On the Identification of Certain Rational Transfer Functions from Truncated Autocovariance Sequences

F. Desbouvries and Ph. Loubaton

***Abstract*— Let $H(z) = \Sigma_{k=0}^{\infty} H_k z^{-k}$ be a $q \times p$ rational causal minimum phase transfer function, and let $(R_n)_{n\in\mathbb{Z}}$ be the autocovariance sequence associated to the spectral density $H(z)H^T(z^{-1})$. In this paper, the authors address the identification problem of $H(z)$ from the truncated autocovariance sequence $(R_n)_{n\neq 0}$ for the case $q > p$.**

***Index Terms*— MIMO systems, rational transfer function matrices, stochastic realization theory.**

I. Introduction and Statement of the Problem

Let $H(z) = \Sigma_{k=0}^{\infty} H_k z^{-k}$ be a $q \times p$ rational causal stable (i.e., analytic for $|z| \geq 1$) transfer function. We assume that $q > p$, and that $H(z)$ is outer (or equivalently minimum phase), i.e., $\mathrm{Rank}(H(z)) = p, |z| > 1$. Let us denote by $(R_n)_{n\in\mathbb{Z}}$ the autocovariance sequence associated to the "spectral density" $H(z)H^T(z^{-1})$ through the identity

$$H(z)H^T(z^{-1}) = \sum_{n\in\mathbb{Z}} R_n z^{-n}$$

where the series converges in a neighborhood of the unit circle. Stochastic realization theory addresses the identification problem of $H(z)$ from the sequence $(R_n)_{n\in\mathbb{Z}}$. This theory by now has a rich history (see, e.g., [6], [4], [3], and the references therein). It is a well-established fact that $H(z)$ is uniquely defined (up to a constant unitary $p \times p$ matrix) from the autocovariance sequence $(R_n)_{n\in\mathbb{Z}}$. Moreover, a number of identification algorithms for $H(z)$ have been developed. In this paper, we address the identification of $H(z)$ from the knowledge of the truncated sequence $(R_n)_{n\neq 0}$, i.e., R_0 is assumed to be unknown. The hypothesis $q > p$ is of course necessary in this context; otherwise, $H(z)$ is not uniquely defined from the truncated sequence $(R_n)_{n\neq 0}$.

Our motivations for addressing this problem stem from the blind equalization problem in digital communications theory (see [5] and [1]). In that context, p streams of emitted symbols $\{s_n^i\}_{i=1}^p$ are transmitted through a propagation medium and received at an array of q sensors; as q is a design parameter, the assumption $q > p$ is not really restrictive. This results in a q-dimensional observation $y_n = x_n + w_n = [H(z)]s_n + w_n$, where $H(z)$ is a $q \times p$ unknown matrix-valued transfer function which models the effects of the propagation medium, $s_n = [s_n^1 \cdots s_n^p]^T$ is a white noise sequence, and w_n is additive noise independent of s_n. In order to retrieve the emitted sequences $\{s_n^i\}_{i=1}^p$, it is often useful to identify $H(z)$ from the (estimated) statistics of the observation. Recent works have been devoted to the identification of $H(z)$ from the (estimated) autocovariance sequence of y_n (see, e.g., [13], [2]), under the assumption that w_n is white noise, the "spatial" covariance matrix $\Sigma = E(w_n w_n^T) = \sigma^2 I$ of which is a multiple of the identity matrix. In the present paper, we still assume that w_n is a white noise sequence, but we make no assumption about its spatial covariance matrix Σ. The reason why is that, in case the noise is due to the propagation medium (e.g., atmospheric HF noise, underwater acoustics, and others), modeling its spatial covariance matrix Σ properly is difficult. Thus the autocovariance sequences of y_n and of x_n are related via $R_k^y = R_k^x + \Sigma\,\delta_k$ where Σ is unknown. Since R_0^y contains little information, we seek to identify $H(z)$ from the autocovariance sequence of the output, excluding the central term R_0^y.

