the bit error rates achieved by the optimal equalizer, the radial basis function (RBF) equalizer with 64 centers [5], the complex recursive least squares (RLS) fuzzy adaptive equalizer [4], and the complex LMS fuzzy adaptive equalizers for different signal-to-noise ratios, for the given channel with equalizer order $n = 2$ and $m = 1$. These equalizers were first trained with 1 000 symbols from the output of the channel and then we evaluated the bit error rates (BER's) based on 10^6 more received symbols for each realization. We see from Fig. 2 that the BER's of the fuzzy equalizers are very close to the optimal values. For the RLS and LMS fuzzy equalizers, the BER curves are indistinguishable.

V. Conclusions

In this correspondence, we developed a complex fuzzy filter and applied it to linear channel equalization problems with complex components. From simulations, we show that the bit error rates of the fuzzy equalizers were close to that of the optimal equalizer.

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Spherical Trigonometry, Yule's Parcor Identity, and FRLS Fully Normalized Lattice

François Desbouvries

Abstract—**Yule's PARCOR identity is recognized as the cosine law of spherical trigonometry. The six PARCORs propagated by the fullly normalized FRLS lattice filter are the cosines of the six elements of a spherical triangle, and this lattice algorithm is one solution to a spherical triangle problem that arises naturally in navigation and astronomy. Exploiting this new geometric interpretation yields unnoticed (and potentially useful) recursions among FRLS quantities.**

I. Introduction

The basic cell of the prewindowed fully normalized adaptive FRLS lattice algorithm [1] consists of a recursion among three variables only. The incoming entries are the forward and delayed backward "double-," "angle-," or "information-" normalized prediction errors, ε_n^t and η_n^{t-1} at order n together with the $(n+1)$th-order PARCOR ρ_{n+1}^{t-1} at time $t-1$. The algorithm first updates the PARCOR, then computes the errors at order $n+1$ as follows:

$$\begin{aligned}\rho_{n+1}^t =& \varepsilon_n^t(\eta_n^{t-1})^T \\ &+ [I - \varepsilon_n^t(\varepsilon_n^t)^T]^{1/2}\, \rho_{n+1}^{t-1} [I - \eta_n^{t-1}(\eta_n^{t-1})^T]^{T/2} \end{aligned} \quad (1a)$$

$$\begin{aligned}\varepsilon_{n+1}^t =& [I - \rho_{n+1}^t(\rho_{n+1}^t)^T]^{-1/2} \\ &\cdot (\varepsilon_n^t - \rho_{n+1}^t \eta_n^{t-1})\, [I - (\eta_n^{t-1})^T \eta_n^{t-1}]^{-T/2}\end{aligned} \quad (1b)$$

$$\begin{aligned}\eta_{n+1}^t =& [I - (\rho_{n+1}^t)^T \rho_{n+1}^t]^{-1/2} \\ &\cdot [\eta_n^{t-1} - (\rho_{n+1}^t)^T \varepsilon_n^t]\, [I - (\varepsilon_n^t)^T \varepsilon_n^t]^{-T/2}.\end{aligned} \quad (1c)$$

These recursions received an elegant geometric interpretation (see e.g., [2]–[4] and the references therein), when it appeared that (1b) and (1c) as well as a reordering (1d) of (1a)

$$\begin{aligned}\rho_{n+1}^{t-1} =& [I - \varepsilon_n^t(\varepsilon_n^t)^T]^{-1/2} \\ &\cdot [\rho_{n+1}^t - \varepsilon_n^t(\eta_n^{t-1})^T]\, [I - \eta_n^{t-1}(\eta_n^{t-1})^T]^{-T/2}\end{aligned} \quad (1d)$$

