Slepian–Bangs Formula and Cramér–Rao Bound for Circular and Non-Circular Complex Elliptical Symmetric Distributions

Habti Abeida and Jean-Pierre Delmas

***Abstract*—This letter is mainly dedicated to an extension of the Slepian-Bangs formula to non-circular complex elliptical symmetric (NC-CES) distributions, which is derived from a new stochastic representation theorem. This formula includes the non-circular complex Gaussian and the circular CES (C-CES) distributions. Some general relations between the Cramér Rao bound (CRB) under CES and Gaussian distributions are deduced. It is proved in particular that the Gaussian distribution does not always lead to the largest stochastic CRB (SCRB) as many authors tend to believe it. Finally a particular attention is paid to the noisy mixture where closed-form expressions for the SCRBs of the parameters of interest are derived.**

***Index Terms*—Deterministic and stochastic Cramér Rao bound, Slepian-Bangs formula, Fisher information matrix, circular, non-circular complex elliptical symmetric distributions.**

I. Introduction

To ASSESS the performance of many algorithms, it is necessary to derive the CRB, which is a lower bound on the variance of any unbiased estimator and is generally achieved by the maximum likelihood estimator. This CRB is usually computed as the inverse of the Fisher information matrix (FIM) that must be derived for each distribution of the observations. Fortunately a simple closed-form expression called Slepian-Bangs formula has been derived for the real Gaussian distribution in [1] and [2], then extended to the circular complex normal (C-CN) and non-circular CN (NC-CN) case in [3] and [4], respectively. This formula has been recently extended to the C-CES distribution (see e.g., [5]–[7]) in [8] and [9] and then in [10] and [11] in the context of model misspecification and semiparametric distribution, respectively. However in many fields such as communications and array signal processing, the observations are non-circular, such that a closed-form expression of the FIM for NC-CES distributions is also very useful.

Manuscript received April 8, 2019; revised August 12, 2019; accepted August 27, 2019. Date of publication September 5, 2019; date of current version September 17, 2019. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Ioannis D. Schizas. *(Corresponding author: Jean-Pierre Delmas.)*

H. Abeida is with the Department of Electrical Engineering, University of Taif, Al-Haweiah 21974, Saudi Arabia (e-mail: abeida3@yahoo.fr).

J.-P. Delmas is with the Telecom SudParis, UMR CNRS 5157, Université de Paris Saclay, Evry Cedex 91011, France (e-mail: jean-pierre.delmas@it-sudparis.eu).

This letter has supplementary downloadable material available at http://ieeexplore.ieee.org, provided by the authors.

Digital Object Identifier 10.1109/LSP.2019.2939714

After reformulating the definition of the NC-CES distribution (also called generalized complex elliptical distribution GCES)) introduced in [12], this letter extends the Slepian-Bangs formula to NC-CES distributions thanks to a new stochastic representation theorem based on an equivalent definition of the univariate GCES introduced in [13, def. 2]. It is shown that this formula that includes now a pseudo-scatter matrix encompasses the NC-CN and the C-CES distributions. Then, some general relations between the CRB under CES and Gaussian distributions are deduced when the symmetry center on one side and the scatter and pseudo-scatter matrices on the other side, are each parameterized by different parameters, which are frequently encountered in signal processing applications. In particular, we give sufficient conditions for which the Gaussian distribution does not lead to the largest SCRB. Finally a particular attention is paid to the noisy mixture where the column subspace of the mixing matrix characterizes the parameter of interest for which new concentrated closed-form expressions for the SCRBs are derived. The specific case of direction-of-arrival (DOA) including rectilinear sources is also studied. Note, that a preliminary result of this letter has been briefly introduced in the conference paper [14].

II. Non-Circular CES Distributions

This section briefly presents a reformulation of the definition of the nonsingular NC-CES distribution and extends the stochastic representation theorem (see e.g., [7, th. 3]) that allows us to derive in the next section, the FIM of NC-CES distributions. Moving from the real to the complex representation, the definition [12, def. 1] is equivalent under the absolutely continuous case to the following:

Definition 1: A r.v. $\mathbf{z} \in \mathbb{C}^M$ is said to have non-singular NC-CES distribution (denoted as $\mathbf{z} \sim \mathrm{EC}_M(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\Omega}, g)$) if its probability density function (p.d.f.) is of the form

$$p(\mathbf{z}) = c_{M,g}(\det(\widetilde{\boldsymbol{\Gamma}}))^{-1/2} g(\tilde{\eta}), \tag{1}$$

where $\tilde{\eta} \stackrel{\text{def}}{=} \frac{1}{2}(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})^H \widetilde{\boldsymbol{\Gamma}}^{-1}(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})$ with $\tilde{\mathbf{z}} \stackrel{\text{def}}{=} (\mathbf{z}^T, \mathbf{z}^H)^T$, $\tilde{\boldsymbol{\mu}} \stackrel{\text{def}}{=} (\boldsymbol{\mu}^T, \boldsymbol{\mu}^H)^T$ and $\widetilde{\boldsymbol{\Gamma}} \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{\Sigma} & \boldsymbol{\Omega} \\ \boldsymbol{\Omega}^* & \boldsymbol{\Sigma}^* \end{pmatrix}$ where $\boldsymbol{\Sigma}$ and $\boldsymbol{\Omega}$ are $M \times M$ Hermitian positive definite and complex symmetric matrices, respectively called scatter and pseudo-scatter matrices. $c_{M,g}$ is a normalizing constant ensuring that integrates to one and is given by $c_{M,g} \stackrel{\text{def}}{=} 2(s_M \delta_{M,g})^{-1}$ and where $s_M \stackrel{\text{def}}{=}$

$2\pi^M/\Gamma(M)$ is the surface area of $\mathbb{C}S^M$ (unit complex M-sphere) and $g(.)$ is the non-negative density generator function satisfying $\delta_{M,g} = \int_0^\infty u^{M-1} g(u) du < \infty$. We note that for $\mathbf{\Omega} = \mathbf{O}$ and thus for $\widetilde{\mathbf{\Gamma}} = \mathrm{Diag}(\mathbf{\Sigma}, \mathbf{\Sigma}^*)$, (1) reduces to the p.d.f. of the C-CES distribution [7, rel. (16)].

