Detailed proofs of paper [1] Slepian-Bangs formula and Cramér Rao bound for circular and non-circular complex elliptical symmetric distributions

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I. Useful relations and lemma

A. Useful relations

We will make use of the following well known relations which hold for any conformable matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$.

$$\mathrm{vec}(\mathbf{ABC}) = (\mathbf{C}^T \otimes \mathbf{A})\mathrm{vec}(\mathbf{B}), \tag{1}$$

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}, \tag{2}$$

$$\mathrm{Tr}(\mathbf{AB}) = \mathrm{vec}^H(\mathbf{A}^H)\mathrm{vec}(\mathbf{B}), \tag{3}$$

$$\mathrm{Tr}(\mathbf{ABCD}) = \mathrm{vec}^H(\mathbf{A}^H)(\mathbf{D}^T \otimes \mathbf{B})\mathrm{vec}(\mathbf{C}), \tag{4}$$

$$\mathrm{Tr}(\mathbf{A} \otimes \mathbf{B}) = \mathrm{Tr}(\mathbf{A})\mathrm{Tr}(\mathbf{B}), \tag{5}$$

$$\mathrm{Tr}[\mathbf{K}(\mathbf{A} \otimes \mathbf{B})] = \mathrm{Tr}(\mathbf{AB}), \tag{6}$$

where $\mathbf{K}$ is the vec-permutation matrix which transforms $\mathrm{vec}(\mathbf{C})$ to $\mathrm{vec}(\mathbf{C}^T)$ for any square matrix $\mathbf{C}$,

$$(\mathbf{A} + \mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{C}^{-1} + \mathbf{DA}^{-1}\mathbf{B})^{-1}\mathbf{DA}^{-1}, \tag{7}$$

where $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{C}^{-1} + \mathbf{DA}^{-1}\mathbf{B}$ are assumed invertible.

B. Useful lemma for the proof of Result 2

Lemma 1: Let $\widetilde{\mathbf{A}} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2^* & \mathbf{A}_1^* \end{pmatrix}$ and $\widetilde{\mathbf{B}} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_2^* & \mathbf{B}_1^* \end{pmatrix}$ be two $2M \times 2M$ partitioned matrices with $\mathbf{A}_1$ and $\mathbf{B}_1$ are $M \times M$ Hermitian matrices, $\mathbf{A}_2$ and $\mathbf{B}_2$ are $M \times M$ complex symmetric matrices, and suppose that $\mathbf{y} \sim \mathbb{C}\mathcal{N}_M(\mathbf{0}, \mathbf{I})$. Then

$$\mathrm{E}[(\widetilde{\mathbf{y}}^H\widetilde{\mathbf{A}}\widetilde{\mathbf{y}})(\widetilde{\mathbf{y}}^H\widetilde{\mathbf{B}}\widetilde{\mathbf{y}})] = \mathrm{Tr}(\widetilde{\mathbf{A}})\mathrm{Tr}(\widetilde{\mathbf{B}}) + 2\mathrm{Tr}(\widetilde{\mathbf{A}}\widetilde{\mathbf{B}}), \tag{8}$$

where $\widetilde{\mathbf{y}} \stackrel{\text{def}}{=} (\mathbf{y}^T, \mathbf{y}^H)^T$.
Proof:

We get from (4) then (2)

$$\mathrm{E}[(\widetilde{\mathbf{y}}^H\widetilde{\mathbf{A}}\widetilde{\mathbf{y}})(\widetilde{\mathbf{y}}^H\widetilde{\mathbf{B}}\widetilde{\mathbf{y}})] = \mathrm{Tr}[(\widetilde{\mathbf{A}}^T \otimes \widetilde{\mathbf{B}})\mathrm{E}(\widetilde{\mathbf{y}}^*\widetilde{\mathbf{y}}^T \otimes \widetilde{\mathbf{y}}\widetilde{\mathbf{y}}^H)], \tag{9}$$

where from e.g. [2, Appendix B]