This problem was introduced in [7] for the case $p = 1$ and in [5] in the case where $p > 1$ and $H(z)$ is a matrix polynomial. It was shown that the so-called subspace approach, first introduced in [5] for the case $\Sigma = \sigma^2 I$ and $p = 1$, could be extended to the case where Σ is totally unknown; see [1] (for $p = 1$) and [5] (for $p > 1$), in the case where $H(z) = \Sigma_{k=0}^{M} H_k z^{-k}$ is an irreducible polynomial matrix and H_M has full column rank. The subspace approach developed in [5] is based on the observation that, under suitable hypotheses, $H(z)$ can be recovered (up to a constant matrix) from the set $\mathcal{G}_{M-1}$ of all q-dimensional polynomials $x(z)$, with degree less than $M-1$, which satisfy $x^T(z)H(z) = 0$. As for this set $\mathcal{G}_{M-1}$, it can be recovered from the left and the right kernels of the $qM \times qM$ block Hankel matrix $\mathcal{R}_{M-1}$ defined by

$$\mathcal{R}_{M-1} = \begin{bmatrix} R_1 & \cdots & R_M \\ \vdots & \iddots & \\ R_M & & 0 \end{bmatrix}.$$

More precisely, a qM-dimensional vector $x = (x_0^T, \cdots, x_{M-1}^T)^T$ satisfies $[x_{M-1}^T \cdots x_0^T]\mathcal{R}_{M-1} = 0^T$ and $\mathcal{R}_{M-1}[x_0^T \cdots x_{M-1}^T]^T = 0$ if and only $x(z) = \Sigma_{k=0}^{M-1} x_k z^{-k}$ belongs to $\mathcal{G}_{M-1}$.

The present paper will show that this subspace approach can be adapted to the case where $H(z)$ is rational, by developing theoretical identifiability results. In particular, we assume that the exact autocovariance sequence of y_n is known. In practice, this sequence must be estimated from a finite number of observations, and our results need to be adapted to provide a concrete identification algorithm. These statistical considerations are not reported here for want of space; we may, nevertheless, remark that subspace-like methods do work quite well in general. As far as simulations are concerned, we refer the interested reader to the closely related, but more engineering-oriented papers [13], [1] (in the case $p = 1$), or [9] (in the case $p > 1$, with additive temporal and spatial white noise).

II. Outline of the Approach

We now consider the case where $H(z)$ is rational and can thus be written as $H(z) = B(z)A^{-1}(z)$, in which $B(z) = \Sigma_{k=0}^{M} B_k z^{-k}$ and $A(z) = \Sigma_{k=0}^{P} A_k z^{-k}$ are two coprime polynomial matrices. As $H(z)$ is assumed to be stable, we assume of course that $\det(A(z)) \neq 0$ for $|z| \geq 1$. In particular, A_0 is invertible, and thus there is no restriction in setting $A_0 = I$. Moreover, we assume the following.

- C1) The columns $(B_i(z))_{i=1,p}$ of $B(z)$ all have the same degree M.
- C2) $B(z)$ is irreducible, i.e., $\mathrm{Rank}(B(z)) = p$ for each $z \neq 0$ (including $z = \infty$).
- C3) $B(z)$ is column-reduced, i.e., $\mathrm{Rank}(B_M) = p$ due to C1).
- C4) A_P is an invertible matrix.

We note that the conditions on $B(z)$ are those under which the approach of [5] is valid.

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As in [5], the method consists of two steps. Let us first denote by $\mathcal{R}_N$ the $q(N+1) \times q(N+1)$ block Hankel matrix defined by

$$\mathcal{R}_N = \begin{bmatrix} R_1 & \cdots & R_{N+1} \\ \vdots & \iddots & \vdots \\ R_{N+1} & \cdots & R_{2N+1} \end{bmatrix}.$$

Let $x = [x_0^T \cdots x_N^T]^T$ denote a vector of $\mathbb{R}^{q(N+1)}$, $x(z) = \Sigma_{k=0}^N x_k z^{-k}$ the corresponding $q \times 1$ transfer function, and $\mathcal{G}_N$ the set of all q-dimensional polynomials $x(z)$ with degree less than N satisfying $x^T(z)H(z) = 0$. As in [5], we shall show that for certain values of N plus some additional conditions, a vector x satisfies

$$[x_N^T \cdots x_0^T]\mathcal{R}_N = 0^T \tag{1}$$

and

$$\mathcal{R}_N[x_0^T \cdots x_N^T]^T = 0 \tag{2}$$

if and only if $x^T(z)H(z) = 0$. The results are gathered in the following theorem which encompasses and generalizes the results already obtained in the purely polynomial case $P = 0$ (see [5, Lemma 3.1]); the proof is deferred to Section III.

