Generalization of Whittle's formula to compound-Gaussian processes

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***Abstract*—This letter presents an extension of the well-known Whittle's formula for the asymptotic Fisher information matrix (FIM) on the power spectrum parameters of zero-mean stationary Gaussian processes to compound Gaussian processes (CGP). The new formula includes a corrective factor that depends on the considered CG distribution, in addition to the usual Gaussian term.**

***Index Terms*—Whittle's formula, Slepian-Bangs's formula, Cramér-Rao bound, compound-Gaussian process, Elliptical symmetric distributions, Student's t processes.**

I. Introduction

Parametric discrete-time CGP, also known as spherically invariant random processes (SIRP), are widely used in many statistical and engineering applications [1]. The CGP-based estimation problem attracting considerable interest is that of estimating its dependent parameters from a set of n consecutive measurements. The Cramér-Rao bound (CRB), which is typically computed as the inverse of the FIM, enables the evaluation of the performance of various parameter estimation algorithms. It is important to note closed-form expressions of the FIM, known as Slepian-Bangs (SB) formulas have been derived for the real, circular and noncircular complex Gaussian distributions in [2] [3], [4] and [5], respectively. These formulas were recently extended to circular and noncircular complex elliptically symmetric (ES) distributions in [6], [7] and [8], respectively. These latter formulas have been later extended when the density generator is considered as an infinite-dimensional nuisance parameter [9] or parameterized by a nuisance parameter [10].

However, the direct computation of these SB formulas roughly requires a number of operations proportional to n^3 in most applications. Additionally, these formulas only allow numerical values without providing any engineering insight into the role of the different parameters. Alternatively, Whittle's asymptotic formula [11] was used to approximate the FIM for zero-mean non-deterministic stationary Gaussian processes. This approximation has much lower computational complexity and can be easily interpreted due to its spectral expression. To the best of our knowledge, no work has yet addressed Whittle's asymptotic formula for zero-mean stationary CGPs.

This letter presents an alternative approximate expression of the FIM and a limit of the FIM rate for zero-mean stationary CGPs. The Toeplitz structure of the covariance matrix of the measurement is profitably used to derive these results, which generalize Whittle's formula [11]

The remainder of this letter is organized as follows. In Sec. II background on stationary CGPs is given, including Bang's formula and Whittle's formula for Gaussian processes. In Sec. III, some new results on Bangs's formula for CG distributed r.v.s. are brought. After proving the EVD of the covariance matrix under various assumptions on the correlation sequence in Sec. IV, an approximation of the FIM and a limit of the FIM rate generalizing Whittle's formula are given for zero-mean stationary CGPs. Finally, Sec. VI concludes this letter.

The following notations are used in the letter: $x =_d y$ and $x \sim D$ mean that the r.v. x and y have the same distribution and x follows the distribution D, respectively.

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II. Statistical Background

A. Preliminaries on stationary CGPs

Let us first recall that a discrete-time random process $(x_k)_{k\in\mathbb{Z}}$ is said to be Gaussian if every finite collection of $\{x_{k_1},...,x_{k_n}\}$ forms a Gaussian random vector $\mathbf{x} \stackrel{\text{def}}{=} (x_{k_1},...,x_{k_n})^T \in \mathbb{R}^n$, denoted $\mathbf{x} \sim \mathcal{N}_n(\boldsymbol{\mu},\boldsymbol{\Sigma})$, with p.d.f.

$$p(\mathbf{x}) = (2\pi)^{-n/2}|\boldsymbol{\Sigma}|^{-1/2}\exp(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})). \quad (1)$$

It is also natural to consider more general families of ES processes (see e.g., [12]) by replacing the exponential function in (1) by an arbitrary function $g_n(.) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that $\int_0^\infty t^{n/2-1}g_n(t)dt < \infty$ (called density generator) to form the p.d.f. of an n-dimensional ES distributed r.v.

$$p(\mathbf{x}) = |\boldsymbol{\Sigma}|^{-1/2}g_n[(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})]. \quad (2)$$

