ON THE IDENTIFICATION OF CERTAIN NOISY FIR CONVOLUTIVE MIXTURES.

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ABSTRACT

In this paper, we address the identification problem of p–inputs q–outputs MA models corrupted by a white noise with unknown covariance matrix in the case where $p < q$. Under certain additional conditions, we show that the generating function of the MA model is identifiable up to a $p \times p$ constant orthogonal matrix by using the autocovariance function of the observation.

1. INTRODUCTION

Let $(y_n)_{n \in \mathbb{Z}}$ be a q-variate time-series given by

$$y_n = [H(z)]v_n + w_n \tag{1}$$

where $H(z) = \sum_{k=0}^{M} H_k z^{-k}$ is a $q \times p$ finite impulse response transfer function, v_n is a p-dimensional (non-observable) white noise sequence for which $E(v_n v_n^T) = I$, and w_n is an additive q-dimensional white noise (i.e. $E(w(n)w^T(m)) = 0$ if $n \neq m$). It is assumed that $p < q$, i.e., the dimension of y is strictly greater than the dimension of the input v and that the transfer function $H(z)$ is irreducible, i.e. that

$$\text{Rank}(H(z)) = p \text{ for each } z \neq 0 \tag{2}$$

If the observations are noiseless (i.e. if $w = 0$), it is well established that the transfer function $H(z)$ can be consistently estimated from the second order statistics of y_n up to a constant $p \times p$ orthogonal matrix (e.g. [13], [3]). A number of efficient algorithms based on the so-called linear prediction approach were recently derived in this aim (see e.g. [9]). Therefore, under condition (2), the convolutive mixture problem considered here can be reduced to a separation of an instantaneous mixture by using the second order statistics of the observation. The purpose of this paper is to study the behaviour of this approach in the noisy case: in other words, if $\Sigma = E(w_n w_n^T)$ is non zero, is it still possible to estimate consistently $H(z)$ up to a constant orthogonal $p \times p$ matrix by using the second order statistics of y_n ? The answer is known to be positive if the covariance matrix Σ can be written as $\Sigma = \sigma^2 I_q$ where σ^2 is an unknown scalar parameter. This corresponds to the case where the components of the additive noise w_n are decorrelated, and have the same variance. In effect, it is possible to identify σ^2 from the covariance matrix $\mathcal{R}_N$ of the vector $Y_N(n) = (y_n^T, \ldots, y_{n-N}^T)^T$ if the parameter N is chosen greater than M. To explain this, we denote by $T_N(H)$ the so-called $q(N+1) \times p(M+N+1)$ Sylvester matrix associated to $H(z)$ given by

$$T_N(H) = \begin{bmatrix} H_0 & \ldots & H_M & & 0 \\ & \ddots & & \ddots & \\ 0 & & H_0 & \ldots & H_M \end{bmatrix} \tag{3}$$

Then, it is clear that $Y_N(n) = T_N(H)V_{M+N}(n) + W_N(n)$ where $V_{M+N}(n)$ and $W_N(n)$ are defined as $Y_N(n)$. The covariance matrix $\mathcal{R}_N$ is thus equal to

$$\mathcal{R}_N = T_N(H)T_N(H)^T + \sigma^2 I$$

As $N \geq M$ and $p < q$, $T_N(H)$ is tall, and $T_N(H)T_N(H)^T$ is a singular matrix. σ^2 is thus the smallest eigenvalue of $\mathcal{R}_N$, and can be consistently estimated from an empirical estimate of $\mathcal{R}_N$. Therefore, one can estimate consistently the second order statistics of the noiseless signal, and use the above mentionned linear prediction approach to estimate $H(z)$ up to an orthogonal $p \times p$ matrix.

However, the assumption that $\Sigma = \sigma^2 I$ may be restrictive in certain contexts. Very often, the components of y_n represent the signals sampled behind a sensor array. If the noise w is a thermal noise due to the acquisition devices of the sensors, its components are likely to be decorrelated. If the sensors are not identical, the variances of the components of w_n do not necessarily coincide. The noise may even be spatially correlated if it is due to a superposition of a large number of weak independent sources. Such a situation arises e.g. in underwater acoustics. In this paper, we are going to present certain results showing that, under certain conditions, it is possible to identify $H(z)$ up to an orthogonal $p \times p$ matrix from the exact second order statistics of y_n in the case where no a priori information on Σ is available.