$$\mathrm{E}(\widetilde{\mathbf{y}}^*\widetilde{\mathbf{y}}^T \otimes \widetilde{\mathbf{y}}\widetilde{\mathbf{y}}^H) = \mathbf{I} \otimes \mathbf{I} + \mathbf{K}(\mathbf{J}' \otimes \mathbf{J}')(\mathbf{I} \otimes \mathbf{I}) + \mathrm{vec}(\mathbf{I})\mathrm{vec}^T(\mathbf{I}), \tag{10}$$

where $\mathbf{J}' \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}$. Plugging (10) in (9), we get:

$$\begin{aligned} \mathrm{E}[(\widetilde{\mathbf{y}}^H\widetilde{\mathbf{A}}\widetilde{\mathbf{y}})(\widetilde{\mathbf{y}}^H\widetilde{\mathbf{B}}\widetilde{\mathbf{y}})] &= \mathrm{Tr}[(\widetilde{\mathbf{A}}^T\otimes\widetilde{\mathbf{B}})(\mathbf{I}\otimes\mathbf{I})]+\mathrm{Tr}[(\widetilde{\mathbf{A}}^T\otimes\widetilde{\mathbf{B}})\mathbf{K}(\mathbf{J}'\otimes\mathbf{J}')(\mathbf{I}\otimes\mathbf{I})] \\ &+ \mathrm{Tr}[(\widetilde{\mathbf{A}}^T\otimes\widetilde{\mathbf{B}})\mathrm{vec}(\mathbf{I})\mathrm{vec}^T(\mathbf{I})], \end{aligned} \tag{11}$$

where we have successively

$$\mathrm{Tr}[(\widetilde{\mathbf{A}}^T\otimes\widetilde{\mathbf{B}})(\mathbf{I}\otimes\mathbf{I})]=\mathrm{Tr}(\widetilde{\mathbf{A}})\mathrm{Tr}(\widetilde{\mathbf{B}})$$

from (2) and (5),

$$\mathrm{Tr}[(\widetilde{\mathbf{A}}^T\otimes\widetilde{\mathbf{B}})\mathbf{K}(\mathbf{J}'\otimes\mathbf{J}')(\mathbf{I}\otimes\mathbf{I})]=\mathrm{Tr}(\widetilde{\mathbf{A}}\widetilde{\mathbf{B}})$$

from (2), (6) and $\mathbf{J}'\widetilde{\mathbf{A}}^T\mathbf{J}'=\widetilde{\mathbf{A}}$, and

$$\mathrm{Tr}[(\widetilde{\mathbf{A}}^T\otimes\widetilde{\mathbf{B}})\mathrm{vec}(\mathbf{I})\mathrm{vec}^T(\mathbf{I})]=\mathrm{Tr}(\widetilde{\mathbf{A}}\widetilde{\mathbf{B}})$$

from (4). Plugging these three expressions in (11), (8) follows. ∎

II. Proof of Result 1 and Eq. (5) of [1]

Since a linear transform in $\mathbb{R}^{2M}$ is tantamount to $\mathbb{R}$-linear transform in $\mathbb{C}^M$, the definition of GCES given in [3] is equivalent to saying that[1]

$$\mathbf{z}=\boldsymbol{\mu}+\boldsymbol{\Psi}\mathbf{z}_0+\boldsymbol{\Phi}\mathbf{z}_0^*, \tag{12}$$

where $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ are $M\times M$ fixed complex-valued matrices and $\mathbf{z}_0$ is a complex spherical distributed r.v. with stochastic representation $\mathbf{z}_0 =_d \mathcal{R}\mathbf{u}$ [4, th. 3]. Since $\mathrm{E}(\mathbf{u}\mathbf{u}^H)=\frac{1}{M}\mathbf{I}$ and $\mathrm{E}(\mathbf{u}\mathbf{u}^T)=\mathbf{0}$ [4, lemma 1b], we get if $\mathrm{E}(\mathcal{R}^2)<\infty$,

$$\boldsymbol{\Sigma}=\mathbf{A}\mathbf{A}^H=\frac{\mathrm{E}(\mathcal{R}^2)}{N\sigma_c}\left(\boldsymbol{\Psi}\boldsymbol{\Psi}^H+\boldsymbol{\Phi}\boldsymbol{\Phi}^H\right) \quad\text{and}\quad \boldsymbol{\Omega}=\mathbf{A}\boldsymbol{\Delta}_\kappa\mathbf{A}^T=\frac{\mathrm{E}(\mathcal{R}^2)}{N\sigma_c}\left(\boldsymbol{\Psi}\boldsymbol{\Phi}^T+\boldsymbol{\Psi}\boldsymbol{\Phi}^T\right), \tag{13}$$