Theorem 2.1: Let $x = [x_0^T \cdots x_N^T]^T$ denote a vector of $\mathbb{R}^{q(N+1)}$ and $x(z) = \Sigma_{k=0}^N x_k z^{-k}$ the corresponding $q \times 1$ transfer function.

- The case $0 \le P \le M$: Assume that C1)–C4) hold, that $N \le M - 1$, and that $\mathrm{Rank}(\mathcal{R}_N) = Mp$ (its maximum value). Then $x^T(z)H(z) = 0$ if and only if x satisfies both (1) and (2).
- The case $P > M$: Assume that C1)–C4) hold, that $N \le 2P - M - 1$, and that $\mathrm{Rank}(\mathcal{R}_N) = Pp$ (its maximum value). Then $x^T(z)H(z) = 0$ if and only if x satisfies both (1) and (2).

Next, once $\mathcal{G}_N$ is available, $H(z)$ can be identified by using the approaches proposed in [5], which we briefly evoke for the convenience of the reader (see [5] for more details). As we shall see below, the larger N is, the milder the identifiability conditions of $H(z)$ become by the subspace approach. This is of course because $\mathcal{G}_N \subset \mathcal{G}_{N'}$ if $N \le N'$. On the other hand, the sufficient condition on $\mathrm{Rank}(\mathcal{R}_N)$ in Theorem 2.1 becomes less stringent as N increases. Consequently, N is chosen as large as possible, namely $M - 1$ (respectively, $2P - M - 1$) if $0 \le P \le M$ (respectively, $P > M$), due to Theorem 2.1. Two cases have to be considered.

Let us first remark that $x^T(z)H(z) = 0$ if and only if $x^T(z)B(z) = 0$. Let $0 \le M_1^\perp \le \cdots \le M_{q-p}^\perp$ be the so-called Kronecker indexes of the $(q-p)$-dimensional rational subspace $\mathcal{G}$ of all q-variate rational functions $x(z)$ satisfying $x^T(z)H(z) = 0$ [11], [8]. If N is greater than the largest Kronecker index $M_{q-p}^\perp$, it is shown in [5] that $B(z)$ can be identified, up to a right constant $p \times p$ invertible matrix, by solving a linear system. In other words, one identifies a matrix polynomial $\tilde{B}(z) = B(z)D$ for some $p \times p$ invertible matrix D.

In order to explain how to retrieve the whole transfer function $H(z)$, let us denote by $(r_n)_{n\in\mathbb{Z}}$ the covariance sequence associated to the "spectral density" $A^{-1}(z)A^{-T}(z^{-1})$, i.e., $A^{-1}(z)A^{-T}(z^{-1}) = \Sigma_{n\in\mathbb{Z}} r_n z^{-n}$. Let L be an integer, and let J_L be the $q(L+1) \times q(L+1)$ block antidiagonal matrix $\mathrm{antidiag}(I_q, \cdots, I_q)$. It is easy to check that $J_L\mathcal{R}_L = T_L(B)\mathcal{K}_L T_L(B)^T$, where $T_L(B)$ is the $(L+1)q \times (M+L+1)p$ generalized Sylvester matrix, and $\mathcal{K}_L$ the matrix defined, respectively, by

$$T_L(B) = \begin{bmatrix} B_0 & \cdots & B_M & & 0 \\ & \ddots & & \ddots & \\ 0 & & B_0 & \cdots & B_M \end{bmatrix}$$

$$\mathcal{K}_L = \begin{bmatrix} r_{L+1} & \cdots & r_{M+2L+1} \\ \vdots & \iddots & \vdots \\ r_{-(M-1)} & \cdots & r_{L+1} \end{bmatrix}.$$

Having $\tilde{B}(z)$, we may obtain the sequence $(D^{-1}r_nD^{-T})_{n\in\mathbb{Z}}$, i.e., the autocovariance sequence associated to $D^{-1}A^{-1}(z)A^{-T}(z^{-1})D^{-T}$. The classical Yule–Walker equations [10] then allow us to retrieve $\tilde{A}(z) = A(z)D$ up to a $p \times p$ constant orthogonal matrix. Therefore, $H(z) = \tilde{B}(z)\tilde{A}(z)^{-1}$ is identified up to a constant orthogonal matrix, as expected.