In (2), it is assumed that the first and second-order moments exist, with $\mathrm{E}(\mathbf{x}) = \boldsymbol{\mu}$ and $\mathrm{Cov}(\mathbf{x}) = \boldsymbol{\Sigma}$, in order to avoid any scale ambiguity. Similarly, discrete-time ES random processes can be defined. However, to define time-discrete ES processes in such a way, not all ES distributed r.v.s can be used. In fact, the sequence of density generators $g_n(.)$ cannot be arbitrary, since it must satisfy the Kolmogorov consistency condition according to which the ES distribution is closed under marginalization. This condition amounts to the fact that if (2) is the p.d.f. of $\mathbf{x}_n$, then the p.d.f. of any marginal $\mathbf{x}_n$ of $\mathbf{x}_m$ (where $m > n$) is always given by (2). A necessary and sufficient condition for this property to hold is given in [13, Th. 2.2], namely the existence of a positive r.v. τ of cumulative distribution function $F\tau(.)$ such that:

$$g_n(t) = (2\pi)^{-n/2} \int_0^\infty \tau^{-n/2} \exp(-t/2\tau) dF_\tau(\tau). \quad (3)$$

A further equivalent condition [14] is given by the stochastic representation of the ES distribution:

$$\mathbf{x} =_d \boldsymbol{\mu} + \sqrt{\mathcal{Q}_n} \boldsymbol{\Sigma}^{1/2} \mathbf{u}, \quad (4)$$

where $\mathcal{Q}_n$ is a positive r.v. whose distribution depends on n which is independent of the r.v. $\mathbf{u}$ uniformly distributed on the unit n-sphere, which has the particular form [15]:

$$\mathbf{x} =_d \boldsymbol{\mu} + \sqrt{\tau} \boldsymbol{\Sigma}^{1/2} \mathbf{n}_0, \quad (5)$$

where the r.v τ is independent of the r.v. $\mathbf{n}_0 \sim \mathcal{N}_n(\mathbf{0}, \mathbf{I})$. This is equivalent to state that the r.v.s $\mathcal{Q}_n$ and τ in (4)-(5) are related by $\mathcal{Q}_n =_d \tau \chi^2_n$ where τ and χ^2_n are independent. This stochastic representation (5) characterizes the subclass of CG distributions, whose associated discrete-time random processes are called SIRP in the engineering literature. Further equivalent conditions on $g_n(t)$ in (3) are given in [15]–[17].

Throughout this paper, we consider only zero-mean stationary CG processes $(x_k)_{k \in \mathbb{Z}} \in \mathbb{R}$. Similar to Gaussian processes, this property is ensured i.f.f. these processes are zero-mean wide-sense stationary. Their distribution is thus characterized by the distribution of the v.a. τ and by the covariance matrix $\mathrm{Cov}(\mathbf{x}_n) = \boldsymbol{\Sigma}_{\mathbf{x}_n}$ of the r.v. $\mathbf{x}_n = (x_{k+1}, x_{k+2}, ..., x_{k+n})^T$, which is a symmetric Toeplitz positive definite matrix with $(\boldsymbol{\Sigma}_{\mathbf{x}_n})_{k,\ell} = r_x(\ell - k)$ where the sequence $r_x(k) \stackrel{\text{def}}{=} \mathrm{E}(x_\ell x_{\ell+k})$ is supposed absolutely summable. We assume that the spectrum $S_x(f) = \sum_k r_x(k) e^{-i2\pi kf}$ depends on a parameter $\boldsymbol{\theta} = (\theta_1, .., \theta_q)^T \in \mathbb{R}^q$, which is omitted by simplicity. Although it is assumed that the mapping $\boldsymbol{\theta} \mapsto S_x(f)$ is differentiable in the vicinity of the true value of $\boldsymbol{\theta}$.

B. Reminder of results on Bangs's formulas

This is a brief reminder of the Slepian-Bangs formula for zero-mean ES distributed data $\mathbf{x}_n \in \mathbb{R}^n$. The formula is reduced to Bang's formula, where the density generator $g_n(t)$ of the dependent distribution is either known or unknown [10]. This Bang's formula is given by the elementwise FIM under usual regularity conditions on $g_n(t)$:

$$\begin{aligned} (\mathbf{I}_{\mathbf{x}_n}(\boldsymbol{\theta}))_{k,\ell} &= a_{1,n} \mathrm{Tr}(\boldsymbol{\Sigma}^{-1}_{\mathbf{x}_n} \boldsymbol{\Sigma}'_{\mathbf{x}_n,k} \boldsymbol{\Sigma}^{-1}_{\mathbf{x}_n} \boldsymbol{\Sigma}'_{\mathbf{x}_n,\ell}) \\ &+ a_{2,n} \mathrm{Tr}(\boldsymbol{\Sigma}^{-1}_{\mathbf{x}_n} \boldsymbol{\Sigma}'_{\mathbf{x}_n,k}) \mathrm{Tr}(\boldsymbol{\Sigma}^{-1}_{\mathbf{x}_n} \boldsymbol{\Sigma}'_{\mathbf{x}_n,\ell}) \quad (6) \end{aligned}$$

with $\boldsymbol{\Sigma}'_{\mathbf{x}_n,k} \stackrel{\text{def}}{=} \frac{\partial \boldsymbol{\Sigma}_{\mathbf{x}_n}}{\partial \theta_k}$. The weights $a_{1,n}$ and $a_{2,n}$ are free from scalar ambiguity and given by

$$a_{1,n} = \frac{1}{2} \xi_{1,n} \quad (7)$$

for both classic and semiparametric Bangs's formulas, and

$$a_{2,n} = a^{\mathrm{Clas}}_{2,n} = \frac{1}{4}(\xi_{1,n} - 1), \quad a_{2,n} = a^{\mathrm{SePa}}_{2,n} = -\frac{1}{2n} \xi_{1,n} \quad (8)$$

for classic and semiparametric Bangs's formulas, respectively where

$$\xi_{1,n} = \frac{\mathrm{E}[\mathcal{Q}^2_n \phi^2_n(\mathcal{Q}_n)]}{n(n+2)}, \quad (9)$$

with $\phi_n(t) \stackrel{\text{def}}{=} \frac{2}{g_n(t)} \frac{dg_n(t)}{dt}$.

For Gaussian distributions $\tau = 1$ gives $\xi^{\mathrm{G}}_{1,n} = 1$, and for Student's t-distributions $\tau^{-1} \sim \mathrm{Gam}(\nu/2, 2/\nu)$ for which $\xi^{\mathrm{St}}_{1,n}$ has been derived in the circular complex case in [6], and in the real case in [10] using the real-to-complex representation, yielding

$$\xi^{\mathrm{St}}_{1,n} = \frac{n+\nu}{n+\nu+2}, \quad (10)$$

where $\nu > 2$ is the degree of freedom parameter. For the generalized Gaussian (GG) distributions, it was proved in [19] that if the exponent s satisfies $s \in (0, 1]$, then these distributions belong to the subset of the CG distributions, whose p.d.f. of $\sqrt{\tau}$ is given by [19, Th. 1]. The corresponding $\xi^{\mathrm{GG}}_{1,n}$ was derived in the circular complex case in [7], and in the real case in [10] using the real-to-complex representation, giving

$$\xi^{\mathrm{GG}}_{1,n} = \frac{n+2s}{n+2}. \quad (11)$$

C. Whittle's formula for Gaussian processes

The Whittle's formula presented here provides the limit of the FIM rate in the frequency domain for zero-mean purely non-deterministic stationary Gaussian processes [11, rel. (6.3)].

$$\lim_{n\to\infty} \frac{1}{n} (\mathbf{I}_{\mathbf{x}_n}(\boldsymbol{\theta}))_{k,\ell} = \int_0^1 S^{-2}_x(f) S'_{x,k}(f) S'_{x,\ell}(f) df. \quad (12)$$

with $S'_{x,k}(f) \stackrel{\text{def}}{=} \frac{\partial S_x(f)}{\partial \theta_k}$.

Whittle first derived this formula using the asymptotic covariance least squares estimator in the multivariable framework. Porat [30, Th. 5.3] later proved it in the univariable framework using linear predictions.

III. Bangs's formulas for CG distributions

Using $\mathcal{Q}_n =_d \tau \chi^2_n$ for CG distributions, the general expression (9) of $\xi_{1,n}$ can be rewritten in the following new form:

$$\xi_{1,n} = \frac{1}{n(n+2)} \int_0^\infty \int_0^\infty u^2 v^2 \left(\frac{\int_0^\infty \tau^{-n/2-1} \exp(-uv/2\tau) dF_\tau(\tau)}{\int_0^\infty \tau^{-n/2} \exp(-uv/2\tau) dF_\tau(\tau)} \right)^2 dF_\tau(v) p_{\chi^2_n}(u) du, \quad (13)$$

depending on n and the distribution of the r.v. τ, under the constraint $\mathrm{E}(\tau)=1$ to ensure that $\mathrm{Cov}(\mathbf{x}_n)=\boldsymbol{\Sigma}_{\mathbf{x}_n}$. Note that a direct calculation of the FIM was carried out in the case of complex circular CG distributions [18, Eqs. (19b), (20), (21b), (21c)] which leads to a much less compact expression.

