Supporting Material for the paper: Generalization of Whittle's formula to compound-Gaussian processes

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I. Proof of eq. (16)

We get straightforwardly

$$[\mathbf{W}_n^H \mathbf{\Sigma}_{\mathbf{x}_n} \mathbf{W}_n]_{k,k} = \sum_{|p|\leq n-1} \left(1-\frac{|p|}{n}\right) r_x(p) e^{-i2\pi p(k-1)/n}.$$

Consequently, as n tends to ∞, $[\mathbf{W}_n^H \mathbf{\Sigma}_{\mathbf{x}_n} \mathbf{W}_n]_{1,1}$ tends to $S_x(0) = \sum_p r_x(p)$ according to the Cesaro summability property [3, A10]. Therefore, for $k>1$, we obtain:

$$[\mathbf{W}_n^H \mathbf{\Sigma}_{\mathbf{x}_n} \mathbf{W}_n]_{k,k} - S_x(\frac{k-1}{n}) = -\sum_{|p|\leq n-1} \frac{|p|}{n} r_x(p) e^{-i2\pi p(k-1)/n} - \sum_{|p|\geq n} r_x(p) e^{-i2\pi p(k-1)/n}, \quad (27)$$

and the modulus of the two terms of (27) are respectively upper-bounded by $\frac{1}{n}\sum_{|p|\leq n-1}|p||r_x(p)|$ and $\sum_{|p|\geq n}|r_x(p)|$. The first bound tends to zero as a consequence of the Cesaro summability property [3, A10], and the second term also tends to zeros as a reminder of the convergent series $\sum_p |r_x(p)|$. ■

II. Proof of eq. (17)

Suppose that $r_x(p)=0$ for $|p|>P$. We get straightforwardly for $k\neq\ell$ and $n>P$:

$$[\mathbf{W}_n^H \mathbf{\Sigma}_{\mathbf{x}_n} \mathbf{W}_n]_{k,\ell} = \frac{1}{n} r_x(0) e^{i2\pi(k-\ell)/n} \sum_{q=1}^{n} [e^{-i2\pi(k-\ell)/n}]^q + \frac{1}{n} \sum_{0<p\leq P} r_x(p) (e^{i2\pi p(1-\ell)/n} + e^{-i2\pi p(1-\ell)/n}) \left(\sum_{q=1}^{n-p} [e^{-i2\pi(k-\ell)/n}]^{q-1}\right), \quad (28)$$

where $\sum_{q=1}^{n}[e^{-i2\pi(k-\ell)/n}]^q = 0$ and $|\sum_{q=1}^{n-p}[e^{-i2\pi(k-\ell)/n}]^{q-1}| = \frac{|\sin(\pi(k-\ell)(n-p)/n)|}{|\sin(\pi(k-\ell)/n)|}$ tends to p when n tends to ∞. ■

Suppose now that $pr_x(p)$ is summable. This naturally implies (16) and for $k\neq\ell$, the second sum of (28) must be replaced by the unbounded sum $\sum_{0<p<n}$ where

$$|r_x(p)(e^{i2\pi p(1-\ell)/n} + e^{-i2\pi p(1-\ell)/n}) \left(\sum_{q=1}^{n-p} [e^{-i2\pi(k-\ell)/n}]^{q-1}\right)| < 2p|r_x(p)|(1+\epsilon), \quad (29)$$

for $n > N(\epsilon)$, $\forall \epsilon>0$. ■

III. Proof of eqs. (22)-(23)

To prove (22)-(23), the concept of asymptotically equivalent sequences of matrices (denoted by $\sim$), introduced by Gray [2], is used to render Szego's theory [1] more accessible to a broader audience. This is achieved by the stronger assumption that the sequence $r_x(k)$ is absolutely summable (i.e., Wiener case).

Following Gray's notation, let $\mathbf{T}_n(S_x) \stackrel{\text{def}}{=} \mathbf{\Sigma}_{\mathbf{x}_n}$, and $\mathbf{C}_n(S_x)$ be an $n\times n$ circulant matrix with the top row $(c_0^n, ..., c_{n-1}^n)$ where $c_\ell^n \stackrel{\text{def}}{=} \frac{1}{n}\sum_{k=0}^{n-1} S_x(\frac{k}{n}) e^{-i2\pi\frac{k}{n}}$.

Assuming $S_x(f)\geq m>0$, [2, Th. 5.2c] implies that $[\mathbf{T}_n(S_x)]^{-1} \sim \mathbf{T}_n(S_x^{-1})$. Then, it follows from [2, Th. 2.1.3] that

$$[\mathbf{T}_n(S_x)]^{-1}\mathbf{T}_n(S'_{x,k}) \sim \mathbf{T}_n(S_x^{-1})\mathbf{T}_n(S'_{x,k}). \quad (30)$$

Furthermore, it follows from [2, Th. 5.3(a), eq. (5.17)] that

$$\mathbf{T}_n(S_x^{-1})\mathbf{T}_n(S'_{x,k}) \sim \mathbf{C}_n(S_x^{-1}S'_{x,k}) \quad (31)$$

Therefore, (23) follows from [2, Th. 5.3(a), eq. (5.19)] with $s=1$. ■

Applying (30) and [2, Th. 2.1.3], we obtain:

$$[\mathbf{T}_n(S_x)]^{-1}\mathbf{T}_n(S'_{x,k})[\mathbf{T}_n(S_x)]^{-1}\mathbf{T}_n(S'_{x,\ell}) \sim \mathbf{T}_n(S_x^{-1})\mathbf{T}_n(S'_{x,k})\mathbf{T}_n(S_x^{-1})\mathbf{T}_n(S'_{x,\ell}), \quad (32)$$

which implies from [2, Th. 5.3(a), eq. (5.22)]:

$$\mathbf{T}_n(S_x^{-1})\mathbf{T}_n(S'_{x,k})\mathbf{T}_n(S_x^{-1})\mathbf{T}_n(S'_{x,\ell}) \sim \mathbf{C}_n(S_x^{-1}S'_{x,k}S_x^{-1}S'_{x,\ell}) \quad (33)$$

and (22) follows from [2, Th. 5.3(a), eq. (5.23)] with $s=1$. ■

References

[1] U. Grenander and G. Szego, *Toeplitz forms and their applications*, Chelsea Publishing Compagny, New York.

[2] R. M. Gray, *Toeplitz and Circulant Matrices: A Review*, The essence of knowledge, Boston Delft, 2006.

[3] B. Porat *Digital Processing of random variables*, Prentice Hall, Inc, 1993.