Let us denote by $(R_n)_{n \in \mathbb{Z}}$ and by $(R_n^y)_{n \in \mathbb{Z}}$ the autocovariance coefficients of the useful signal $[H(z)]v_n$ and of y_n respectively. As w_n is assumed to be white and $H(z)$ is FIR, it is clear that $R_0^y = R_0 + \Sigma$, $R_n^y = R_n$ for $1 \leq |n| \leq M$, and $R_n^y = R_n = 0$ for $|n| > M$. In particular, R_0^y does not bring any information on R_0. We thus reformulate our problem as follows :

> Let $H(z) = \sum_{k=0}^{M} H_k z^{-k}$ be a $q \times p$ irreducible FIR filter, and let (R_n) be the autocovariance function associated to the "spectral density" $S(z) = H(z)H^T(z^{-1})$. How to identify $H(z)$ (up to a $p \times p$ orthogonal matrix) from the knowledge of the truncated sequence $(R_n)_{1 \leq n \leq M}$?

This problem was first introduced in [7] if $p = 1$. It is shown that the unknown $q \times 1$ transfer function $H(z)$ is not necessarily identifiable if $q = 2$. In case of identifiability, an identification procedure based on the stochastic realization theory is proposed. However, it is based on a difficult non convex optimization problem, for which no satisfying solution has been proposed. Later, still in the case $p = 1$, [2] showed that the SIMO FIR subspace identification method introduced in [12] if $\Sigma = \sigma^2 I$ could be generalized if $q \geq 3$. The case $p > 1$ was first considered in [5] in the FIR case, and some results in the IIR case are to published ([6]).

We now precise the content of the paper. In section 2, we first present the results provided by an approach aiming at identifying the unknown coefficient R_0. Subsection 2-1 outlines the approach, subsection 2-2 reformulates some results of [5], and subsection 2-3 shows how they can be improved. Finally, we introduce in section 3 a new identifiability result based on the so-called Wiener-Hopf factorization theory.

In this paper, we concentrate on identifiability results based on the knowledge of the true autocovariance coefficients $(R_n)_{1 \leq n \leq M}$. However, concrete estimation algorithms can be derived immediately from the results of section 2. The practical use of the material presented in section 3 is more involved, and is out of the scope of this paper.

2. IDENTIFICATION OF R_0.

2.1. Outline of the results.

The results of this section are based on the following observation. As $q > p$, it exists certain FIR $1 \times q$ filters $g(z) = \sum_{k=0}^{N} g_k z^{-k}$ (where N is to be determined) for which

$$g(z)H(z) = 0 \text{ for each } z \tag{4}$$

or equivalently for which $g(z)H(z)H(z^{-1})^T = 0$ for each z. Let us assume that a family of degree N FIR filters $(g_1(z), \ldots, g_r(z))$ satisfying (4) is available, and let us set $G(z) = (g_1(z)^T, \ldots, g_r(z)^T)^T = \sum_{n=0}^{N} G_n z^{-n}$. Put $S(z) = \sum_{n=-M}^{M} R_n z^{-n} = R_0 + T(z)$, where $T(z)$ is supposed to be known. As $S(z) = H(z)H(z^{-1})^T$, it is clear that

$$G(z)R_0 = -G(z)T(z) \tag{5}$$

Equating the coefficients of both sides, this equation allows to compute the matrix $\mathcal{G}R_0$ where $\mathcal{G} = (G_0^T, \ldots, G_N^T)^T$. If $\mathcal{G}$ is full column rank, R_0 can be retrieved from (5), and $H(z)$ can be identified from the whole sequence $(R_n)_{n=0,M}$.

This approach is based on the FIR filters satisfying (5). Therefore, it is useful to to present some of their properties. for this, we have to recall some well known results related to rational subspaces.