Moreover, it follows from [20, Appendix A] for complex circular case, that for real-valued ES distributions

$$\frac{\mathrm{E}[\mathcal{Q}_n^2\phi_n^2(\mathcal{Q}_n)]}{n(n+2)}=1-2\frac{\mathrm{E}[\mathcal{Q}_n^2\phi_n^{'}(\mathcal{Q}_n)]}{n(n+2)}, \qquad (14)$$

with $\phi_n^{'}(t)\stackrel{\text{def}}{=}\frac{d\phi_n(t)}{dt}$. We get for CG distributions, using $\phi_n^{'}(t)\geq 0$ proved in the Appendix, that $0<\xi_{1,n}\leq 1$, and $\xi_{1,n}=1$ is equivalent to $\mathrm{E}[\mathcal{Q}_n^2\phi_n^{'}(\mathcal{Q}_n)]=0 \Leftrightarrow \mathcal{Q}_n^2\phi_n^{'}(\mathcal{Q}_n)=0$ a.s. $\Leftrightarrow \phi_n^{'}(\mathcal{Q}_n)=0$ because $\mathcal{Q}_n>0$ a.s., i.e., $\mathbf{x}_n$ is Gaussian distributed. This allows us to state the new result:

Result 1: For CG distributions, the parameters $(a_{1,n},a_{2,n})$ of the FIM satisfy the relations: $0\leq a_{1,n}\leq 1/2$ and $a_{2,n}\leq 0$, and $\mathbf{x}_n$ is Gaussian distributed i.f.f. $(a_{1,n},a_{2,n})=(1/2,0)$.

This property proves that the Gaussian distribution always leads to the smallest stochastic CRB for CG distributions, but not in the larger family of the ES distributions as shown by the GG distributions associated with $s>1$ (11). In contrast, for parameterized mean, the Gaussian distribution always yields the largest stochastic CRB for all second-order distributions [21].

The asymptotic behavior of the sequence $\xi_{1,n}$ is generally difficult to analyze from the intricate expression (13), However, in the next section, we will prove that $\lim_{n\to\infty}\xi_{1,n}=1$.

IV. Whittle's formula for CG processes

A. Approximation formula

The following approximate eigenvalue decomposition (EVD) of the Toeplitz structured matrix $\boldsymbol{\Sigma}_{\mathbf{x}_n}$ has been derived and used for large n compared to the correlation length of x_k without rigorous theoretical support (see e.g., [22, eq. (9)], [23, eq. (5)] [24, p.186]):

$$\mathbf{W}_n^H\boldsymbol{\Sigma}_{\mathbf{x}_n}\mathbf{W}_n\approx \mathrm{Diag}(S_x(0),S_x(\frac{1}{n}),...,S_x(\frac{n-1}{n})), \qquad (15)$$

where $\mathbf{W}_n\in\mathbb{C}^{n\times n}$ is the discrete Fourier transform (DFT) unitary matrix defined by $[\mathbf{W}_n]_{k,\ell}\stackrel{\text{def}}{=}\frac{1}{\sqrt{n}}e^{i2\pi(k-1)(\ell-1)/n}$.

But this approximation (15) can be justified by the following limits for a (k,ℓ)-th fixed element of $\mathbf{W}_n^H\boldsymbol{\Sigma}_{\mathbf{x}_n}\mathbf{W}_n$, which are proved in the Supporting Material under the following conditions: For absolutely summable $r_x(k)$, one obtains

$$\lim_{n\to\infty}[\mathbf{W}_n^H\boldsymbol{\Sigma}_{\mathbf{x}_n}\mathbf{W}_n]_{k,k}-S_x(\frac{k-1}{n})=0, \qquad (16)$$

and, additionally, for banded Toeplitz matrices $\boldsymbol{\Sigma}_{\mathbf{x}_n}$