The condition $N \ge M_{q-p}^\perp$ may be quite restrictive, in particular if $N = M - 1$ (see [5]). Therefore, the authors developed in [5] an improved subspace approach. This approach can be adapted immediately to the case considered in this paper ($H(z)$ is rational) and is therefore omitted here.

III. Proof of Theorem 2.1

The aim of this final section is to give a proof of Theorem 2.1, to which we now return. We want to show how to identify $\mathcal{G}_N$ from $\mathcal{R}_N$ for the largest possible values of N. The approach we adopt is simpler than that previously developed in [5] and is valid when $H(z)$ is rational. We first have to indicate for which minimal value of N the set $\mathcal{G}_N$ is not reduced to $\{0\}$. For this, we remark that $x^T(z)H(z) = 0$ if and only if $x^T(z)B(z) = 0$, which, in turn, is equivalent to $(x_0^T \cdots x_N^T)T_N(B) = 0$. Therefore, $\mathcal{G}_N$ is not reduced to zero if and only if the left kernel of $T_N(B)$ is nontrivial. As $B(z)$ is irreducible and column reduced, the dimension of this kernel depends on the Kronecker indexes $\{M_i^\perp\}_{i=1}^{q-p}$ of $\mathcal{G}$; we refer the reader to [8], [12], or [5] for more details. In particular, the left kernel of $T_N(B)$ is nontrivial if and only if $N \ge M_1^\perp$. Making use of the identity $\Sigma_{i=1}^{q-p} M_i^\perp = pM$ [8], we get $(q-p)M_1^\perp \le pM$. Therefore, if $(q-p)N \ge pM$, N is greater than $M_1^\perp$ and the set $\mathcal{G}_N$ is not reduced to zero.

As in [5], we want to show that under specific conditions, a vector x satisfies (1) and (2) if and only if $x^T(z)H(z) = 0$. As the cases $0 \le P \le M$ and $P > M$ lead to different results, we consider these two situations separately.

A. The Case $0 \le P \le M$

First, we remark that conditions C1)–C4) imply that the McMillan degree δ_H of $H(z)$ is equal to Mp. Since $H(z)$ is outer, $H(z)$ is a minimal degree causal spectral factor of the "spectral density" $H(z)H^T(z^{-1})$. Due to a standard result [6], the McMillan degree of the transfer function $\Sigma_{i=1}^\infty R_i z^{-i}$ also coincides with Mp. Let us now assume that N is chosen in such a way that $\mathcal{R}_N$ "captures" the McMillan degree, i.e., that

$$\mathrm{Rank}(\mathcal{R}_N) = Mp.$$

Let us factor $\mathcal{R}_N$ as

$$\mathcal{R}_N = \begin{bmatrix} H_1 & \cdots & H_M & \cdots \\ \vdots & & \vdots & \\ H_{N+1} & \cdots & & \end{bmatrix} \begin{bmatrix} H_0^T & & 0 \\ \vdots & \ddots & \\ H_N^T & \cdots & H_0^T \\ H_{N+1}^T & \cdots & H_1^T \\ \vdots & & \vdots \end{bmatrix}$$

$$= \mathcal{H}_l\mathcal{H}_r^T. \tag{3}$$

From the Sylvester inequality, $\mathrm{Rank}(\mathcal{H}_l) \ge Mp$. On the other hand, $\mathrm{Rank}(\mathcal{H}_l) \le \delta_H = Mp$, and thus $\mathrm{Rank}(\mathcal{H}_l) = Mp$. Next, using the identity $H(z)A(z) = B(z)$, we get

$$\begin{bmatrix} H_1 & \cdots & H_M \\ \vdots & & \vdots \\ H_{N+1} & \cdots & H_{N+M} \\ \vdots & \iddots & \vdots \end{bmatrix} \begin{bmatrix} 0 \\ -A_P \\ \vdots \\ -A_1 \end{bmatrix} = \begin{bmatrix} H_{M+1} \\ \vdots \\ H_{N+M+1} \\ \vdots \end{bmatrix}.$$