were three particular applications of a general identity among partial correlation coefficients, which could be traced back [2] to an old paper by Yule [5]. More precisely, let us set ourselves in the space R^N of N-dimensional vectors, and let Y, A, B, and C denote matrices with N rows and an arbitrary number of columns. Let P_Y (resp. $P_Y^\perp$) stand for the orthogonal projection operator onto the subspace of R^N spanned by the columns of Y (resp. onto its orthogonal complement), and let $M^{1/2}$ denote the Cholesky factor (either lower or upper triangular) of the symmetric positive definite matrix M (we assume the regular case for simplicity). The partial correlation (PARCOR) coefficient $\rho_Y(A, B)$ is defined as the "inner product" of two normalized projection residuals $\rho_Y(A, B) \triangleq (\overline{P_Y^\perp A})^T \cdot \overline{P_Y^\perp B}$ where $\overline{P_Y^\perp A} \triangleq P_Y^\perp A(A^T P_Y^\perp A)^{-T/2}$. Yule's PARCOR Identity expresses $\rho_{Y,A}(C, B)$ in terms of $\rho_Y(A, C)$, $\rho_Y(B, A)$, and $\rho_Y(C, B)$:

$$\begin{aligned}\rho_{Y,A}(C, B) =& [I - \rho_Y(C, A)\rho_Y(A, C)]^{-1/2} \\ &\cdot [\rho_Y(C, B) - \rho_Y(C, A)\rho_Y(A, B)] \\ &\cdot [I - \rho_Y(B, A)\rho_Y(A, B)]^{-T/2}\end{aligned} \quad (2)$$

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The author is with Département Signal et Image, Institut National des Télécommunications, Evry, France (e-mail: desbou@int-evry.fr).

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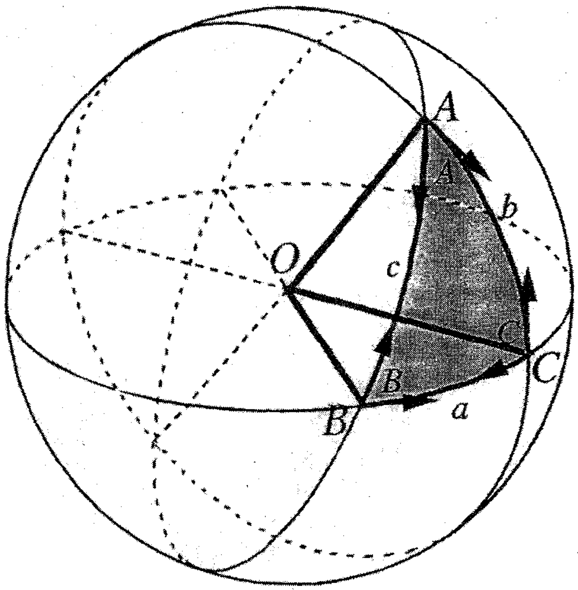

Fig. 1.

As a particular application, the lattice (1) is obtained by replacing (A, B, C) by the permutations of (y_t, y_{t-n-1}, σ) where Y, y_t, y_{t-n-1}, and σ are defined in Table I.

The main result of this paper consists of recognizing that (a reorganized version of) Yule's PARCOR identity coincides with the cosine law of spherical trigonometry. This unexpected bridge between FRLS adaptive filtering and spherical trigonometry (a branch of classical trigonometry) sheds new light on the fully normalized FRLS lattice; in this new geometrical framework, the recursions of Lee *et al.* happen to be one particular solution to an important spherical triangle problem that also arises naturally in air or sea navigation and in astronomy. This link with spherical trigonometry is exhibited in Section II and exploited in Section III, since some spherical trigonometry formulae happen to admit PARCOR analogs which, in turn, imply new (and potentially useful) relations among least-squares quantities.

II. FRLS Fully Normalized Lattice and Spherical Trigonometry

In the introduction, $\rho_Y(A, B)$ was introduced as the inner product of two normalized projection residuals and can thus be interpreted (at least in the scalar case) as the cosine of the angle between the two vectors $\overline{P_Y^\perp A}$ and $\overline{P_Y^\perp B}$. This well-known result is a first trigonometric interpretation of PARCORs. In this section, we introduce a new trigonometric interpretation, but now both of PARCORs and of Yule's PARCOR identity.

Let A, B and C be three points on a sphere of unit radius. By definition, the *spherical triangle* ABC consists of the three arcs AB, AC, and BC of "great circles" obtained by intersecting the sphere and the three planes OAB, OAC and OBC passing through its center O (see Fig. 1).