$$\lim_{n\to\infty}[\mathbf{W}_n^H\boldsymbol{\Sigma}_{\mathbf{x}_n}\mathbf{W}_n]_{k,\ell}-\delta_{k,\ell}S_x(\frac{k-1}{n})=0, \qquad (17)$$

where is $\delta_{k,\ell}$ the Kronecker notation. Likewise, (17) also remains valid for the strongest absolutely summable condition $kr_x(k)$. Using $\mathbf{W}_n^H\boldsymbol{\Sigma}^{'}{}_{\mathbf{x}_n,k}\mathbf{W}_n\approx \mathrm{Diag}(S_{x,k}^{'}(0),S_{x,k}^{'}(\frac{1}{n}),...,S_{x,k}^{'}(\frac{n-1}{n}))$ deduced from (15), we get if $S_x(f)>0$ for $f\in[0,1)$:

$$\begin{aligned}&\mathrm{Tr}(\boldsymbol{\Sigma}_{\mathbf{x}_n}^{-1}\boldsymbol{\Sigma}_{\mathbf{x}_n,k}^{'}\boldsymbol{\Sigma}_{\mathbf{x}_n}^{-1}\boldsymbol{\Sigma}_{\mathbf{x}_n,\ell}^{'})=\\ &\quad\mathrm{Tr}[(\mathbf{W}_n\boldsymbol{\Sigma}_{\mathbf{x}_n}^{-1}\mathbf{W}_n^H)(\mathbf{W}_n\boldsymbol{\Sigma}_{\mathbf{x}_n,k}^{'}\mathbf{W}_n^H)(\mathbf{W}_n\boldsymbol{\Sigma}_{\mathbf{x}_n}^{-1}\mathbf{W}_n^H)\\ &(\mathbf{W}_n^H\boldsymbol{\Sigma}_{\mathbf{x}_n,\ell}^{'}\mathbf{W}_n)]\approx n\sum_{p=0}^{n-1}S_x^{-2}(\frac{p}{n})S_{x,k}^{'}(\frac{p}{n})S_{x,\ell}^{'}(\frac{p}{n})\frac{1}{n}. \qquad (18)\end{aligned}$$

We obtain for n sufficiently large, by replacing the Riemann sum at the points for $\{0,\frac{1}{n},...,\frac{n-1}{n}\}$ (18) with an integral, the approximation

$$\frac{1}{n}\mathrm{Tr}(\boldsymbol{\Sigma}_{\mathbf{x}_n}^{-1}\boldsymbol{\Sigma}_{\mathbf{x}_n,k}^{'}\boldsymbol{\Sigma}_{\mathbf{x}_n}^{-1}\boldsymbol{\Sigma}_{\mathbf{x}_n,\ell}^{'})\approx\int_0^1 S_x^{-2}(f)S_{x,k}^{'}(f)S_{x,\ell}^{'}(f)df. \qquad (19)$$

Similarly, we have the following approximation

$$\frac{1}{n}\mathrm{Tr}(\boldsymbol{\Sigma}_{\mathbf{x}_n}^{-1}\boldsymbol{\Sigma}_{\mathbf{x}_n,k}^{'})\approx\int_0^1 S_x^{-1}(f)S_{x,k}^{'}(f)df. \qquad (20)$$

Inserting (19) and (20) in (6), we immediately deduce the following new result:

Result 2: The entries of the FIM for zero-mean stationary CG processes are given for large n by the approximation:

$$\begin{aligned}\frac{1}{n}\left(\mathbf{I}_{\mathbf{x}_n}(\boldsymbol{\theta})\right)_{k,\ell} &\approx a_{1,n}\int_0^1 S_x^{-2}(f)S_{x,k}^{'}(f)S_{x,\ell}^{'}(f)df\\ &+na_{2,n}\int_0^1 S_x^{-1}(f)S_{x,k}^{'}(f)df\int_0^1 S_x^{-1}(f)S_{x,\ell}^{'}(f)df, \qquad (21)\end{aligned}$$

where $a_{1,n}$ and $a_{2,n}$ are respectively given by (7) and (8), which are expressed in terms of $\xi_{1,n}$ in (9).