Thus one can see easily that

$$\mathrm{Rank}(\mathcal{H}_l) = \mathrm{Rank}\left(\begin{bmatrix} H_1 & \cdots & H_M \\ \vdots & & \vdots \\ H_{N+1} & & H_{N+M} \end{bmatrix}\right) = Mp. \quad (4)$$

We may now show how to compute $\mathcal{G}_N$ from $\mathcal{R}_N$. Let x satisfy both (1) and (2). We first claim that (1) implies that

$$[x_N^T \cdots x_0^T] \begin{bmatrix} H_1 & H_2 & \cdots & \cdots \\ \vdots & \iddots & & \\ H_{N+1} & & & \end{bmatrix} = [0 \quad \cdots \quad 0 \quad \cdots]. \quad (5)$$

To see this, we recall that the two sequences $(R_k)_{k\geq 1}$ and $(H_k)_{k\geq 1}$ can be realized in minimal state space form with the same observability pair, i.e., there exist minimal dimension matrices such that $R_k = CA^{k-1}G$ and $H_k = CA^{k-1}B$ for $k \geq 1$ [6]. Thus $\mathcal{R}_N$ admits the minimal factorization

$$\mathcal{R}_N = \begin{bmatrix} C \\ \vdots \\ CA^N \end{bmatrix} [G \quad \cdots \quad A^N G].$$

Hence, (1) implies that $[x_N^T \cdots x_0^T][C^T \cdots (CA^N)^T]^T = 0$, which, in turn, implies (5) because (A, C) is an observability pair of the sequence $(H_k)_{k\geq 1}$. On the other hand, due to (5) and (3), (2) reduces to

$$\begin{bmatrix} H_1 & \cdots & H_{N+1} \\ \vdots & \iddots & \vdots \\ H_{N+1} & \cdots & H_{2N+1} \end{bmatrix} \begin{bmatrix} H_0^T & & 0 \\ \vdots & \ddots & \\ H_N^T & \cdots & H_0^T \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_N \end{bmatrix} = 0. \quad (6)$$

Due to (4), the first matrix on the left-hand side of (6) has full column-rank $(N+1)p$ if and only $N+1 \leq M$. In this case, it admits a left inverse, so that (5) and (6) can finally be gathered into $x^T(z)H(z) = 0$. Conversely, it is obvious, due to (3), that $x^T(z)H(z) = 0$ implies (1) and (2). Gathering these results, we get the first part of Theorem 2.1.

B. The Case $P > M$

In this case, the McMillan degree of $H(z)$ is equal to Pp. Assume that $\mathrm{rank}(\mathcal{R}_N)$ attains its maximum value Pp. Equation $H(z)A(z) = B(z)$ now gives

$$\begin{bmatrix} 0 & & H_0 & \cdots & H_M \\ & \iddots & & & \vdots \\ H_0 & \cdots & & \cdots & H_{P-1} \\ H_1 & \cdots & & \cdots & H_P \\ \vdots & & & & \vdots \\ H_{N+1} & \cdots & & \cdots & H_{N+P} \\ \vdots & & & & \vdots \end{bmatrix} \begin{bmatrix} -A_P \\ \vdots \\ -A_1 \end{bmatrix} = \begin{bmatrix} H_{M+1} \\ \vdots \\ H_P \\ H_{P+1} \\ \vdots \\ \vdots \end{bmatrix} \quad (7)$$

so that (4) becomes

$$\mathrm{Rank}(\mathcal{H}_l) = \mathrm{Rank}\left(\begin{bmatrix} H_1 & \cdots & H_P \\ \vdots & & \vdots \\ H_{N+1} & & H_{N+P} \end{bmatrix}\right) = Pp. \quad (8)$$

Let us assume that (1) and (2) hold. Then (5) and (6) hold as well; repeating the steps of Section III-A, and using (8), we see that $x^T(z)H(z) = 0$ if $N+1 \leq P$. So Theorem 2.1 still holds in the case $P > M$, provided M is replaced by P.