where σ_c is defined by $\mathrm{E}[(\mathbf{z}-\boldsymbol{\mu})(\mathbf{z}-\boldsymbol{\mu})^H]=\sigma_c\boldsymbol{\Sigma}$ and $\mathrm{E}[(\mathbf{z}-\boldsymbol{\mu})(\mathbf{z}-\boldsymbol{\mu})^T]=\sigma_c\boldsymbol{\Omega}$ whose value is $\mathrm{E}(\mathcal{R}^2)/N$ [4, (14)]. Consequently (13) reduces to

$$\mathbf{A}\mathbf{A}^H=\boldsymbol{\Psi}\boldsymbol{\Psi}^H+\boldsymbol{\Phi}\boldsymbol{\Phi}^H \quad\text{and}\quad \mathbf{A}\boldsymbol{\Delta}_\kappa\mathbf{A}^T=\boldsymbol{\Psi}\boldsymbol{\Phi}^T+\boldsymbol{\Psi}\boldsymbol{\Phi}^T. \tag{14}$$

By the one to one change of variable (because $\mathbf{A}$ is nonsingular): $\boldsymbol{\Psi}'=\mathbf{A}\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}'=\mathbf{A}\boldsymbol{\Phi}$, (14) is equivalent to:

$$\mathbf{I}=\boldsymbol{\Psi}'\boldsymbol{\Psi}'^H+\boldsymbol{\Phi}\boldsymbol{\Phi}'^H \quad\text{and}\quad \boldsymbol{\Delta}_\kappa=\boldsymbol{\Psi}'\boldsymbol{\Phi}'^T+\boldsymbol{\Psi}'\boldsymbol{\Phi}'^T. \tag{15}$$

It is clear that the solution of (15) is not unique, but we can look for solutions in real-valued diagonal form $(\boldsymbol{\Psi},\boldsymbol{\Phi})=(\boldsymbol{\Delta}_1,\boldsymbol{\Delta}_2)$ with

$$\mathbf{I}=\boldsymbol{\Delta}_1^2+\boldsymbol{\Delta}_2^2 \quad\text{and}\quad \boldsymbol{\Delta}_\kappa=2\boldsymbol{\Delta}_1\boldsymbol{\Delta}_2, \tag{16}$$

whose solutions are $\boldsymbol{\Delta}_1=\frac{\boldsymbol{\Delta}_+ + \boldsymbol{\Delta}_-}{2}$ and $\boldsymbol{\Delta}_2=\frac{\boldsymbol{\Delta}_+ - \boldsymbol{\Delta}_-}{2}$ where $\boldsymbol{\Delta}_+\stackrel{\text{def}}{=}\sqrt{\mathbf{I}+\boldsymbol{\Delta}_\kappa}$ and $\boldsymbol{\Delta}_-\stackrel{\text{def}}{=}\sqrt{\mathbf{I}-\boldsymbol{\Delta}_\kappa}$. Consequently

$$\mathbf{z}=_d\boldsymbol{\mu}+\mathcal{R}[\boldsymbol{\Psi}\mathbf{u}+\boldsymbol{\Phi}\mathbf{u}^*]=\boldsymbol{\mu}+\mathcal{R}\mathbf{A}[\boldsymbol{\Delta}_1\mathbf{u}+\boldsymbol{\Delta}_2\mathbf{u}^*]. \tag{17}$$

If $\mathrm{E}(\mathcal{R}^2)$ is not finite, the scatter and pseudo-scatter matrices of $\mathbf{z}$ given by (17) are also $\boldsymbol{\Sigma}=\mathbf{A}\mathbf{A}^H$ and $\boldsymbol{\Omega}=\mathbf{A}\boldsymbol{\Delta}_\kappa\mathbf{A}^T$, respectively. ∎