B. Asymptotic formula

We consider here asymptotic properties of Toeplitz matrices depending on specific conditions on the spectra of x_k. Under the condition that there exists an m such that $S_x(f)\geq m>0$ and thanks to the notion of asymptotic equivalence of sequences of Toeplitz and circulant structured matrices with absolutely summable elements [27], [28], the following limits are proved in the Supporting Material of this paper:

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{Tr}(\boldsymbol{\Sigma}_{\mathbf{x}_n}^{-1}\boldsymbol{\Sigma}_{\mathbf{x}_n,k}^{'}\boldsymbol{\Sigma}_{\mathbf{x}_n}^{-1}\boldsymbol{\Sigma}_{\mathbf{x}_n,\ell}^{'})$$

$$=\int_0^1 S_x^{-2}(f)S_{x,k}^{'}(f)S_{x,\ell}^{'}(f)df \qquad (22)$$

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{Tr}(\boldsymbol{\Sigma}_{\mathbf{x}_n}^{-1}\boldsymbol{\Sigma}_{\mathbf{x}_n,k}^{'}) = \int_0^1 S_x^{-1}(f)S_{x,k}^{'}(f)df. \qquad (23)$$

Limits (22) and (23) are also proved in [26, Th. 5.1] for long-range dependent Gaussian processes, whose spectrum $S_x(f)$ and $(S'_{x,k}(f))_{k=1,..,q}$ satisfy some conditions [26, A2-A7]. However, since the proof only requires second-order properties of x_k, (22) and (23) remain valid for CG ARMA processes.

Finally, for Gaussian ARMA processes, whose spectra $S_x(f) = \sigma^2 \left|\frac{A(e^{-i2\pi f})}{B(e^{-i2\pi f})}\right|^2$ such that $A(z) = \sum_{k=0}^{p} a_k z^k$ and $B(z) = \sum_{k=0}^{q} b_k z^k$ are both bounded away from zero for $|z| \leq 1$, the following stronger limits are proved in [25, Th. 1]:

$$\frac{1}{n}\mathrm{Tr}(\mathbf{\Sigma}_{\mathbf{x}_n}^{-1}\mathbf{\Sigma}_{\mathbf{x}_n,k}^{'}\mathbf{\Sigma}_{\mathbf{x}_n}^{-1}\mathbf{\Sigma}_{\mathbf{x}_n,\ell}^{'}) = \int_0^1 S_x^{-2}(f)S_{x,k}^{'}(f)S_{x,\ell}^{'}(f)df + O(\frac{1}{n}) \quad (24)$$

and

$$\frac{1}{n}\mathrm{Tr}(\mathbf{\Sigma}_{\mathbf{x}_n}^{-1}\mathbf{\Sigma}_{\mathbf{x}_n,k}^{'}) = \int_0^1 S_x^{-1}(f)S_{x,k}^{'}(f)df + O(\frac{1}{n}). \quad (25)$$

These limits are still valid for CG ARMA processes for the same reasons as previously stated.

Now consider, the FIM rate $\frac{1}{n}\mathbf{I}_{\mathbf{x}_n}(\boldsymbol{\theta})$, whose limit as n tends to ∞ exists from (6), (22) and (23) i.f.f $\lim_{n\to\infty} a_{1,n}$ and $\lim_{n\to\infty} na_{2,n}$ exist. Proving these latter limits from (13) is rather challenging. However, it is proven in [29, rel. 32] that $\lim_{n\to\infty}\frac{1}{n}\mathbf{I}_{\mathbf{x}_n}(\boldsymbol{\theta})$ exists for stationary processes with finite Markov order (i.e., where there exists $P \in \mathbb{N}$ such that $p(x_n/x_{n-1}, x_{n-2}, ...) = p(x_n/x_{n-1}, ..., x_{n-P})$).

Thus, we conclude from $na_{2,n}^{\mathrm{Clas}} = \frac{n}{4}(\xi_{1,n} - 1)$ (8) that for these processes $\lim_{n\to\infty}\xi_{1,n} = 1$, which implies from (7) that $\lim_{n\to\infty} a_{1,n} = 1/2$, and that $\lim_{n\to\infty} na_{2,n}^{\mathrm{Clas}} = c \leq 0$ from Result 1 and $\lim_{n\to\infty} na_{2,n}^{\mathrm{SePa}} = -1/2$ from (8). And furthermore, since more knowledge about the density generator leads to a larger FIM, we get $c \geq -1/2$. This allows us to state the following new result, which is an extension of Whittle's formula [11, rel. (6.3)], [30, Th. 5.3]:

Result 3: For stationary CG processes of finite Markov order, the FIM rate limit, which is a generalization of Whittle's formula for purely non-deterministic stationary Gaussian processes, has the following expression

$$\lim_{n\to\infty}\frac{1}{n}\left(\mathbf{I}_{\mathbf{x}_n}(\boldsymbol{\theta})\right)_{k,\ell} = \frac{1}{2}\int_0^1 S_x^{-2}(f)S_{x,k}^{'}(f)S_{x,\ell}^{'}(f)df + c\int_0^1 S_x^{-1}(f)S_{x,k}^{'}(f)df\int_0^1 S_x^{-1}(f)S_{x,\ell}^{'}(f)df, \quad (26)$$

where $c = -1/2$ when $g_n(.)$ is unknown, and $-1/2 \leq c \leq 0$, depending on the CG distribution when $g_n(.)$ is known.

From the values of $\xi_{1,n}$ given in Sec. II-B, $c = 0$, $c = -1/2$, and $c = -(1-s)/2$ for the Gaussian, Student's t and GG distributions with $s \in (0, 1]$, respectively. We will see in the next section that $c = 0$ for the ϵ-contaminated Gaussian distribution.

V. Numerical illustrations

In this section, we assume that x_k is a zero-mean ϵ-contaminated Gaussian distributed with $P(\tau = \tau_1) = 1 - \epsilon$ and $P(\tau = \tau_0) = \epsilon$ under the constraint $\mathrm{E}(\tau) = (1 - \epsilon)\tau_1 + \epsilon\tau_0 = 1$, where (τ_0, ϵ) are parameters that control the heaviness of the tails relative the Gaussian distribution. We assume an AR(1) spectrum, i.e., $S_x(f) = \frac{\sigma_x^2(1-a^2)}{|1-ae^{-i2\pi f}|^2}$ associated with $r_x(k) = \sigma_x^2 a^{|k|}$ and $|a| < 1$, where $\theta_1 \stackrel{\mathrm{def}}{=} a$ is the only unknown parameter.

From numerical calculations, we can state that the approximations (19) and (20) are valid with a relative accuracy of 1% for arbitrary values of $|a| < 1$, and that $\lim_{n\to\infty} na_{2,n} = 0$ as shown e.g. in Fig.1 for $\epsilon = 0.1$ and some values of τ_0. As τ_0 increases, distribution of $\mathbf{x}_n$ deviates more from Gaussian distribution and $na_{2,n}$ converges more slowly to 0.

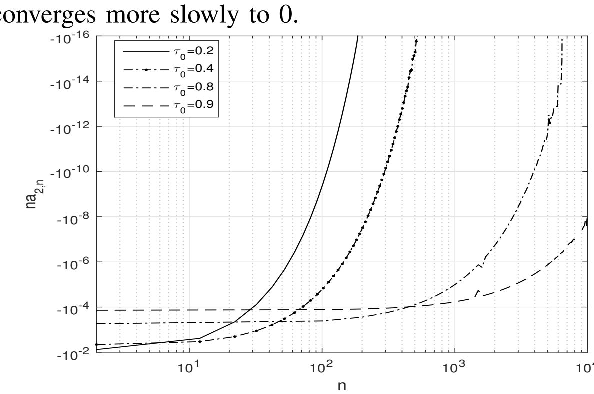

Fig.1 Coefficients $na_{2,n}$ as a function of n.

VI. Conclusion

This letter presents complementary results on Bangs's formula for CG distributed r.v.s. It provides an approximation of the FIM and a limit of the FIM rate, which is a generalization of Whittle's formula, for zero-mean stationary CG distributed random processes. Research is underway on extending Whittle's formula to continuous-time and multivariate stationary CG distributed random processes and giving some applications.

Appendix

A. Proof of $\phi_n^{'}(t) \geq 0$

The first and second derivatives of $g_n(t)$ can be obtained using (3), which yields $g_n^{(k)}(t) = (-2)^{-k}(2\pi)^{-n/2} \int_0^\infty \tau^{-n/2-k}\exp(-t/2\tau)dF_\tau(\tau), k = 1, 2$. Thus, from the definition of $\phi_n(t)$, $\phi_n^{'}(t) \geq 0 \Leftrightarrow g_n^{(2)}(t)g_n(t) \geq [g_n^{(1)}(t)]^2$, which holds thanks to Cauchy-Schwarz inequality. ■

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