From the structures of $\mathbf{\Sigma}$ and $\mathbf{\Omega}$, there exists an $M \times M$ non-singular matrix $\mathbf{A}$ such that $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^H$ and $\mathbf{\Omega} = \mathbf{A}\mathbf{\Delta}_\kappa \mathbf{A}^T$ where $\mathbf{\Delta}_\kappa = \mathrm{Diag}(\kappa_1, \ldots, \kappa_M)$ is a real diagonal matrix with non-negative real entries $(\kappa_k)_{k=1,\ldots,M}$ [15, Corollary 4.6.12(b)]. Furthermore, it has been proved in [16], that $0 \leq \kappa_k \leq 1$. This parameterization allows us to state the stochastic representation theorem proved in the Appendix:

Result 1: $\mathbf{z} \sim \mathrm{EC}_M(\boldsymbol{\mu}, \mathbf{\Sigma}, \mathbf{\Omega}, g)$ with $\mathrm{rank}(\mathbf{\Sigma}) = M$ if and only if it admits the stochastic representation:

$$\mathbf{z} =_d \boldsymbol{\mu} + \mathcal{R}\mathbf{A}\mathbf{v}, \tag{2}$$

where $\mathbf{v}$ is defined by

$$\mathbf{v} = \mathbf{\Delta}_1 \mathbf{u} + \mathbf{\Delta}_2 \mathbf{u}^*, \tag{3}$$

with $\mathbf{u} \sim U(\mathbb{C}S^M)$, $\mathbf{\Delta}_1 \stackrel{\text{def}}{=} \frac{\mathbf{\Delta}_+ + \mathbf{\Delta}_-}{2}$ and $\mathbf{\Delta}_2 \stackrel{\text{def}}{=} \frac{\mathbf{\Delta}_+ - \mathbf{\Delta}_-}{2}$ where $\mathbf{\Delta}_+ \stackrel{\text{def}}{=} \sqrt{\mathbf{I} + \mathbf{\Delta}_\kappa}$ and $\mathbf{\Delta}_- \stackrel{\text{def}}{=} \sqrt{\mathbf{I} - \mathbf{\Delta}_\kappa}$.

Note that (2)–(3) is a multivariate extension of the univariate generation of NC-CES r.v. presented in [13, sec. IV.C], and $\mathrm{E}(\mathbf{v}\mathbf{v}^H) = \mathrm{E}(\mathbf{u}\mathbf{u}^H) = \frac{1}{M}\mathbf{I}$ and $\mathrm{E}(\mathbf{v}\mathbf{v}^T) = \frac{1}{M}\mathbf{\Delta}_\kappa$. The p.d.f. of the 2nd-order modular variate $\mathcal{Q} \stackrel{\text{def}}{=} \mathcal{R}^2$ is always given by

$$p(\mathcal{Q}) = \delta_{M,g}^{-1} \mathcal{Q}^{M-1} g(\mathcal{Q}). \tag{4}$$

and it is also proved in the Appendix that

$$\frac{1}{2}(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})^H \widetilde{\mathbf{\Gamma}}^{-1} (\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}}) =_d \mathcal{Q}. \tag{5}$$

III. FIM for Circular and Non-Circular CES Distributions

A. Slepian-Bangs Formula

If $\boldsymbol{\mu}$, $\mathbf{\Sigma}$ and $\mathbf{\Omega}$ are parameterized by the real-valued parameter $\boldsymbol{\alpha}$, the following result is proved in the Appendix.

Result 2: The FIM corresponding to the NC-CES distributed is given (elemenwise) by

$$[\mathbf{I}_{\mathrm{CES}}^{\mathrm{NC}}]_{k,l} = \xi_1 \widetilde{\boldsymbol{\mu}}_k^H \widetilde{\mathbf{\Gamma}}^{-1} \widetilde{\boldsymbol{\mu}}_l + \frac{\xi_2}{2} \mathrm{Tr}[\widetilde{\mathbf{\Gamma}}_k \widetilde{\mathbf{\Gamma}}^{-1} \widetilde{\mathbf{\Gamma}}_l \widetilde{\mathbf{\Gamma}}^{-1}] + \frac{(\xi_2 - 1)}{4} \mathrm{Tr}[\widetilde{\mathbf{\Gamma}}_k \widetilde{\mathbf{\Gamma}}^{-1}] \mathrm{Tr}[\widetilde{\mathbf{\Gamma}}_l \widetilde{\mathbf{\Gamma}}^{-1}], \tag{6}$$

where $\widetilde{\boldsymbol{\mu}}_k \stackrel{\text{def}}{=} \frac{\partial \widetilde{\boldsymbol{\mu}}}{\partial \alpha_k}$ and $\widetilde{\mathbf{\Gamma}}_k \stackrel{\text{def}}{=} \frac{\partial \widetilde{\mathbf{\Gamma}}}{\partial \alpha_k}$ with $\xi_1 \stackrel{\text{def}}{=} \frac{\mathrm{E}[\mathcal{Q}\phi^2(\mathcal{Q})]}{M}$ and $\xi_2 \stackrel{\text{def}}{=} \frac{\mathrm{E}[\mathcal{Q}^2\phi^2(\mathcal{Q})]}{M(M+1)}$ where $\phi(t) \stackrel{\text{def}}{=} g'(t)/g(t)$.

This FIM (6) allows us to derive the FIM associated with T i.i.d. snapshots $\mathbf{z}_t \sim \mathrm{EC}_M(\boldsymbol{\mu}_t, \mathbf{\Sigma}, \mathbf{\Omega}, g)$ by summation of the associated FIM (6), which reduces to [8, eq. (20)] for C-CES distributions for which $\mathbf{\Omega} = \mathbf{0}$ and thus $\widetilde{\mathbf{\Gamma}} = \mathrm{Diag}(\mathbf{\Sigma}, \mathbf{\Sigma}^*)$. Note also that (6) reduces to

$$[\mathbf{I}_{\mathrm{CN}}^{\mathrm{NC}}]_{k,l} = \widetilde{\boldsymbol{\mu}}_k^H \widetilde{\mathbf{\Gamma}}^{-1} \widetilde{\boldsymbol{\mu}}_l + \frac{1}{2} \mathrm{Tr}[\widetilde{\mathbf{\Gamma}}_k \widetilde{\mathbf{\Gamma}}^{-1} \widetilde{\mathbf{\Gamma}}_l \widetilde{\mathbf{\Gamma}}^{-1}], \tag{7}$$

for NC-CN distributions [4, Rel (2.1)] where $g(t) = e^{-t}$ for which $\xi_1 = \xi_2 = 1$.

Finally, note that if $\boldsymbol{\alpha}$ is partitioned as $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1^T, \boldsymbol{\alpha}_2^T)^T$ where $\boldsymbol{\mu}$ and $\widetilde{\mathbf{\Gamma}}$ depend only on $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$, respectively, (which is very frequently encountered in signal processing modeling), the FIM (6) continues to be block diagonal partitioned as for the circular Gaussian Slepian-Bangs formula [3], Consequently, the associated CRB on the parameters $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are decoupled.