There are six *elements* a, b, c, A, B, C in a spherical triangle. The angle BOC, say, is equal to the length of arc BC and is denoted by a. The dihedral angle A between planes OAB and OAC is defined as the planar angle between two straight lines orthogonal to OA and belonging to OAB and OAC, respectively. Note that A is equal to the planar angle formed by tangents to the side of the angle at vertex A. The remaining elements are defined similarly. a, b, c are classically referred to as the three *sides*, and A, B, C as the three *angles*, of the spherical triangle.

Spherical trigonometry consists of deriving all the relations among the elements of a spherical triangle ABC. This tool is of outstanding importance in areas such as geodesy and astronomy, and in sea and air navigation. We now show that it can also be applied to FRLS adaptive filtering.

There are three degrees of freedom in a spherical triangle: Any three elements determine the three remaining ones. Thus, there can be no more than three distinct relationships among the six elements. One such set [3(a), (b), (c)] is obtained by permuting variables[1] into *the fundamental spherical trigonometry law of cosines* (3a),

$$\boxed{\cos a = \cos b \cos c + \sin b \cos A \sin c} \tag{3a}$$

$$\cos b = \cos a \cos c + \sin a \cos B \sin c \tag{3b}$$

$$\cos c = \cos a \cos b + \sin a \cos C \sin b. \tag{3c}$$

We recognize that *the law of cosines* [(3a), say] *is formally equal*[2] to Yule's PARCOR identity (2) (and thus to the lattice recursions (1), as a particular case) via the double identification of variables of Table I. Consequently the six PARCORs in (1), which could already be seen as cosines of (planar) angles between projection residuals, are also the cosines of the six elements of a spherical triangle in the 3-D space.

Let us briefly explain the underlying reasons for this surprising result (see [6] for more details). The PARCOR $\rho_Y(A, B)$ is defined in the plane spanned by $\overline{P_Y^\perp A}$ and $\overline{P_Y^\perp B}$. Now, the FRLS transversal and lattice recursions can be derived from the three particular planar biorthogonalization steps (i.e., both $\overline{P_{Y,B}^\perp A}$ and $\overline{P_{Y,A}^\perp B}$ are a linear combination of $\overline{P_Y^\perp A}$ and $\overline{P_Y^\perp B}$) obtained by replacing A and B by any two elements out of the set (y_t, y_{t-n-1}, σ) of Table I. They can thus be described in a constrained set (and not just any set) of three planar triangles OAB, OAC, OBC (where $\vec{OA} = \overline{P_Y^\perp y_t}$, $\vec{OB} = \overline{P_Y^\perp y_{t-n-1}}$, and $\vec{OC} = \overline{P_Y^\perp \sigma}$), which together form the 3-D tetrahedron $OABC$ of Fig. 1.

Now, tetrahedrons are very particular geometrical figures. They play the same fundamental role in solid geometry as triangles in planar geometry [7]. If, moreover, we require that the tetrahedron have "unit length" (meaning $OA = OB = OC = 1$), which is the case here, then the three vertices A, B, and C lie on a 3-D sphere of unit radius. Thus, these three points determine not only the tetrahedron $OABC$, but also the spherical triangle ABC. Now, in the same way that trigonometry consists of deriving relations among the sides and angles of a planar triangle (hence the first, classical trigonometric interpretation of PARCORs), spherical trigonometry consists of deriving relations among the sides and angles of a unit-length tetrahedron or, equivalently, within a spherical triangle ABC; hence, we get this new trigonometric interpretation.

Let us now revisit (1a)–(1c) in this new framework. We are given two sides b and c, plus the angle in between A (actually given by their cosines): $\cos b = \varepsilon_n^t$, $\cos c = \eta_n^{t-1}$ and $\cos A = \rho_{n+1}^{t-1}$; and we want to compute $\cos a = \rho_{n+1}^t$, $\cos B = \varepsilon_{n+1}^t$, and $\cos C = \eta_{n+1}^t$. This very "spherical triangle problem" (a spherical triangle problem [8], [9] consists in determining three elements out of the six given the three others) is a particularly important one. It arises often in widely different contexts, such as navigation on ships

[1] This constant feature of spherical trigonometry formulas should be kept in mind in the sequel, where a set of formulas in which each element can be deduced from each other simply by permuting variables, will be described by one generic equation.