A review of rational subspaces Let us first recall that the set $\mathcal{F}_q$ of all $q \times 1$ rational transfer functions is a q-dimensional subspace over the field $\mathcal{F}_1$ of all scalar rational transfer functions. Let $\mathcal{S}$ be a p-dimensional ($p < q$) subspace of $\mathcal{F}_q$. It therefore admits bases $F(z) = (F_1(z), \ldots, F_p(z))$ characterized by the fact that Rank$(F(z)) = p$ for almost all z : the rational matrix valued function $F(z)$ is said to have a normal rank equal to p. $\mathcal{S}$ admits polynomial bases. A polynomial basis $(F_1(z), \ldots, F_p(z))$ is said to be minimal if $\sum_{i=1}^{p} \deg(F_i(z))$ is minimum (see [10] for more details). All minimal bases share the same degrees $(M_i)_{i=1,p}$, and are characterized by the well known criterion (see [8], [10]) :

Proposition 1 *The polynomial basis $(F_1(z), \ldots, F_p(z))$ is minimal if and only if the matrix polynomial $F(z) = (F_1(z), \cdots, F_p(z))$ is irreducible and column reduced.*

Usually, the minimal degrees $(M_i)_{i=1,p}$ are called the Kronecker indices associated to $\mathcal{S}$. The "orthogonal" $\mathcal{B}$ of $\mathcal{S}$ is the $(q-p)$-dimensional subspace of all $1 \times q$ rational transfer functions $g(z)$ satisfying $g(z)f(z) = 0$ for each $f \in \mathcal{S}$. $\mathcal{B}$ admits Kronecker indices denoted $(M_j^{\perp})_{j=1,q-p}$, which satisfy the important equality:

$$\sum_{i=1}^{p} M_i = \sum_{j=1}^{q-p} M_j^{\perp} \tag{6}$$

Finally, we recall that if $F(z) = (F_1(z), \cdots, F_p(z))$ is a minimal polynomial basis of $\mathcal{S}$, then the rank of the Sylvester matrix $T_N(F)$ is given by [11] :

$$\text{Rank}(T_N(F)) = q(N+1) - \sum_{j, M_j^{\perp} \leq N} (N + 1 - M_j^{\perp}) \tag{7}$$

As $H(z)$ satisfies condition (2), the rational space generated by its columns is p–dimensional. From now on, this subspace is denoted by $\mathcal{S}$ and its $(q-p)$–dimensional dual space by $\mathcal{B}$. By the very definition of $\mathcal{B}$, it is clear that a degree N $1 \times q$ FIR filter $g(z)$ satisfies (4) if and only if $g(z)$ is a degree N polynomial of $\mathcal{B}$. If we denote by $(M_j^{\perp})_{j=1,q-p}$ the dual Kronecker indices of $\mathcal{B}$, it turns out that if $N < M_1^{\perp}$, then $g(z)H(z) = 0$ holds if and only if $g(z) = 0$ for each z, while if $M_s^{\perp} \leq N < M_{s+1}^{\perp}$, then it exists exactly s linearly independant degree N FIR filters $g(z)$ for which $g(z)H(z) = 0$. In particular, if $N \geq M_{q-p}^{\perp} = \max_j M_j^{\perp}$, then it exists a polynomial basis of degree N FIR filters in $\mathcal{B}$.

2.2. The use of the block-Hankel matrix associated to the sequence $(R_n)_{1 \leq n \leq M}$.

In this paragraph, we propose an approach which allows to compute degree $M-1$ FIR filters of $\mathcal{B}$ from $(R_n)_{1 \leq n \leq M}$. In order the approach to be effective, we assume in this subsection that $H(z)$ satisfies the following extra-assumptions :

- The columns of $H(z)$ share the same degree M
- Rank$(H_M) = p$

Let $\mathcal{H}$ be the $qM \times qM$ block Hankel matrix given by

$$\mathcal{H} = \begin{bmatrix} R_1 & \cdots & R_M \\ \vdots & & \\ R_M & & 0 \end{bmatrix} \tag{8}$$

$\mathcal{H}$ can be factored as

$$\mathcal{H} = \begin{bmatrix} H_1 & \cdots & H_M \\ \vdots & & \\ H_M & & 0 \end{bmatrix} \begin{bmatrix} H_0^T & & 0 \\ \vdots & \ddots & \\ H_{M-1}^T & \cdots & H_0^T \end{bmatrix} = \mathcal{O}\mathcal{C}^T \tag{9}$$