B. General CRB Properties

In the previous conditions, we can deduce some general CRB properties. Under finite 2nd-order moments, $\mathbf{\Sigma}$ and $\mathbf{\Omega}$ are proportional to the covariance $\mathbf{R} \stackrel{\text{def}}{=} \mathrm{E}[(\mathbf{z} - \boldsymbol{\mu})(\mathbf{z} - \boldsymbol{\mu})^H]$ and pseudo-covariance $\mathbf{C} \stackrel{\text{def}}{=} \mathrm{E}[(\mathbf{z} - \boldsymbol{\mu})(\mathbf{z} - \boldsymbol{\mu})^T]$, respectively with the same factor. Consequently, it is possible to impose in this case $\mathbf{\Sigma} = \mathbf{R}$ and $\mathbf{\Omega} = \mathbf{C}$ by using similarly as for C-CES distributions, the constraint [7, rel. (20)] on g such that

$$\delta_{M+1,g}/\delta_{M,g} = M. \tag{8}$$

Under this constraint, it is proved in the Appendix that

$$\xi_1 \geq 1. \tag{9}$$

Consequently, we deduce from (6) and the decoupling between $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ in the CRB, the following result:

Result 3: For C-CES and NC-CES distributions with finite 2nd-order moments, the SCRB on the parameter $\boldsymbol{\alpha}_1$ is upper bounded by the SCRB under the Gaussian distributions of same moments $\boldsymbol{\mu}$, $\mathbf{\Sigma}$ and $\mathbf{\Omega}$.

$$\mathrm{SCRB}_{\mathrm{CES}}(\boldsymbol{\alpha}_1) \leq \mathrm{SCRB}_{\mathrm{CN}}(\boldsymbol{\alpha}_1). \tag{10}$$

In particular for noisy linear mixtures where $\mathbf{s}_t$ defined in (15) are considered as nuisance deterministic parameters, the CRB concentrated on the parameter of interest $\boldsymbol{\theta}$ is denoted as the deterministic CRB (DCRB) on $\boldsymbol{\theta}$. By taking the principal submatrix of the FIM (6) on the parameter $\boldsymbol{\alpha}_1$ associated with the parameter $\boldsymbol{\theta}$, we deduce from (9) in the circular and non-circular case:

$$\mathrm{DCRB}_{\mathrm{CES}}(\boldsymbol{\theta}) \leq \mathrm{DCRB}_{\mathrm{CN}}(\boldsymbol{\theta}). \tag{11}$$

These results have been proved by many authors (e.g., [3, B.3.26]) under various conditions in the circular case.

Consider now the parameter $\boldsymbol{\alpha}_2$ of $\mathbf{\Sigma}$ and $\mathbf{\Omega}$. Writing the associated FIM (6) $\mathbf{I}_{\mathrm{CES}}^{\mathrm{NC}}(\boldsymbol{\alpha}_2)$ in matrix form:

$$\mathbf{I}_{\mathrm{CES}}^{\mathrm{NC}}(\boldsymbol{\alpha}_2) = \left(\frac{d\mathrm{vec}(\widetilde{\mathbf{\Gamma}})}{d\boldsymbol{\alpha}_2^T}\right)^H \left(\frac{\xi_2}{2}(\widetilde{\mathbf{\Gamma}}^{-T} \otimes \widetilde{\mathbf{\Gamma}}^{-1}) + \frac{\xi_2 - 1}{4} \mathrm{vec}(\widetilde{\mathbf{\Gamma}}^{-1}) \mathrm{vec}^H(\widetilde{\mathbf{\Gamma}}^{-1})\right) \left(\frac{d\mathrm{vec}(\widetilde{\mathbf{\Gamma}})}{d\boldsymbol{\alpha}_2^T}\right), \tag{12}$$

we prove the following result in the Appendix:

Result 4: For C-CES and NC-CES distributions, the SCRB on the parameter $\boldsymbol{\alpha}_2$ is upper or lower bounded by the SCRB under the Gaussian distributions of same scatter $\boldsymbol{\Sigma}$ and pseudo-scatter $\boldsymbol{\Omega}$ matrices, according to $\xi_2 \geq 1$ and $\xi_2 \leq 1$, respectively.

$$\mathrm{SCRB}_{\mathrm{CES}}(\boldsymbol{\alpha}_2) \leq \mathrm{SCRB}_{\mathrm{CN}}(\boldsymbol{\alpha}_2) \;\text{ if }\; \xi_2 \geq 1 \tag{13}$$

$$\mathrm{SCRB}_{\mathrm{CES}}(\boldsymbol{\alpha}_2) \geq \mathrm{SCRB}_{\mathrm{CN}}(\boldsymbol{\alpha}_2) \;\text{ if }\; \xi_2 \leq 1. \tag{14}$$

This property proves that the Gaussian distributions do not always lead to the largest stochastic CRB as may authors tend to believe it. For example, for the complex Student t distribution of ν degree of freedom ($0 < \nu < \infty$) with $\nu > 2$ to have finite 2nd-order moments (see e.g., [7, sec. IV.A]), $\xi_2 = \frac{\nu/2+M}{\nu/2+M+1} < 1$ [8]. For the complex generalized Gaussian distribution with exponent $s > 0$ (see e.g., [7, sec. IV.B]), $\xi_2 = \frac{s+M}{M+1}$ [9] and $\xi_2 < 1, \xi_2 = 1$ and $\xi_2 > 1$ for $s < 1$ (sub-Gaussian case), $s = 1$ (Gaussian case) and $s > 1$ (super-Gaussian case), respectively.

IV. Stochastic CRB for Noisy Linear Mixture

A. General Model

Consider the following model

$$\mathbf{z}_t = \mathbf{A}_\theta \mathbf{s}_t + \mathbf{n}_t \in \mathbb{C}^M \quad t = 1, \ldots, T, \tag{15}$$

where $(\mathbf{z}_t)_{t=1,\ldots,T}$ are independent zero-mean with finite 2nd-order moments C-CES or NC-CES distributed r.v.s such that $\mathbf{z}_t \sim \mathrm{EC}_M(\mathbf{0}, \boldsymbol{\Sigma}, g)$ or $\mathbf{z}_t \sim \mathrm{EC}_M(\mathbf{0}, \boldsymbol{\Sigma}, \boldsymbol{\Omega}, g)$, respectively. $\mathbf{s}_t$ and $\mathbf{n}_t$ are independent zero-mean r.v.s. $\mathbf{n}_t$ is circular with $\mathrm{E}(\mathbf{n}_t\mathbf{n}_t^H) = \sigma_n^2\mathbf{I}$ and $\mathbf{s}_t \in \mathbb{C}^K$ is either circular with $\mathrm{E}(\mathbf{s}_t\mathbf{s}_t^H) = \mathbf{R}_s$ nonsingular or non-circular with $\mathrm{E}(\tilde{\mathbf{s}}_t\tilde{\mathbf{s}}_t^H) = \mathbf{R}_{\tilde{s}}$ nonsingular with $\tilde{\mathbf{s}}_t \stackrel{\mathrm{def}}{=} (\mathbf{s}_t^T, \mathbf{s}_t^H)^T$ and $\mathrm{E}(\mathbf{s}_t\mathbf{s}_t^T) = \mathbf{C}_s$. Consequently, the scatter and pseudo-scatter matrices of $\mathbf{z}_t$ are given by $\boldsymbol{\Sigma} = \mathbf{A}_\theta\mathbf{R}_s\mathbf{A}_\theta^H + \sigma_n^2\mathbf{I}$ and $\boldsymbol{\Omega} = \mathbf{A}_\theta\mathbf{C}_s\mathbf{A}_\theta^T$, where we assume that the real-valued parameter of interest $\boldsymbol{\theta}$ is characterized by the subspace generated by the columns of the full column rank $M \times K$ matrix $\mathbf{A}_\theta$ with $K < M$ [17] The vector of nuisance parameters is defined by $\boldsymbol{\alpha}_n \stackrel{\mathrm{def}}{=} (\boldsymbol{\rho}^T, \sigma_n^2)^T$ where $\boldsymbol{\rho}$ contains the terms $[\mathbf{R}_s]_{i,j}$ for $1 \leq i \leq j \leq K$ [resp., the terms $[\mathbf{R}_s]_{i,j}$ and $[\mathbf{C}_s]_{i,j}$ for $1 \leq i \leq j \leq K$] in the circular [resp., non-circular] case. Under these conditions, the following result is proved in the Appendix:

Result 5: The SCRB on the parameter $\boldsymbol{\theta}$ alone for the general model (15) is given by:

$$\mathrm{SCRB}_{\mathrm{CES}}(\boldsymbol{\theta}) = \frac{1}{\xi_2}\mathrm{SCRB}_{\mathrm{CN}}(\boldsymbol{\theta}), \tag{16}$$

where $\mathrm{CRB}_{\mathrm{CN}}(\boldsymbol{\theta})$ is the SCRB derived under the Gaussian distributions of $\mathbf{z}_t$, given by [18]:

$$\mathrm{SCRB}_{\mathrm{CN}}(\boldsymbol{\theta}) = \frac{\sigma_n^2}{2T}\left[\mathrm{Re}\left(\frac{d\mathbf{a}_\theta^H}{d\boldsymbol{\theta}}(\mathbf{H}^T \otimes \boldsymbol{\Pi}_{\mathbf{A}_\theta}^\perp)\frac{d\mathbf{a}_\theta}{d\boldsymbol{\theta}}\right)\right]^{-1}, \tag{17}$$

where $\mathbf{a}_\theta \stackrel{\mathrm{def}}{=} \mathrm{vec}(\mathbf{A}_\theta)$, $\boldsymbol{\Pi}_{\mathbf{A}_\theta}^\perp \stackrel{\mathrm{def}}{=} \mathbf{I} - \mathbf{A}_\theta(\mathbf{A}_\theta^H\mathbf{A}_\theta)^{-1}\mathbf{A}_\theta^H$ is the ortho-complement of the projection matrix on the columns of $\mathbf{A}_\theta$ and $\mathbf{H}$ is given by the Hermitian matrices $\mathbf{R}_s\mathbf{A}_\theta^H\boldsymbol{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s$ and $[\mathbf{R}_s\mathbf{A}_\theta^H, \mathbf{C}_s\mathbf{A}_\theta^T]\widetilde{\boldsymbol{\Gamma}}^{-1}\left[\begin{smallmatrix}\mathbf{A}_\theta\mathbf{R}_s\\ \mathbf{A}_\theta^*\mathbf{C}_s^*\end{smallmatrix}\right]$ in the circular and non-circular cases, with $K < M$ and $K < 2M$, respectively.

We note that (17) extends the CRB compact expressions [19, rel. (5)] and [4, rel. (3.9)], in the circular and non-circular cases, respectively, given for the DOA modeling with scalar-sensors for one parameter per source for which:

$$\mathrm{SCRB}_{\mathrm{CN}}(\boldsymbol{\theta}) = \frac{\sigma_n^2}{2T}\left\{\mathrm{Re}\left((\mathbf{D}_\theta^H\boldsymbol{\Pi}_{\mathbf{A}_\theta}^\perp\mathbf{D}_\theta) \odot \mathbf{H}^T\right)\right\}^{-1}, \tag{18}$$

where $\mathbf{A}_\theta \stackrel{\mathrm{def}}{=} [\mathbf{a}_1, \ldots, \mathbf{a}_K]$ and $(\mathbf{a}_k)_{k=1,\ldots,K}$ are respectively, the steering matrix and vectors parameterized by the DOA θ_k with $\boldsymbol{\theta} \stackrel{\mathrm{def}}{=} (\theta_1, \ldots, \theta_K)^T$, and $\mathbf{D}_\theta \stackrel{\mathrm{def}}{=} [\frac{d\mathbf{a}_1}{d\theta_1}, \ldots, \frac{d\mathbf{a}_K}{d\theta_K}]$. But (17) encompasses DOA modeling with vector-sensors for an arbitrary number of parameters per source $s_{t,k}$ (with $\mathbf{s}_t \stackrel{\mathrm{def}}{=} (s_{t,1}, .., s_{t,K})^T$ [20] and many other modelings as the SIMO [21] and MIMO [22] modelings.

B. Stochastic CRB for DOA With Rectilinear Sources

Consider now the DOA modeling with scalar-sensors for one parameter per source and for rectilinear sources, i.e., $s_{t,k} = e^{i\phi_k}r_{t,k}$ where ϕ_k are unknown fixed parameters and $r_{t,k}$ is real-valued with $K < 2M$. Following similar steps as used in proving Result 5 by replacing $\mathbf{A}_\theta$ by $\widetilde{\mathbf{A}}_\omega = \left(\begin{smallmatrix}\mathbf{A}_\theta\boldsymbol{\Delta}_\phi\\ \mathbf{A}_\theta^*\boldsymbol{\Delta}_\phi^*\end{smallmatrix}\right)$ where $\boldsymbol{\Delta}_\phi \stackrel{\mathrm{def}}{=} \mathrm{Diag}(e^{i\phi_1}, \ldots, e^{i\phi_K})$ and $\boldsymbol{\omega} \stackrel{\mathrm{def}}{=} (\boldsymbol{\theta}^T, \boldsymbol{\phi}^T)^T$ with $\boldsymbol{\phi} \stackrel{\mathrm{def}}{=} (\phi_1, \ldots, \phi_K)^T$ and the methodology used in proving [23, th. 1], we obtain the following result proved in the appendix:

Result 6: The SCRB on the parameter $\boldsymbol{\omega}$ is given by

$$\mathrm{SCRB}_{\mathrm{CES}}(\boldsymbol{\omega}) = \frac{1}{\xi_2}\mathrm{SCRB}_{\mathrm{CN}}(\boldsymbol{\omega}), \tag{19}$$

where $\mathrm{SCRB}_{\mathrm{CN}}(\boldsymbol{\omega})$ is the CRB derived under the Gaussian distribution, given by [23, rel. (11)]:

$$\mathrm{SCRB}_{\mathrm{CN}}(\boldsymbol{\omega}) = \frac{\sigma_n^2}{T}\left((\widetilde{\mathbf{D}}_\omega^H\boldsymbol{\Pi}_{\widetilde{\mathbf{A}}_\omega}^\perp\widetilde{\mathbf{D}}_\omega) \odot \left(\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \otimes \widetilde{\mathbf{H}}\right)\right)^{-1}, \tag{20}$$

with $\boldsymbol{\Pi}_{\widetilde{\mathbf{A}}_\omega}^\perp \stackrel{\mathrm{def}}{=} \mathbf{I} - \widetilde{\mathbf{A}}_\omega(\widetilde{\mathbf{A}}_\omega^H\widetilde{\mathbf{A}}_\omega)^{-1}\widetilde{\mathbf{A}}_\omega^H, \widetilde{\mathbf{D}}_\omega \stackrel{\mathrm{def}}{=} [\widetilde{\mathbf{D}}_\theta, \widetilde{\mathbf{D}}_\phi]$ where $\widetilde{\mathbf{D}}_\theta \stackrel{\mathrm{def}}{=} [\frac{\partial\tilde{\mathbf{a}}_1}{\partial\theta_1}, \ldots, \frac{\partial\tilde{\mathbf{a}}_K}{\partial\theta_K}]$, $\widetilde{\mathbf{D}}_\phi \stackrel{\mathrm{def}}{=} [\frac{\partial\tilde{\mathbf{a}}_1}{\partial\phi_1}, \ldots, \frac{\partial\tilde{\mathbf{a}}_K}{\partial\phi_K}]$ and $\tilde{\mathbf{a}}_k \stackrel{\mathrm{def}}{=} (\mathbf{a}_k^Te^{i\phi_k}, \mathbf{a}_k^He^{-i\phi_k})^T$, and $\widetilde{\mathbf{H}} \stackrel{\mathrm{def}}{=} \mathbf{R}_r\widetilde{\mathbf{A}}_\omega^H\widetilde{\boldsymbol{\Gamma}}^{-1}\widetilde{\mathbf{A}}_\omega\mathbf{R}_r$ where $\mathbf{R}_r \stackrel{\mathrm{def}}{=} \mathrm{E}(\mathbf{r}_t\mathbf{r}_t^T)$ with $\mathbf{r}_t \stackrel{\mathrm{def}}{=} (r_{t,1}, .., r_{t,K})^T$.

V. Conclusion

An extension of the Slepian-Bang formula under NC-CES distributions is derived thanks to a new stochastic representation theorem. Comparisons between CRB under CES and Gaussian distributions are presented. In particular conditions are given for which the Gaussian distribution does not lead to the largest stochastic CRB. Finally new closed-form expressions of the CRB on the parameter of interest of noisy mixtures under CES distributions are proved.

Appendix

Detailed proofs are available at [24].

Proof of Result 1 and Eq. (5): Since a linear transform in $\mathbb{R}^{2M}$ is tantamount to $\mathbb{R}$-linear in $\mathbb{C}^M$, the definition of GCES given in [12] is equivalent to saying that

$$\mathbf{z} = \boldsymbol{\mu} + \boldsymbol{\Psi}\mathbf{z}_0 + \boldsymbol{\Phi}\mathbf{z}_0^*, \tag{21}$$

where $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ are $M \times M$ fixed complex-valued matrices and $\mathbf{z}_0$ is a complex spherical distributed r.v. with stochastic representation $\mathbf{z}_0 =_d \mathcal{R}\mathbf{u}$ [7, th. 3]. Since $\mathrm{E}(\mathbf{u}\mathbf{u}^H) = \frac{1}{M}\mathbf{I}$ and $\mathrm{E}(\mathbf{u}\mathbf{u}^T) = \mathbf{0}$, we get:

$$\mathbf{A}\mathbf{A}^H = \boldsymbol{\Psi}\boldsymbol{\Psi}^H + \boldsymbol{\Phi}\boldsymbol{\Phi}^H \text{ and } \mathbf{A}\boldsymbol{\Delta}_\kappa\mathbf{A}^T = \boldsymbol{\Psi}\boldsymbol{\Phi}^T + \boldsymbol{\Psi}\boldsymbol{\Phi}^T, \tag{22}$$

where it is easy to prove that $(\boldsymbol{\Psi}, \boldsymbol{\Phi}) = (\mathbf{A}\boldsymbol{\Delta}_1, \mathbf{A}\{\boldsymbol{\Delta_2})$ is a solution. ■

It is easy to prove that $\widetilde{\boldsymbol{\Gamma}}^{1/2} = \left(\begin{smallmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}^*\end{smallmatrix}\right)\left(\begin{smallmatrix}\boldsymbol{\Delta}_1 & \boldsymbol{\Delta}_2\\ \boldsymbol{\Delta}_2 & \boldsymbol{\Delta}_1\end{smallmatrix}\right)$ and thus $\widetilde{\mathbf{z}} =_d \widetilde{\boldsymbol{\mu}} + \mathcal{R}\widetilde{\boldsymbol{\Gamma}}^{1/2}\widetilde{\mathbf{u}}$ with $\widetilde{\mathbf{u}} \stackrel{\text{def}}{=} (\mathbf{u}^T, \mathbf{u}^H)^T$. Consequently, $\frac{1}{2}(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})^H\tilde{\boldsymbol{\Gamma}}^{-1}(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}}) =_d \frac{1}{2}\mathcal{R}^2\|\widetilde{\mathbf{u}}\|^2 = \mathcal{Q}$. ■

Proof of Result 2: The proof follows the steps of proof in [8, Sec. 3] where $\boldsymbol{\Sigma}$ and η are replaced by $\widetilde{\boldsymbol{\Gamma}}$ and $\tilde{\eta}$, respectively. For example, using now $\widetilde{\mathbf{z}} =_d \widetilde{\boldsymbol{\mu}} + \mathcal{R}\widetilde{\boldsymbol{\Gamma}}^{1/2}\widetilde{\mathbf{u}}$, it follows $\frac{1}{2}(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})^H\widetilde{\boldsymbol{\Gamma}}^{-1}\widetilde{\boldsymbol{\Gamma}}_k\widetilde{\boldsymbol{\Gamma}}^{-1}(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}}) =_d \frac{1}{2}\mathcal{Q}\tilde{\mathbf{u}}^H\widetilde{\mathbf{H}}_k\tilde{\mathbf{u}}$, where $\widetilde{\mathbf{H}}_k \stackrel{\text{def}}{=} \widetilde{\boldsymbol{\Gamma}}^{-1/2}\widetilde{\boldsymbol{\Gamma}}_k\widetilde{\boldsymbol{\Gamma}}^{-1/2}$. Then after proving that the "regularity" condition $\mathrm{E}(\frac{\partial \log p(\mathbf{z};\boldsymbol{\alpha})}{\partial \alpha_k}) = 0$ also applies by proving that $\mathrm{E}(\phi(\tilde{\eta})\frac{\partial\tilde{\eta}}{\partial\alpha_k}) = \frac{1}{2}\mathrm{Tr}(\widetilde{\boldsymbol{\Gamma}}^{-1}\widetilde{\boldsymbol{\Gamma}}_k)$, it follows that the FIM [8, rel. (14] becomes

$$[\mathbf{I}_{\mathrm{CES}}^{\mathrm{NC}}]_{k,l} = -\frac{1}{4}\mathrm{Tr}(\widetilde{\boldsymbol{\Gamma}}^{-1}\widetilde{\boldsymbol{\Gamma}}_k)\mathrm{Tr}(\widetilde{\boldsymbol{\Gamma}}^{-1}\widetilde{\boldsymbol{\Gamma}}_l) + \mathrm{E}\left(\phi^2(\mathcal{Q})\frac{\partial\tilde{\eta}}{\partial\alpha_k}\frac{\partial\tilde{\eta}}{\partial\alpha_l}\right).$$