[2] Strictly speaking, the equality $\cos a = \rho_Y(C, B)$ holds in the scalar case only. However, the analogy extends further to the general multidimensional case since the spectral norm of a PARCOR ρ is bounded by 1 [2] whatever the dimensions of ρ.

TABLE I
CORRESPONDING OF THE VARIABLES

$$Y = \begin{bmatrix} 0 & \cdots & 0 \\ y_0^T & \ddots & \vdots \\ & \ddots & 0 \\ \vdots & & y_0^T \\ & & \vdots \\ \vdots & & \vdots \\ y_{t-1}^T & \cdots & y_{t-n}^T \end{bmatrix}, \quad \begin{matrix} y_t \\ \updownarrow \\ C \end{matrix} = \begin{bmatrix} y_0^T \\ y_1^T \\ \vdots \\ \\ \\ \vdots \\ y_t^T \end{bmatrix}, \quad \begin{matrix} y_{t-n-1} \\ \updownarrow \\ B \end{matrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ y_0^T \\ \vdots \\ y_{t-n-1}^T \end{bmatrix}, \quad \begin{matrix} \sigma \\ \updownarrow \\ A \end{matrix} = \begin{bmatrix} 0 \\ \vdots \\ \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

ρ_{n+1}^t	$\leftrightarrow$	$\rho_Y(C,B)$	$\leftrightarrow$	$\cos a$	ρ_{n+1}^{t-1}	$\leftrightarrow$	$\rho_{Y,A}(C,B)$	$\leftrightarrow$	$\cos A$
ε_n^t	$\leftrightarrow$	$\rho_Y(C,A)$	$\leftrightarrow$	$\cos b$	ε_{n+1}^t	$\leftrightarrow$	$\rho_{Y,B}(C,A)$	$\leftrightarrow$	$\cos B$
η_n^{t-1}	$\leftrightarrow$	$\rho_Y(B,A)$	$\leftrightarrow$	$\cos c$	η_{n+1}^t	$\leftrightarrow$	$\rho_{Y,C}(B,A)$	$\leftrightarrow$	$\cos C$

or airplanes, when we solve the "terrestrial triangle" [9, p. 196]; and in astrometry, when we convert the horizontal coordinates of a star into its hour coordinates or vice versa, or its equatorial coordinates into its ecliptic coordinates or vice versa [9, pp. 222–231], [10]. The solution (1a)–(1c) found by Lee *et al.* makes use of the cosine law only. First, $\cos a = \rho_{n+1}^t$ is computed through (3a) = (1a). Then, $\cos B = \varepsilon_{n+1}^t$ and $\cos C = \eta_{n+1}^t$ are extracted by rearranging (3b) into (1b) and (3c) into (1c), respectively.

III. Contributions of Spherical Trigonometry to the FRLS Problem

Spherical trigonometry first arose in ancient Greece and has been developed considerably since then [11]. It was still a research subject two centuries ago when Gauss and then Riemann discovered that the geometry on the sphere could be understood in the framework of Riemannian (or elliptic) geometry [12]. Therefore, it is unexpected that the connection of Section II should enlighten and extend this classical subject.

Conversely, spherical trigonometry becomes a new element in the toolbox used to derive FRLS algorithms: Any standard spherical trigonometry textbook contains a variety of formulas, and the first question that arises naturally is whether they have a PARCOR analog. It happens that the formulas involving multiples or fractions of the elements cannot be adapted directly. This is due to the major difference between the two frameworks: In spherical trigonometry we know elements, in FRLS adaptive filtering we know their cosines. Among the formulae that can be adapted, one is already well known (the scalar sine law), and then only a geometrical interpretation is brought, but the other formulae seem original.

We begin by evoking the "duality principle" of spherical trigonometry. Let A' be the pole (w.r.t. the equator passing through B and C) in the same hemisphere as A. Define B' and C' similarly. $A'B'C'$ is the polar spherical triangle of ABC. In $A'B'C'$, $a' = \pi - A$ and $A' = \pi - a$. This is similar for the other elements [7, pp. 980–981]. As a consequence, any spherical trigonometry formula admits for free a dual relation, obtained simply by replacing the variables (a, b, c, A, B, C) by $(\pi - A, \pi - B, \pi - C, \pi - a, \pi - b, \pi - c)$, respectively.