As H_M and H_0 are full rank column, it is clear that $\mathcal{O}$ and $\mathcal{C}$ have are also full column rank Mp. Therefore, the rank of $\mathcal{H}$

is also equal to Mp. Let J be the $q-$block exchange matrix : $J = J_{M\times M} \otimes I_q$, where $\otimes$ denotes the Kronecker product, and $(J_{M\times M}(i,j) = \delta_{i+j-(M+1)})_{i,j=1}^{M}$. It is easy to check that a row qM–dimensional vector $g = (g_0, \dots, g_{M-1})$ (where each g_k is q–dimensional) satisfies

$$g\mathcal{H} = 0 \text{ and } \mathcal{H}Jg^T = 0 \tag{10}$$

if and only if $gT_{M-1}(H) = 0$. Therefore, $\text{Ker}^l(T_{M-1}(H)) = \text{Ker}^l(\mathcal{H}) \cap \text{Ker}^l(J\mathcal{H}^T)$ (Ker^l stands for the left Kernel). Let $g(z) = \sum_{k=0}^{M-1} g_k z^{-k}$ be the degree $M-1$ FIR filter associated to g. Then, it is easily seen that $gT_{M-1}(H) = 0$ if and only if $g(z)H(z) = 0$ for each z. Therefore, there is a one to one correspondance between the space $\text{Ker}^l(\mathcal{H}) \cap \text{Ker}^l(J\mathcal{H}^T)$ and the set of all degree less than or equal to $M-1$ FIR filters of $\mathcal{B}$. We have first to check if this subset of $\mathcal{B}$ is not reduced to $\{0\}$. For this, we have to compare $M-1$ with the dual Kronecker indices of $\mathcal{S}$. As $H(z)$ is irreducible (condition (2)) and column reduced (because $\text{Rank}(H_M) = p$), $H(z)$ is a minimal polynomial basis $\mathcal{S}$. The Kronecker indices of $\mathcal{S}$ all coincide therefore with M. Hence, by relation (6),

$$pM = \sum_{j=1}^{q-p} M_j^{\perp}$$

If $(M-1) < M_1^{\perp}$, we get that $pM > (q-p)(M-1)$, i.e. $p(2M-1) > q(M-1)$, or equivalently $q < 2p + p/(M-1)$. Consequently, if $q \geq 2p + p/(M-1)$, it exists at least a non zero degree $M-1$ FIR filter $g(z)$ satisfying $g(z)H(z) = 0$ for each z. This condition, which is assumed to hold from now on in this paragraph, implies in particular that $q > 2p$. It remains to investigate under which conditions it exist degree $M-1$ FIR filters $g_1(z), \dots, g_r(z)$ such that the matrix $\mathcal{G}$ associated to $G(z) = (g_1^T(z), \dots, g_r^T(z))^T$ is full rank column. For this, we first note that $\mathcal{G}$ full rank is equivalent to the condition

$$G(z)x_0 = 0 \text{ for each } z \Rightarrow x_0 = 0 \tag{11}$$

In order to guarantee the validity of this condition, the number of rows r of $G(z)$ have of course to be as large as possible. However, what is really important is the number of linearly independent (over the field $\mathcal{F}$) rows of $G(z)$: if say $g_r(z)$ belongs to the rational space generated by the first $r-1$ rows of $G(z)$, $\mathcal{G}$ is full rank column if and only if the matrix $\mathcal{G}'$ associated to $G'(z) = (g_1^T(z), \dots, g_{r-1}^T(z))^T$ is itself full column rank. The number s of linearly independent degree $M-1$ FIR filters of $\mathcal{B}$ is defined by the fact that

$$M_s^{\perp} \leq M-1 < M_{s+1}^{\perp}$$

The maximum value of s is of course equal to $q-p$ (the dimension of $\mathcal{B}$, and is reached if $(M-1) \geq M_{q-p}^{\perp} = \max_{j=1,q-p} M_j^{\perp}$. More precisely, we have the following result :

Theorem 1 *Let us assume that $(M-1) \geq M_{q-p}^{\perp}$ (in particular $M \geq 1$) and let $G(z) = (g_1(z)^T, \dots, g_{q-p}(z)^T)^T$ be $q-p$ linearly independent degree $M-1$ FIR filters satisfying $g_i(z)H(z) = 0$ for each z. Then, the matrix $\mathcal{G} = (G_0^T, \dots, G_{M-1}^T)^T$ associated to $G(z)$ is full rank column, so that R_0 can be identified. from relation (5).*