However, it is possible to establish the equivalence between (1), (2), and $x^T(z)H(z) = 0$ for larger values of N. So, let us assume that $(N+1) > P$. From (8), we see that the first matrix of the left-hand side of (6) admits the full rank factorization

$$\begin{bmatrix} H_1 & \cdots & H_{N+1} \\ \vdots & \iddots & \vdots \\ H_{N+1} & \cdots & H_{2N+1} \end{bmatrix} = \begin{bmatrix} H_1 & \cdots & H_P \\ \vdots & & \vdots \\ H_{N+1} & \cdots & H_{N+P} \end{bmatrix} [I_{Pp\times Pp} \quad V]$$

for some matrix V, so that (6) implies

$$[I_{Pp\times Pp} \quad V] \begin{bmatrix} H_0^T & & 0 \\ \vdots & \ddots & \\ H_N^T & \cdots & H_0^T \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_N \end{bmatrix} = 0_{Pp\times 1}.$$

If we show that

$$\begin{bmatrix} H_0^T & & 0 \\ \vdots & \ddots & \\ H_N^T & \cdots & H_0^T \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_N \end{bmatrix} = 0_{(N+1)p\times 1} \quad (9)$$

then $x^T(z)H(z) = 0$ will follow from (5). To establish (9), we introduce the $(N+1-P)p$-dimensional vector y defined by

$$\begin{bmatrix} I_{Pp\times Pp} & V \\ 0 & I_{\substack{(N+1-P)p\times \\ (N+1-P)p}} \end{bmatrix} \begin{bmatrix} H_0^T & & 0 \\ \vdots & \ddots & \\ H_N^T & \cdots & H_0^T \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} 0_{Pp\times 1} \\ y \end{bmatrix}. \quad (10)$$

It is sufficient to show that, under some conditions on N, y is equal to zero. From (7), we easily deduce, after some computations, that

$$\begin{bmatrix} 0 & & H_0 & \cdots & H_M \\ & \iddots & \vdots & & \vdots \\ H_0 & & & & \\ \vdots & & \vdots & & \vdots \\ H_P & \cdots & H_{P+(P-M-1)} & \cdots & H_{2P-1} \end{bmatrix} = \begin{bmatrix} H_1 & \cdots & \cdots & H_P \\ \vdots & & & \vdots \\ & & & \\ \vdots & & & \vdots \\ H_{2P-M} & \cdots & \cdots & H_{3P-M-1} \end{bmatrix} \times E^{-1} \quad (11)$$

where

$$E = \begin{bmatrix} 0 \\ I_{Mp\times Mp} \end{bmatrix}, a, \cdots, \mathcal{A}^{P-M-1}a \Big]$$

$$= \begin{bmatrix} 0 \\ I_{Mp\times Mp} \end{bmatrix}, \begin{bmatrix} -A_P & & 0 \\ \vdots & \ddots & \\ -A_{M+1} & & -A_P \\ \vdots & & \vdots \\ -A_1 & \cdots & -A_{P-M} \end{bmatrix}$$

$$\cdot \begin{bmatrix} I & \cdots & \alpha_{P-M-1} \\ & \ddots & \vdots \\ 0 & & I \end{bmatrix} \Bigg]$$

$$a = \begin{bmatrix} -A_P \\ \vdots \\ \vdots \\ -A_1 \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & \cdots & 0 & -A_P \\ I & & 0 & \vdots \\ & \ddots & & \vdots \\ 0 & & I & -A_1 \end{bmatrix}$$

and $\{\alpha_i\}_{i=0}^{\infty}$ (with $\alpha_0 = I$) are the coefficient matrices of the development of $A^{-1}(z)$ about $z = \infty$. Thus E is indeed invertible because A_P is invertible. If $N+1-P \le P-M \Leftrightarrow N+1 \le 2P-M$, then reading out the $(N+1-P)$ first p-block columns of (11), we get

$$\begin{bmatrix} & 0_{((2P-M)-(N+1))q\times (N+1-P)p} & \\ & & H_0 \\ & \iddots & \vdots \\ H_0 & & H_{N-P} \\ \vdots & & \vdots \\ H_P & \cdots & H_N \end{bmatrix}$$

$$= \begin{bmatrix} H_1 & \cdots & H_P \\ H_{2P-M-N-1} & & H_{3P-M-N-2} \\ H_{2P-M-N} & \cdots & H_{3P-M-N-1} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ H_{2P-M} & \cdots & H_{3P-M-1} \end{bmatrix}$$

$$\cdot E^{-1} \begin{bmatrix} I_{(N+1-P)p\times (N+1-P)p} \\ 0_{(2P-(N+1))p\times (N+1-P)p} \end{bmatrix}. \tag{12}$$

Injecting (12) into (10), then for some matrix F we have

$$y = F \begin{bmatrix} H_{2P-M}^T & \cdots & H_{2P-M-N}^T \\ \vdots & & \vdots \\ H_{3P-M-1}^T & \cdots & H_{3P-M-N-1}^T \end{bmatrix} \begin{bmatrix} x_0 \\ \vdots \\ x_N \end{bmatrix}$$

and thus $y = 0$, because of (5).