Among any four elements there exists one and only one relation. These 15 relations are usually classified into four different groups [13], which all admit a PARCOR analog.

1) The three cosine laws express one side, given the two other sides plus the angle in between. They admit a dual group of three formulas (the cosine law in the polar triangle), which express one angle in terms of the other two angles plus the side in between.
2) The sine law is a self-dual group of three formulas, relating two angles and the sides opposite to them.
3) Lastly, the cotangent law is a self-dual group of six formulas, relating four consecutive elements.

On the other hand, there are many different relations among any five or six elements, and it seems always possible to find new ones. Thus, we will not try to furnish an exhaustive list, even simply of those that admit a PARCOR analog. We just mention the useful "five elements formula."

Let us now list these relations. For want of space, we shall omit all the proofs, as well as the scalar PARCOR formulae, since they can be obtained without ambiguity via Table I from the associated spherical trigonometry equation. This is no longer the case for multidimensional formulae, which in turn need to be written down.

A. The Cosine Law in the Polar Triangle

In the polar triangle, the cosine law reads

$$\cos A = -\cos B \cos C + \sin B \cos a \sin C. \tag{4}$$

One can show [6] that (4) also holds for scalar PARCORs; consequently, in the same way that $\rho_{Y,A}(C, B) = F[\rho_Y(C, B), \rho_Y(C, A), \rho_Y(A, B)]$ where $F[x, y, z] = (1 - y^2)^{-1/2}(x - yz)(1 - z^2)^{-1/2}$, we have $-\rho_Y(C, B) = F[-\rho_{Y,A}(C, B), -\rho_{Y,B}(C, A), -\rho_{Y,C}(A, B)]$. In the multidimensional case, the formula extends to the "polar version" of Yule's PARCOR identity:

$$\begin{aligned}\tilde{\rho}_{Y,A}(C, B) = &-\tilde{\rho}_{Y,B}(C, A)\tilde{\rho}_{Y,C}(A, B) \\ &+ [I - \rho_{Y,B}(C, A)\rho_{Y,B}(A, C)]^{T/2}\, \tilde{\rho}_Y(C, B) \\ &\cdot [I - \rho_{Y,C}(B, A)\rho_{Y,C}(A, B)]^{1/2}\end{aligned} \tag{5}$$

in which the matrix $\tilde{\rho}$ is defined for any PARCOR ρ as

$$\begin{aligned}\tilde{\rho} &\triangleq (I - \rho\rho^T)^{T/2}\rho(I - \rho^T\rho)^{-T/2} \\ &= (I - \rho\rho^T)^{-1/2}\rho(I - \rho^T\rho).^{1/2}\end{aligned} \tag{6}$$

Note that $\rho = \tilde{\rho}$ in the scalar case. In the multidimensional case, ρ and $\tilde{\rho}$ are different but they share the same singular values. The FRLS application of (5) is the time-decreasing fully normalized lattice algorithm.

B. The Sine Law

In the scalar case, the PARCOR analog of the spherical trigonometry self-dual sine law

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \quad (7)$$

is true, and yields a known relation [14, relation (12)] when applied to the FRLS framework. In the multidimensional case, (7) does not have a PARCOR analog; however, $\sin c \sin B = \sin b \sin C$ reads

$$\begin{aligned}\{I - \rho_Y(A, [B, C])\rho_Y([B, C], A)\}^{1/2} &= [I - \rho_Y(A, B)\rho_Y(B, A)]^{1/2} \\ &\quad \cdot [I - \rho_{Y,B}(A, C)\rho_{Y,B}(C, A)]^{1/2} \\ &= [I - \rho_Y(A, B)\rho_Y(B, A) - \rho_Y(A, C)\rho_Y(C, A)]^{1/2} \\ &= [I - \rho_Y(A, C)\rho_Y(C, A)]^{1/2} \quad (8)\end{aligned}$$