Proof. In order to establish that $\mathcal{G}$ is full rank column, we consider a vector x_0 such that $G(z)x_0 = 0$ for each z. As the $q-p$ rows of $G(z)$ are linearly independent, $G(z)$ is a basis of the rational space $\mathcal{B}$. Therefore, the condition $G(z)x_0 = 0$ for each z holds if and only if the constant FIR filter $x(z) = x_0$ belongs to the dual space of $\mathcal{B}$, i.e. to $\mathcal{S}$. But, as $M \geq 1$, the space $\mathcal{S}$ does not contain non zero constant vectors. This in turn implies that $x_0 = 0$. ■

In practice, if the conditions of the theorem are met, this result is equivalent to the following property : let $U = (U_0, \dots, U_{M-1})$ be an orthonormal basis of $\text{Ker}^l(\mathcal{H}) \cap \text{Ker}^l(J\mathcal{H}^T)$. Then, the matrix $\mathcal{U} = (U_0^T, \dots, U_{M-1}^T)^T$ is full rank column, so that R_0 can be identified from the product $\mathcal{U}R_0$. This can lead immediately to a concrete estimation algorithm based on the empirical autocovariance coefficients of the observation. We also note that the dimension of $\text{Ker}^l(\mathcal{H}) \cap \text{Ker}^l(J\mathcal{H}^T)$ is generally much greater than the dimension $q-p$ of $\mathcal{B}$. This is because the linear independence of the rows of U does not imply the linear independence over the field $\mathcal{F}$ of their associated $1 \times q$ FIR filters. In other words, the rows of $U(z) = \sum_{k=0}^{M-1} U_k z^{-k}$ span $\mathcal{B}$, but are not linearly independent over $\mathcal{F}$.

Finally, we remark that the condition $M-1 \geq M_{q-p}^{\perp}$ can be restrictive. Consider for example the case $p=2$, $q=5$ and $M=5$. Among the 14 triples $M_1^{\perp}, M_2^{\perp}, M_3^{\perp}$ satisfying $0 \leq M_1^{\perp} \leq M_2^{\perp} \leq M_3^{\perp}$ and $\sum_{j=1}^{3} M_j^{\perp} = 10$, only 2 satisfy the above condition. If $M-1 < M_{q-p}^{\perp}$, one can extract $s < (q-p)$ linearly independent degree $M-1$ FIR filters of $\mathcal{B}$. However, we have no reasonnable condition guaranteeing that the corresponding matrices $\mathcal{G}$ are full rank column.

2.3. The use of the derivative of $S(z)$.

The main limitation of the approach presented subsection 2-2 follows from the fact that it just provides degree $M-1$ FIR filters of $\mathcal{B}$. Here, we propose an alternative approach in order to overcome this drawback. It is based on the use of the derivative $S'(z)$ w.r.t. z^{-1} of $S(z)$. $S(z) = H(z)H(z^{-1})^T$ is also given by

$$S(z) = R_0 + \sum_{n=-M,M,n\neq 0} R_n z^{-n}$$

Therefore, $S'(z) = \sum_{n=-M}^{-1} nR_n z^{-(n-1)} + \sum_{n=1}^{M} -nR_n z^{-(n-1)}$ does not depend on R_0, and is therefore known. On the other hand,

$$S'(z) = \begin{bmatrix} H'(z) & H(z) \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & z^2 I_p \end{bmatrix} \begin{bmatrix} H^T(z^{-1}) \\ -H'^T(z^{-1}) \end{bmatrix} \tag{12}$$

where $H'(z)$ stands for the derivative of $H(z)$ w.r.t. z^{-1}. In this subsection, we assume the following condition on $H(z)$:

$$\text{Normal Rank}(H'(z), H(z)) = 2p \tag{13}$$

i.e. the dimension of the rational space $\mathcal{S}'$ generated by the columns of $(H'(z), H(z))$ is equal to $2p$, or equivalently, $\text{Rank}(H'(z), H(z)) = 2p$ for almost all z. This of course implies that $q > 2p$.

Let us outline the proposed approach. As $S'(z)$ is known, one can extract for each N the set of all degree N $1 \times q$ FIR filters $g(z) = \sum_{k=0}^{N} g_k z^{-k}$ satisfying $g(z)S'(z) = 0$ for each z: in effect, $g(z)S'(z) = 0$ holds if and only if the row vector $g = (g_0, \dots, g_N)$ belongs to the left kernel of the Sylvester matrix associated to $S'(z)$. As condition (13) holds, we get from formula (12) that $g(z)S'(z) = 0$ if and only if

$$g(z)(H'(z), H(z)) = 0$$

This in particular implies that $g(z)H(z) = 0$ for each z. Therefore, using the derivative of $S(z)$ allows to extract certain FIR degree N filters of $\mathcal{B}$, whatever N is. However, these elements do not span in general the whole set $\mathcal{B}$ because they generate the dual space $\mathcal{B}'$ of $\mathcal{S}'$ which has dimension $q-2p < q-p$. We now check if one can build from these elements of $\mathcal{B}$ a matrix polynomial $G(z)$ for which $\mathcal{G}$ is full column rank.