Using now $\widetilde{\boldsymbol{\Gamma}}^{-1/2}(\widetilde{\mathbf{z}} - \widetilde{\boldsymbol{\mu}}) =_d \sqrt{\mathcal{Q}}\ \ \widetilde{\mathbf{u}}$, $\mathrm{E}(\tilde{\mathbf{u}}\tilde{\mathbf{u}}^H) = \frac{1}{M}\mathbf{I}$, $\mathrm{E}(\tilde{\mathbf{u}}\tilde{\mathbf{u}}^T) = \frac{1}{M}\mathbf{J}'$, $\mathbf{J}'\widetilde{\boldsymbol{\Gamma}}^{1/2}\mathbf{J}' = \widetilde{\boldsymbol{\Gamma}}^{*1/2}$ and $\mathbf{J}'\tilde{\boldsymbol{\mu}}_l = \tilde{\boldsymbol{\mu}}_l^*$, where $\mathbf{J}' \stackrel{\text{def}}{=} \left(\begin{smallmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{smallmatrix}\right)$, the evaluation of $\mathrm{E}(\phi^2(\mathcal{Q})\frac{\partial\tilde{\eta}}{\partial\alpha_k}\frac{\partial\tilde{\eta}}{\partial\alpha_l})$ goes through slight modifications w.r.t. [8, rels. (15), (16), (17)] by extending the key relations in [8, Sec. 3], in particular [8, rel. (18a)]) as $\mathrm{E}[(\tilde{\mathbf{y}}^H\widetilde{\mathbf{A}}\tilde{\mathbf{y}})(\tilde{\mathbf{y}}^H\widetilde{\mathbf{B}}\tilde{\mathbf{y}})] = \mathrm{Tr}(\widetilde{\mathbf{A}})\mathrm{Tr}(\widetilde{\mathbf{B}}) + 2\mathrm{Tr}(\widetilde{\mathbf{A}}\widetilde{\mathbf{B}}))$, where $\mathbf{y} =_d \|\mathbf{y}\|\mathbf{u}$ when $\mathbf{y} \sim \mathbb{C}\mathcal{N}_M(\mathbf{0}, \mathbf{I})$ with $\tilde{\mathbf{y}} \stackrel{\text{def}}{=} (\mathbf{y}^T, \mathbf{y}^H)^T$. With some further simple calculations (6) is derived. ■

Proof of Eq. (9): It follows from Cauchy-Schwarz inequality using $\mathrm{E}(\mathcal{Q}\phi(\mathcal{Q})) = -M$, that $M^2 = (\mathrm{E}(\mathcal{Q}\phi(\mathcal{Q})))^2 \leq \mathrm{E}(\mathcal{Q})$ $\mathrm{E}(\mathcal{Q}\phi^2(\mathcal{Q})) = \mathrm{E}(\mathcal{Q})M\xi_1$. Next, note that $\mathrm{E}(\mathcal{Q}) = \int_0^\infty \delta_{M,g}^{-1}$ $\mathcal{Q}^M g(\mathcal{Q})d\mathcal{Q} = \delta_{M,g}^{-1}\delta_{M+1,g}\int_0^\infty \delta_{M+1,g}^{-1}\mathcal{Q}^M g(\mathcal{Q})d\mathcal{Q} = \delta_{M,g}^{-1}$ $\delta_{M+1,g} = M$. ■

Proof of Result 4: Because $\xi_2 = 1$ for Gaussian distributions, we get for NC-CES distributions:

$$\mathbf{I}_{\mathrm{CES}}^{\mathrm{NC}}(\boldsymbol{\alpha}_2) - \mathbf{I}_{\mathrm{CN}}^{\mathrm{NC}}(\boldsymbol{\alpha}_2) = \frac{\xi_2 - 1}{2}\left(\frac{d\mathrm{vec}(\widetilde{\boldsymbol{\Gamma}})}{d\boldsymbol{\alpha}_2^T}\right)^H \left((\widetilde{\boldsymbol{\Gamma}}^{-T} \otimes \widetilde{\boldsymbol{\Gamma}}^{-1}) + \frac{1}{2}\mathrm{vec}(\widetilde{\boldsymbol{\Gamma}}^{-1})\mathrm{vec}^H(\widetilde{\boldsymbol{\Gamma}}^{-1})\right)\frac{d\mathrm{vec}(\widetilde{\boldsymbol{\Gamma}})}{d\boldsymbol{\alpha}_2^T} \tag{23}$$

where $(\widetilde{\boldsymbol{\Gamma}}^{-T} \otimes \widetilde{\boldsymbol{\Gamma}}^{-1}) + \frac{1}{2}\mathrm{vec}(\widetilde{\boldsymbol{\Gamma}}^{-1})\mathrm{vec}^H(\widetilde{\boldsymbol{\Gamma}}^{-1})$ is positive definite. The proof is identical for C-CES distributions. ■