From the eigenvalue decomposition $\begin{pmatrix}\mathbf{I} & \boldsymbol{\Delta}_\kappa\\ \boldsymbol{\Delta}_\kappa & \mathbf{I}\end{pmatrix}=\left[\frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{I} & \mathbf{I}\\ \mathbf{I} & -\mathbf{I}\end{pmatrix}\right]\begin{pmatrix}\mathbf{I}+\boldsymbol{\Delta}_\kappa & \mathbf{0}\\ \mathbf{0} & \mathbf{I}+\boldsymbol{\Delta}_\kappa\end{pmatrix}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{I} & \mathbf{I}\\ \mathbf{I} & -\mathbf{I}\end{pmatrix}\right]$, we deduce from $\widetilde{\boldsymbol{\Gamma}}=\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}^*\end{pmatrix}\begin{pmatrix}\mathbf{I} & \boldsymbol{\Delta}_\kappa\\ \boldsymbol{\Delta}_\kappa & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{A}^H & \mathbf{0}\\ \mathbf{0} & \mathbf{A}^T\end{pmatrix}$ that $\widetilde{\boldsymbol{\Gamma}}^{1/2}=\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}^*\end{pmatrix}\begin{pmatrix}\boldsymbol{\Delta}_1 & \boldsymbol{\Delta}_2\\ \boldsymbol{\Delta}_2 & \boldsymbol{\Delta}_1\end{pmatrix}$. Consequently,

[1] Note that if $\boldsymbol{\Phi}=\mathbf{0}$, $\mathbf{z}$ is C-CES distributed.

the stochastic representation $\mathbf{z} =_d \boldsymbol{\mu} + \mathcal{R}\mathbf{A}\mathbf{v}$ is equivalent to

$$\widetilde{\mathbf{z}} =_d \widetilde{\boldsymbol{\mu}} + \mathcal{R}\widetilde{\boldsymbol{\Gamma}}^{1/2}\widetilde{\mathbf{u}} \tag{18}$$

with $\widetilde{\mathbf{u}} \stackrel{\text{def}}{=} (\mathbf{u}^T, \mathbf{u}^H)^T$. It follows directly $\frac{1}{2}(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})^H \tilde{\boldsymbol{\Gamma}}^{-1}(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}}) =_d \frac{1}{2}\mathcal{R}^2\|\widetilde{\mathbf{u}}\|^2 = \mathcal{Q}$. ■

III. PROOF OF RESULT 2

To prove this result, we follows the different steps of [5, sec. 3]. First, we cheek that the p.d.f. $p(\mathbf{z};\boldsymbol{\alpha})$ satisfies the "regularity" condition

$$\mathrm{E}\left(\frac{\partial \log p(\mathbf{z};\boldsymbol{\alpha})}{\partial \alpha_k}\right) = 0. \tag{19}$$

Taking the derivative of the p.d.f. [1, (1)] w.r.t. α_k, yields

$$\frac{\partial \log p(\mathbf{z};\boldsymbol{\alpha})}{\partial \alpha_k} = -\frac{1}{2}\mathrm{Tr}(\widetilde{\boldsymbol{\Gamma}}^{-1}\widetilde{\boldsymbol{\Gamma}}_k) + \phi(\tilde{\eta})\frac{\partial \tilde{\eta}}{\partial \alpha_k}. \tag{20}$$

It follows from the definition of $\tilde{\eta}$ that

$$\frac{\partial \tilde{\eta}}{\partial \alpha_k} = -\mathrm{Re}\left(\tilde{\boldsymbol{\mu}}_k^H \widetilde{\boldsymbol{\Gamma}}^{-1}(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})\right) - \frac{1}{2}(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})^H \widetilde{\boldsymbol{\Gamma}}^{-1}\widetilde{\boldsymbol{\Gamma}}_k\widetilde{\boldsymbol{\Gamma}}^{-1}(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}}), \tag{21}$$

where $\widetilde{\boldsymbol{\mu}}_k \stackrel{\text{def}}{=} \frac{\partial \tilde{\boldsymbol{\mu}}}{\partial \alpha_k}$ and $\widetilde{\boldsymbol{\Gamma}}_k \stackrel{\text{def}}{=} \frac{\partial \widetilde{\boldsymbol{\Gamma}}}{\partial \alpha_k}$. Making use of the extended stochastic representation (18), the second term of (21) is given by