C. The Cotangent Formulas

These are the six self-dual formulas obtained by permuting variables in the generic equation

$$\cos C \cos a = \cot b \sin a - \sin C \cot B. \quad (9)$$

Following [7, pp. 987–988], one can show that the scalar PARCOR analog of (9) holds. In the multidimensional case it reads

$$\begin{aligned}\rho_{Y,C}(A, B)\tilde{\rho}_Y(B, C) &= \{[I - \rho_Y(A, C)\rho_Y(C, A)]^{-1/2}\rho_Y(A, C)\} \\ &\quad \cdot [I - \rho_Y(C, B)\rho_Y(B, C)]^{1/2} \\ &\quad - [I - \rho_{Y,C}(A, B)\rho_{Y,C}(B, A)]^{1/2} \\ &\quad \cdot \{[I - \rho_{Y,B}(A, C)\rho_{Y,B}(C, A)]^{-1/2}\rho_{Y,B}(A, C)\}. \quad (10)\end{aligned}$$

D. The Five-Element Formulas

The six five-element formulas are described by (11a), the polar version of which is (11b):

$$\cos b \sin c = \sin b \cos A \cos c + \sin a \cos B \quad (11a)$$

$$\cos B \sin C = -\sin B \cos a \cos C + \sin A \cos b. \quad (11b)$$

One can show that the PARCOR analogs hold, and in the multidimensional case they read

$$\begin{aligned}\rho_Y(C, A)[I - \rho_Y(A, B)\rho_Y(B, A)]^{1/2} &= [I - \rho_Y(C, A)\rho_Y(A, C)]^{1/2}\rho_{Y,A}(C, B)\tilde{\rho}_Y(B, A) \\ &\quad + [I - \rho_Y(C, B)\rho_Y(B, C)]^{1/2}\rho_{Y,B}(C, A) \quad (12a)\end{aligned}$$

$$\begin{aligned}\tilde{\rho}_{Y,B}(C, A)[I - \rho_{Y,C}(A, B)\rho_{Y,C}(B, A)]^{T/2} &= -[I - \rho_{Y,B}(C, A)\rho_{Y,B}(A, C)]^{T/2}\tilde{\rho}_Y(C, B)\rho_{Y,C}(B, A) \\ &\quad + [I - \rho_{Y,A}(C, B)\rho_{Y,A}(B, C)]^{T/2}\tilde{\rho}_Y(C, A). \quad (12b)\end{aligned}$$

E. Alternatives to the Solution of Lee et al.

We now come back to the end of Section II. Surprisingly, the algorithm of Lee *et al.* (using the cosine law only) does not seem to be a classical solution of the underlying spherical triangle problem. More commonly used are "Napier's analogies" [7]–[9], [13] or the cosine law for computing a plus the cotangent formulas for computing B and C [13]. Other variants, using the five elements formulas [13], the sine law [7], [13] or right spherical triangles [9] are also available.

Some of these solutions can be adapted to the FRLS framework. For instance, [(1b) and (1c)] can be replaced by the following rearrangement of two five-elements formulas:

$$\begin{aligned}\varepsilon_{n+1}^t &= [I - \rho_{n+1}^t(\rho_{n+1}^t)^T]^{-1/2} \\ &\quad \cdot \{\varepsilon_n^t[I - (\eta_n^{t-1})^T\eta_n^{t-1}]^{1/2} \\ &\quad - [I - \varepsilon_n^t(\varepsilon_n^t)^T]^{1/2}\rho_{n+1}^{t-1}\tilde{\eta}_n^{t-1}\} \quad (13a)\\ \eta_{n+1}^t &= [I - (\rho_{n+1}^t)^T\rho_{n+1}^t]^{-1/2} \\ &\quad \cdot \{\eta_n^{t-1}[I - (\varepsilon_n^t)^T\varepsilon_n^t]^{1/2} \\ &\quad - [I - \eta_n^{t-1}(\eta_n^{t-1})^T]^{1/2}(\rho_{n+1}^{t-1})^T\tilde{\varepsilon}_n^t\} \quad (13b)\end{aligned}$$

and there are probably other possibilities. However, though it admits alternatives, the original solution found by Lee *et al.* will probably remain the simplest of all.

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