REFERENCES

[1] K. Abed-Meraim, Y. Hua, Ph. Loubaton, and E. Moulines, "Subspace method for blind identification of multichannel FIR systems in noise field with unknown spatial covariance," *IEEE Signal Processing Lett.*, vol. 4, pp. 135–137, May 1997.

[2] K. Abed-Meraim, Ph. Loubaton, and E. Moulines, "A subspace algorithm for certain blind identification problems," *IEEE Trans. Inform. Theory*, vol. 43, pp. 499–511, Mar. 1997.

[3] M. Aoki, *State Space Modeling of Time Series*. New York: Springer-Verlag, 1990.

[4] P. E. Caines, *Linear Stochastic Systems*. New York: Wiley, 1988.

[5] F. Desbouvries, I. Fijalkow, and P. Loubaton, "On the identification of noisy MA models," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1810–1814, Dec. 1996.

[6] P. Faurre, M. Clerget, and F. Germain, *Opérateurs Rationnels Positifs*. Paris, France: Dunod, 1979.

[7] I. Fijalkow and P. Loubaton, "Identification of rank one rational spectral densities from noisy observations: A stochastic realization approach," *Syst. Contr. Lett.*, vol. 24, pp. 201–205, 1995.

[8] G. D. Forney, "Minimal bases of rational vector spaces, with applications to multivariable linear systems," *SIAM J. Contr.*, vol. 13, no. 3, pp. 493–520, May 1975.

[9] A. Gorokhov, Ph. Loubaton, "Subspace based techniques for blind separation of convolutive mixtures with temporally correlated sources," *IEEE Trans. Circuits Syst.*, vol. 44, pp. 813–820, Sept. 1997.

[10] E. J. Hannan and M. Deistler, *The Statistical Theory of Linear Systems*. New York: Wiley, 1988.

[11] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.

[12] S. Kung, T. Kailath, and M. Morf, "A generalized resultant matrix for polynomial matrices," in *Proc. IEEE Conf. Decision and Control*, 1976, pp. 892–895.

[13] E. Moulines, P. Duhamel, J. F. Cardoso, and S. Mayrargue, "Subspace methods for the blind identification of multichannel FIR filters," *IEEE Trans. Signal Processing*, vol. 43, pp. 516–525, Feb. 1995.

Impulsive Control

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***Abstract*—In this paper, the concept and principles of impulsive control are presented. Impulsive control can be applied to a class of systems whose state variables are changeable in a very short time period. In particular, the asymptotic stability of the impulsive control of a nonlinear system is given.**

***Index Terms*—Impulsive control, impulsive differential equation.**

I. INTRODUCTION

In this section, we address the issues which concern the necessity of impulsive control.

Definition 1—Impulsive Control: Given a *plant* $\mathcal{P}$ whose state variable is denoted by $X \in \boldsymbol{R}^n$, a set of *control instants* $T = \{\tau_k\}$, $\tau_k \in \boldsymbol{R}$, $\tau_k < \tau_{k+1}$, $k = 1, 2, \cdots$, and *control laws* $U(k, X) \in \boldsymbol{R}^n, k = 1, 2, \cdots$. At each τ_k, X is changed impulsively by $X(\tau_k^+) = X(\tau_k^-) + U(k, X)$ such that the output $Y = f(X)$, f: $\boldsymbol{R}^n \mapsto \boldsymbol{R}^m$, $Y \in \boldsymbol{R}^m$, approaches a goal $Y^* \in \boldsymbol{R}^m$ as $k \to \infty$.

Remarks:

1) At least one state variable in a plant $\mathcal{P}$ can be changed instantaneously to any value which is given by a control law. In this sense, not all physical systems can be controlled by impulsive control schemes. For example, if state variables are voltages across capacitors, it is hard to imagine that one can

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