Proof of Result 5: In the circular case, all the steps of the proofs given for the DOA model in [19] are applied here. It follows from (12), that $\boldsymbol{\Delta}$ in [19, rel. (10)] and its partition in [19, rel. (13)], and $\mathbf{G}$ in [19, rel. (10)] are replaced by $\boldsymbol{\Delta} \stackrel{\text{def}}{=} \mathbf{T}_i^{1/2}(\boldsymbol{\Sigma}^{-T/2} \otimes \boldsymbol{\Sigma}^{-1/2})\frac{\partial\mathrm{vec}(\boldsymbol{\Sigma})}{\partial\boldsymbol{\alpha}_n^T} = \mathbf{T}_i^{1/2}(\boldsymbol{\Sigma}^{-T/2} \otimes \boldsymbol{\Sigma}^{-1/2})[\frac{\partial\mathrm{vec}(\boldsymbol{\Sigma})}{\partial\boldsymbol{\rho}^T} \mid \frac{\partial\mathrm{vec}(\boldsymbol{\Sigma})}{\partial\sigma_n^2}] \stackrel{\text{def}}{=} [\mathbf{V} \mid \mathbf{u}_n]$ and $\mathbf{G} \stackrel{\text{def}}{=} \mathbf{T}_i^{1/2}(\boldsymbol{\Sigma}^{-T/2} \otimes \boldsymbol{\Sigma}^{-1/2})\frac{\partial\mathrm{vec}(\boldsymbol{\Sigma})}{\partial\boldsymbol{\theta}^T}$ where $\mathbf{T}_i \stackrel{\text{def}}{=} \xi_2\mathbf{I} + (\xi_2 - 1)\mathrm{vec}(\mathbf{I})\mathrm{vec}^T(\mathbf{I})$. Thus, the SCRB of DOA alone in [19, rel. (12)], with dependent matrix [19, rel. (14)], are preserved. Letting $\mathbf{A}'_{\theta_k} \stackrel{\text{def}}{=} \frac{\partial\mathbf{A}_\theta}{\partial\theta_k}$, [19, rel. (16)] is replaced by

$$\frac{\partial\boldsymbol{\Sigma}}{\partial\theta_k} = \mathbf{A}'_{\theta_k}\mathbf{R}_s\mathbf{A}_\theta^H + \mathbf{A}_\theta\mathbf{R}_s\mathbf{A}'^H_{\theta_k}, \tag{24}$$

and the term $\mathbf{g}_k$ in [19, rel. (17)] is multiplied by $\mathbf{T}_i^{1/2}$ and where the term $\mathbf{A}_\theta\mathbf{c}_k\mathbf{d}_k^H$ in [19, rel. (18)] and [19, rel. (27)] is replaced by the term $\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}'^H_{\theta_k}$. It follows, using $\mathrm{vec}(\mathbf{R}_s) = \mathbf{J}\boldsymbol{\rho}$ where $\mathbf{J}$ is defined in [19], that $\mathbf{V}$ in [19, rel. (19)] can be expressed as $\mathbf{V} = \mathbf{T}_i^{1/2}\mathbf{W}\mathbf{J}$ where $\mathbf{W} \stackrel{\text{def}}{=} \boldsymbol{\Sigma}^{-T/2}\mathbf{A}_\theta^* \otimes \boldsymbol{\Sigma}^{-1/2}\mathbf{A}_\theta$. Then, it follows from the matrix inverse lemma that $\boldsymbol{\Pi}_{\mathbf{V}}^\perp$ in [19, rel. (20)] becomes $\boldsymbol{\Pi}_{\mathbf{V}}^\perp = \mathbf{I} - \mathbf{T}_i^{1/2}\mathcal{B}\mathbf{T}_i^{1/2}$ with $\mathcal{B} \stackrel{\text{def}}{=} \frac{1}{\xi_2}(\mathbf{H}_1^* \otimes \mathbf{H}_1) - \eta\mathrm{vec}(\mathbf{H}_1)(\mathrm{vec}(\mathbf{H}_1))^H$ where $\mathbf{H}_1 \stackrel{\text{def}}{=} \boldsymbol{\Sigma}^{-1/2}\mathbf{A}_\theta\mathbf{U}^{-1}\mathbf{A}_\theta^H\boldsymbol{\Sigma}^{-1/2}$ and where $\mathbf{U} \stackrel{\text{def}}{=} \mathbf{A}_\theta^H\boldsymbol{\Sigma}^{-1}\mathbf{A}_\theta$ and $\eta \stackrel{\text{def}}{=} \frac{\xi_2 - 1}{\xi_2^2(1 + \frac{\xi_2 - 1}{\xi_2}K)}$. By replacing $\mathbf{u}$ in [19, rel. (22)] by $\mathbf{u}_n = \mathbf{T}_i^{1/2}\mathrm{vec}(\boldsymbol{\Sigma}^{-1})$, and through some tedious algebraic manipulations, one finds $\mathbf{u}_n^H\boldsymbol{\Pi}_{\mathbf{V}}^\perp\mathbf{g}_k = 0$, implying that $\frac{1}{T}[\mathrm{SCRB}_{\mathrm{CES}}^{-1}(\boldsymbol{\theta})]_{k,l} = \mathbf{g}_k^H\boldsymbol{\Pi}_{\mathbf{V}}^\perp\mathbf{g}_l$. By further calculations we get $\frac{1}{T}[\mathrm{SCRB}_{\mathrm{CES}}^{-1}(\boldsymbol{\theta})]_{k,l} = \frac{2\xi_2}{\sigma_n^2}\mathrm{Re}(\mathrm{Tr}(\boldsymbol{\Pi}_{\mathbf{A}_\theta}^\perp\mathbf{A}'_{\theta_k}\mathbf{H}\mathbf{A}'^H_{\theta_l}))$ which can also be written in the matrix form (17) using [19, rel. (6)].

In the noncircular case, the proof follows the similar above steps by replacing $\mathbf{T}_i$ by $\widetilde{\mathbf{T}}_i \stackrel{\text{def}}{=} \frac{\xi_2}{2}\mathbf{I} + \frac{\xi_2 - 1}{4}\mathrm{vec}(\mathbf{I})\mathrm{vec}^T(\mathbf{I})$, and $\boldsymbol{\Sigma}$ by $\widetilde{\boldsymbol{\Gamma}}$ where (24) is replaced by $\frac{\partial\widetilde{\boldsymbol{\Gamma}}}{\partial\theta_k} = \tilde{\mathbf{A}}'_{\theta_k}\mathbf{R}_{\tilde{s}}\tilde{\mathbf{A}}_\theta^H + \tilde{\mathbf{A}}_\theta\mathbf{R}_{\tilde{s}}\tilde{\mathbf{A}}'^H_{\theta_k}$ with $\tilde{\mathbf{A}}_\theta \stackrel{\text{def}}{=} \mathrm{Diag}(\mathbf{A}_\theta, \mathbf{A}_\theta^*)$ and $\tilde{\mathbf{A}}'_{\theta_k} \stackrel{\text{def}}{=} \frac{\partial\tilde{\mathbf{A}}_\theta}{\partial\theta_k}$. ■