Theorem 2 *Assume that the rational space $\mathcal{S}'$ does not contain non zero constant vectors, i.e. that its smallest Kronecker index M_1' is non zero. Then, if $N \geq \sum_{i=1}^{p}(deg(H_i(z) + deg(H_i'(z))$, it exists $q-2p$ degree N linearly independent FIR filters $G(z) = (g_1(z)^T, \dots, g_{q-2p}(z)^T)^T$ for which the associated matrix $\mathcal{G}$ is full rank column.*

Proof. Denote by $\mathcal{B}'$ the $q-2p$ dimensional dual space of $\mathcal{S}'$, and by $(M_i')_{i=1,p}$ and $(M_j^{\perp\prime})_{j=1,q-2p}$ its Kronecker and dual Kronecker indices. It is clear that

$$\begin{aligned} \sum_{j=1}^{q-2p} M_j^{\perp\prime} &= \sum_{i=1}^{p} M_i' \\ &\leq \sum_{i=1}^{p}(\deg(H_i(z) + \deg(H_i'(z)) \end{aligned}$$

Therefore, $M_{q-2p}^{\perp\prime} \leq \sum_{i=1}^{p}(\deg(H_i(z) + \deg(H_i'(z)$. If N is chosen as in the statement of the theorem, N is greater than $M_{q-2p}^{\perp\prime}$. It thus exists a polynomial basis $G(z) = (g_1(z)^T, \dots, g_{q-2p}(z)^T)^T$ for which $\deg(g_i(z)) \leq N$ for each $i = 1, q-2p$. Let $\mathcal{G}$ be the corresponding matrix. If $\mathcal{G}$ is not full column rank, it exists a non zero q–dimensional constant vector x_0 for which $G(z)x_0 = 0$ for each z. As $G(z)$ is a basis of $\mathcal{B}'$, this holds if and only if the constant polynomial vector $x(z) = x_0$ belongs to the dual space of $\mathcal{B}'$, i.e. to $\mathcal{S}'$. As $M_1' \geq 1$, x_0 must be reduced to 0, i.e. $\mathcal{G}$ is full rank column. ■

This result shows that if N is chosen large enough, then a basis of the left Kernel of $T_N(S')$ allows to identify the matrix R_0 provided that $M_1' \geq 1$. This condition implies in particular that $\deg(H_i(z)) \geq 2$ for each i. We note however that this new approach needs less restrictive assumptions on $H(z)$ than the exploitation of the block Hankel matrix $\mathcal{H}$. In particular, we do not need that $\deg(H_i(z) = M$ for each i and the quite restrictive assumption $M - 1 \geq M_{q-p}^{\perp}$.

3. A WIENER-HOPF FACTORIZATION BASED APPROACH.

We finally present briefly an alternative approach. It needs the following extra assumption :

$$\text{Rank}(H'(z), H(z)) = 2p \text{ for each } z, |z| \geq 1 \qquad (14)$$

Put $W_+(z) = (H'(z), H(z))$, $\Lambda(z) = \begin{pmatrix} I_p & 0 \\ 0 & z^2 I_p \end{pmatrix}$, and $W_-(z) = \begin{pmatrix} H^T(z^{-1}) \\ -H'^T(z^{-1}) \end{pmatrix}$. It is clear that condition (14) implies that $\text{Rank}(W_-(z)) = 2p$ for $|z| \leq 1$. Therefore, the factorization (12) $S'(z) = W_+(z)\Lambda(z)W_-(z)$ is a so-called Wiener-Hopf factorization ([4]) of $S'(z)$, and its 3 factors $W_+(z)$, $\Lambda(z)$, and $W_-(z)$ are uniquely defined up to certain non trivial indeterminacies. More precisely, the following result holds (see [4] for more details).

Theorem 3 *Let $S'(z) = W_+'(z)\Lambda'(z)W_-'(z)$ be an another Wiener-Hopf factorization of $S'(z)$, i.e. $W_+'(z)$ is analytic and has rank $2p$ in $|z| \geq 1$, $W_-'(z)$ is analytic and has rank $2p$ in $|z| \leq 1$, and $\Lambda'(z)$ is a diagonal matrix with diagonal entries $(z^{k_1}, \dots, z^{k_{2p}})$, where the indices $k_1 \leq \dots \leq k_{2p}$ belong to $\mathbb{Z}$. Then, $\Lambda'(z) = \Lambda(z)$, $W_+'(z) = W_+(z)C_+^{-1}(z)$, and $W_-'(z) = \Lambda^{-1}C_+(z)\Lambda(z)W_-(z)$ where $C_+(z)$ belongs to the multiplicative group $\mathcal{C}_+$ of $2p \times 2p$ block upper triangular matrices*

$$C_+(z) = \begin{bmatrix} C_{1,1} & C_{1,2}(z) \\ 0 & C_{2,2} \end{bmatrix}, \qquad (15)$$

satisfying : $C_{1,1}$ and $C_{2,2}$ are constant regular matrices, and $C_{1,2}(z) = \sum_{i=0}^{2} C_{1,2}^{i} z^{-i}$ is a polynomial matrix in z^{-1} of degree at most 2.

This shows that the matrix $W_+(z) = (H'(z), H(z))$ is identifiable from $S'(z)$ up to certain non trivial indeterminacies. We now show that taking into account the particular structure of $W_+(z)$ (i.e. its first block is the derivative of its second block) allows to raise the indeterminacies.

Theorem 4 *Assume that the matrix (H_1, H_2) is full rank column. Let $W_+'(z)$ be a left Wiener-Hopf factor of $S'(z)$. Then, the entries of the matrix $C_+(z)$ of $\mathcal{C}_+$ defined by $W_+'(z) = W_+(z)C_+(z)$ are defined up to a constant invertible $p \times p$ matrix.*

Proof. Put $W_+'(z) = (P_1(z), P_2(z))$. Then,

$$[H'(z), H(z)] = [P_1(z), P_2(z)] \begin{bmatrix} C_{1,1}^{-1} & -C_{1,1}^{-1}C_{1,2}(z)C_{2,2}^{-1} \\ 0 & C_{2,2}^{-1} \end{bmatrix} \qquad (16)$$

Therefore,

$$\left(P_2(z)C_{2,2}^{-1} - P_1(z)C_{1,1}^{-1}C_{1,2}(z)C_{2,2}^{-1}\right)' = P_1(z)C_{1,1}^{-1} \qquad (17)$$

Equating the coefficients of both sides of (17) leads immediately to the matrix equation :

$$\begin{bmatrix} 0 & P_{1,0} & P_{1,1} & P_{1,0} \\ 2P_{1,0} & 2P_{1,1} & 2P_{1,2} & P_{1,1} \\ 3P_{1,1} & 3P_{1,2} & 3P_{1,3} & P_{1,2} \\ 4P_{1,2} & 4P_{1,3} & 4P_{1,4} & P_{1,3} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} C_{1,1}^{-1}C_{1,2}^{(0)} \\ C_{1,1}^{-1}C_{1,2}^{(1)} \\ C_{1,1}^{-1}C_{1,2}^{(2)} \\ -C_{1,1}^{-1}C_{2,2} \end{bmatrix} = \begin{bmatrix} P_{2,1} \\ 2P_{2,2} \\ 3P_{2,3} \\ 4P_{2,4} \\ \vdots \end{bmatrix} \qquad (18)$$

As $P_1(z) = H'(z)C_{1,1}$, it is easy to check that the first matrix of the left hand side of (18) is equal to

$$\begin{bmatrix} 0 & H_1 & 2H_2 & H_1 \\ 2H_1 & 4H_2 & 6H_3 & 2H_2 \\ \vdots & & & \vdots \end{bmatrix} \begin{bmatrix} C_{1,1} & & 0 \\ & \ddots & \\ 0 & & C_{1,1} \end{bmatrix}$$

If (H_1, H_2) if full rank column, this last matrix is also full rank column. Therefore, equation (18) allows to calculate matrices $C_{1,1}^{-1}C_{1,2}^{(0)}, C_{1,1}^{-1}C_{1,2}^{(1)}, C_{1,1}^{-1}C_{1,2}^{(2)}, C_{1,1}^{-1}C_{2,2}$. This means that one can recover the matrix $H(z)C_{1,1}$ from any left Wiener-Hopf factor of $S'(z)$. ■

This identifiability result can be used in order to derive a concrete estimation algorithm of $H(z)$ (up to a constant factor). This algorithm of course requires, at least implicitely the calculation of a left Wiener-Hopf factor of $S'(z)$. This non obvious problem is out of the scope of the present paper, and is under investigation.