Proof of Result 6: As we said before presenting this result, its proof follows similar steps as the proof of Result 5 based on [23, th. 1] by replacing $\boldsymbol{\Sigma}$ by $\widetilde{\boldsymbol{\Gamma}} = \tilde{\mathbf{A}}_\omega\mathbf{R}_r\tilde{\mathbf{A}}_\omega^H + \sigma_n^2\mathbf{I}$, $\mathbf{A}_\theta$ by $\widetilde{\mathbf{A}}_\omega$, and also by pointing out that $\mathbf{R}_r \in \mathbb{R}^{K\times K}$ is symmetric which lead us to replace $\mathbf{J}$ by $\mathbf{D}_\rho$ defined in [23, th. 1] to get $\mathrm{vec}(\mathbf{R}_r) = \mathbf{D}_\rho\boldsymbol{\rho}$. Thus, $\mathbf{V}$ becomes $\mathbf{V} = \widetilde{\mathbf{T}}_i^{1/2}\mathbf{W}\mathbf{D}_\rho$. Hence $\boldsymbol{\Pi}_{\mathbf{V}}^\perp$ in [23, th. 1] takes here the following key form expression: $\boldsymbol{\Pi}_{\mathbf{V}}^\perp = \mathbf{I} - \widetilde{\mathbf{T}}_i^{1/2}\mathcal{B}\widetilde{\mathbf{T}}_i^{1/2}$ with $\mathcal{B} = \frac{2}{\xi_2}\mathbf{W}(\mathbf{U}^{-1} \otimes \mathbf{U}^{-1})\mathbf{N}_K\mathbf{W}^H - \tilde{\eta}\mathrm{vec}(\mathbf{H}_1)\mathrm{vec}^H(\mathbf{H}_1)$ where $\mathbf{N}_K$ is defined in [23, th. 1] and $\tilde{\eta} \stackrel{\text{def}}{=} \frac{\xi_2 - 1}{\xi_2^2(1 + \frac{\xi_2 - 1}{2\xi_2}K)}$. The rest of the proof follows the same line of arguments as that of the proof of Result 5. ■

REFERENCES

[1] D. Slepian, "Estimation of signal parameters in the presence of noise," *Trans. IRE Professional Group Inf. Theory*, vol. 3, pp. 68–89, 1954.

[2] W. J. Bangs, "Array processing with generalized beamformers," Ph.D. dissertation, Dept. Elect. Eng., Yale Univ., New Haven, CT, USA, 1971.

[3] P. Stoica and R. Moses, *Introduction to Spectral Analysis*. Upper Saddle River, NJ, USA: Prentice-Hall, 1997.

[4] J. P. Delmas and H. Abeida, "Stochastic Cramer-Rao bound for noncircular signals with application to DOA estimation," *IEEE Trans. Signal Process.*, vol. 52, no. 11, pp. 3192–3199, Nov. 2004.

[5] P. R. Krishnaiah and J. Lin, "Complex elliptically symmetric distributions," *Commun. Statist., Theory Methods*, vol. 15, pp. 3693–3718, 1986.

[6] C. D. Richmond, “Adaptive array signal processing and performance analysis in non-Gaussian environments,” Ph.D. dissertation, Dept. Elect. Eng. Comput. Sci., Massachusetts Inst. Technol., Cambridge, MA, USA, 1996.

[7] E. Ollila, D. Tyler, V. Koivunen, and H. Poor, “Complex elliptically symmetric distributions: Survey, new results and applications,” *IEEE Trans. Signal Process.*, vol. 60, no. 11, pp. 5597–5625, Nov. 2012.

[8] O. Besson and Y. I. Abramovich, “On the Fisher information matrix for multivariate elliptically contoured distributions,” *IEEE Signal Process. Lett.*, vol. 20, no. 11, pp. 1130–1133, Nov. 2013.

[9] M. Greco and F. Gini, “Cramér-Rao lower bounds on covariance matrix estimation for complex elliptically symmetric distributions,” *IEEE Trans. Signal Process.*, vol. 61, no. 24, pp. 6401–6409, Dec. 2013.

[10] A. Mennad, S. Fortunati, M. N. El Korso, A. Younsi, A. M. Zoubir, and A. Renaux, “Bangs-type formulas and the related misspecified Cramér-Rao bounds for complex elliptically symmetric distributions,” *Signal Process.*, vol. 142, pp. 320–329, Jan. 2018.

[11] S. Fortunati, F. Gini, M. S. Greco, A. M. Zoubir, and M. Rangaswamy, “Semiparametric CRB and Slepian-Bangs formulas for complex elliptically symmetric distributions,” *IEEE Trans. Signal Process.*, submitted for publication.

[12] E. Ollila and V. Koivunen, “Generalized complex elliptical distributions,” in *Proc. Sensor Array Multichannel Signal Workshop*, Jul. 2004, pp. 460–464.

[13] E. Ollila, J. Eriksson, and V. Koivunen, “Complex elliptically symmetric random variables-generation, characterization, and circularity tests,” *IEEE Trans. Signal Process.*, vol. 59, no. 1, pp. 58–69, Jan. 2011.

[14] H. Abeida, “Fisher information matrix for non-circular complex elliptical symmetric distributions with application to DOA estimation,” in *Proc. Eur. Signal Process. Conf.*, Kos Island, Greece, Sep. 2017, pp. 1485–1489.

[15] R. A. Horn and C. A. Johnson, *Matrix Analysis*. New York, NY, USA: Cambridge Univ. Press, 1985.

[16] J. Eriksson and V. Koivunen, “Complex random vectors and ICA models: Identifiability, uniqueness, and separability,” *IEEE Trans. Inf. Theory*, vol. 52, no. 3, pp. 1017–1029, Mar. 2006.

[17] H. Abeida and J. P. Delmas, “Robustness of subspace-based algorithms with respect to the distribution of the noise: Application to DOA estimation,” *Signal Process.*, vol. 164, pp. 313–319, Jun. 2019.

[18] J. P. Delmas and H. Abeida, “Survey and some new results on performance analysis of complex-valued parameter estimators,” *Signal Process.*, vol. 111, pp. 210–221, Jun. 2015.

[19] P. Stoica, A. G. Larsson, and A. B. Gershman, “The stochastic CRB for array processing: A textbook derivation,” *IEEE Signal Process. Lett.*, vol. 8, no. 5, pp. 148–150, May 2001.

[20] M. N. El Korso, R. Boyer, A. Renaux, and S. Marcos, “Statistical resolution limit for multiple parameters of interest and for multiple signals,” in *Proc. Int. Conf. Acoust., Speech, Signal Process.*, Dallas, TX, USA, 2010, pp. 3602–3605.

[21] E. Moulines, P. Duhamel, J. F. Cardoso, and S. Mayrargue, “Subspace methods for the blind identification of multichannel FIR filters?” *IEEE Trans. Signal Process.*, vol. 43, no. 2, pp. 516–525, Feb. 1995.

[22] K. Abed-Meraim and Y. Hua, “Blind identification of multi-input multi-output system using minimum noise subspace,” *IEEE Trans. Signal Process.*, vol. 45, no. 1, pp. 254–258, Jan. 1997.

[23] H. Abeida and J. P. Delmas, “Direct derivation of the stochastic CRB of DOA estimation for rectilinear sources,” *IEEE Signal Process. Lett.*, vol. 24, no. 10, pp. 1522–1526, Oct. 2017.

[24] H. Abeida and J. P. Delmas, Detailed proofs of the different results. 2019. [Online]. Available: https://arxiv.org/pdf/1908.03544.pdf and http://www-public.imtbs-tsp.eu/∼delmas/