4. CONCLUSION

In this paper, we have proposed two second order based statistics approaches to blindly identify up to a constant matrix a noisy MA model. The problem has been formulated as the identification of the filter $H(z)$ from the truncated autocovariance sequence $(R_n)_{n\neq 0}$ associated to the spectral density $S(z) = H(z)H^T(z^{-1})$. The first approach consists in identifying directly R_0 by calculating certain FIR filters $1 \times q$ filters satisfying $g(z)H(z) = 0$. The extraction of these filters use either the block Hankel matrix associated to the sequence $(R_n)_{n\geq 1}$, either the derivative $S'(z)$ of $S(z)$ w.r.t. the variable z^{-1}. The use of $S'(z)$ require less restrictive assumptions on $H(z)$. Finally, we have introduced an alternative approach based on the Wiener-Hopf factorization theory. In this context, an identifiability result has been shown. The practical use of the Wiener-Hopf factorization approach is under study.

5. REFERENCES

[1] K.Abed Meraim, ”Second Order Blind Identification/Equalization: Algorithms and Statistical Performance”, PHD Thesis, Ecole Nationale Supérieure des Télécommunications, Paris, July 1995

[2] K. Abed-Meraim, Y. Hua, Ph. Loubaton, E. Moulines, ”Subspace method for blind identification of multichannel FIR systems in noise field with unknown spatial covariance”, *IEEE Signal Processing Letters*, vol. 4, N. 5, pp. 135–137, May 1997

[3] K. Abed-Meraim, Ph. Loubaton, E. Moulines, ”A subspace algorithm for certain blind identification problems”, *IEEE Transactions on Information Theory*, vol. 43, N. 2, pp. 499–511, March 1997

[4] K. Clancey and I. Gohberg, *Factorization of Matrix Functions and Singular Integral Operators*, Operator Theory: Advances and Applications, Vol. 3, Birkhäuser Verlag, Basel, 1981

[5] F. Desbouvries, I. Fijalkow and P. Loubaton, ”On the identification of noisy MA models”, *IEEE Transactions on Automatic Control*, Vol. 41, N. 12, pp. 1810–14, December 1996

[6] F. Desbouvries and P. Loubaton, ”On the identification of certain rational transfer functions from truncated autocovariance sequences”, To appear in *IEEE Transactions on Automatic Control*, 1999.

[7] I. Fijalkow and P. Loubaton, ”Identification of rank one rational spectral densities from noisy observations: a stochastic realization approach”, *System and Control Letters* 24, pp. 201–205, 1995

[8] G. D. Forney, ”Minimal bases of rational vector spaces, with applications to multivariable linear systems”, *SIAM Journal on Control*, vol 13, N. 3, pp. 493–520, May 1975

[9] A. Gorokhov, Ph. Loubaton, “Blind identification of MIMO-FIR systems : a generalized linear prediction approach”, to appear in *Signal Processing*, 1999.

[10] T. Kailath, *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ, 1980

[11] S. Kung, T. Kailath and M. Morf, ”A generalized resultant matrix for polynomial matrices”, *Proceedings of the IEEE Conference on Decision and Control, Florida* pp. 892–95, 1976

[12] E. Moulines, P. Duhamel, J.F. Cardoso, S. Mayrargue, ”Subspace methods for the blind identification of multichannel FIR filters”, *IEEE Transactions on Signal Processing*, vol. 43, N. 2, pp. 516–525, February 1995

[13] D. Slock, “Blind joint equalization of multiple synchroneous mobile users using oversampling and/or multiple antennas”, *Proc. Asilomar Conf.*, October 1994.

[14] L. Tong, G. Xu, T. Kailath, ”A new approach to blind identification and equalization of multipath channels”, *Proc. Asilomar Conf.*, 1991, pp. 856-860.