$$\frac{1}{2}(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}})^H \widetilde{\boldsymbol{\Gamma}}^{-1}\widetilde{\boldsymbol{\Gamma}}_k\widetilde{\boldsymbol{\Gamma}}^{-1}(\tilde{\mathbf{z}} - \tilde{\boldsymbol{\mu}}) =_d \frac{1}{2}\mathcal{Q}\tilde{\mathbf{u}}^H\widetilde{\mathbf{H}}_k\tilde{\mathbf{u}} \tag{22}$$

where $\widetilde{\mathbf{H}}_k \stackrel{\text{def}}{=} \widetilde{\boldsymbol{\Gamma}}^{-1/2}\widetilde{\boldsymbol{\Gamma}}_k\widetilde{\boldsymbol{\Gamma}}^{-1/2}$. Thus using $\tilde{\eta} =_d \mathcal{Q}$ [1, (5)], we get:

$$\mathrm{E}\left(\phi(\tilde{\eta})\frac{\partial \tilde{\eta}}{\partial \alpha_k}\right) = -\mathrm{E}\left(\mathcal{Q}^{1/2}\phi(\mathcal{Q})\mathrm{Re}(\tilde{\boldsymbol{\mu}}_k^H\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}})\right) - \frac{1}{2}\mathrm{E}[\mathcal{Q}\phi(\mathcal{Q})\tilde{\mathbf{u}}^H\widetilde{\mathbf{H}}_k\tilde{\mathbf{u}}]. \tag{23}$$

Since $\mathcal{Q}$ and $\mathbf{u}$ are independent, $\mathcal{Q}$ and $\tilde{\mathbf{u}}$ are also independent. It follows then from $\mathrm{E}(\tilde{\mathbf{u}}) = \mathbf{0}$, $\mathrm{E}(\tilde{\mathbf{u}}\tilde{\mathbf{u}}^H) = \frac{1}{M}\mathbf{I}$ and $\mathrm{E}(\mathcal{Q}\phi(\mathcal{Q})) = -M$ [5, (11)] that

$$\mathrm{E}\left(\mathcal{Q}^{1/2}\phi(\mathcal{Q})\mathrm{Re}(\tilde{\boldsymbol{\mu}}_k^H\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}})\right) = 0$$

and

$$\mathrm{E}[\mathcal{Q}\phi(\mathcal{Q})\tilde{\mathbf{u}}^H\widetilde{\mathbf{H}}_k\tilde{\mathbf{u}}] = \mathrm{E}[\mathcal{Q}\phi(\mathcal{Q})]\mathrm{Tr}[\widetilde{\mathbf{H}}_k\mathrm{E}(\tilde{\mathbf{u}}\tilde{\mathbf{u}}^H)] = -\mathrm{Tr}(\widetilde{\mathbf{H}}_k) = -\mathrm{Tr}(\widetilde{\boldsymbol{\Gamma}}^{-1}\widetilde{\boldsymbol{\Gamma}}_k).$$

Thus

$$\mathrm{E}\left(\phi(\tilde{\eta})\frac{\partial \tilde{\eta}}{\partial \alpha_k}\right) = \frac{1}{2}\mathrm{Tr}(\widetilde{\boldsymbol{\Gamma}}^{-1}\widetilde{\boldsymbol{\Gamma}}_k), \tag{24}$$

which proves (19).

Now, we evaluate the elements of the FIM. It follows from (20), using (24), that

$$[\mathbf{I}_{\mathrm{CES}}^{\mathrm{NC}}]_{k,l} = \mathrm{E}\left(\frac{\partial \log p(\mathbf{z};\boldsymbol{\alpha})}{\partial \alpha_k}\frac{\partial \log p(\mathbf{z};\boldsymbol{\alpha})}{\partial \alpha_l}\right) = -\frac{1}{4}\mathrm{Tr}(\widetilde{\boldsymbol{\Gamma}}^{-1}\widetilde{\boldsymbol{\Gamma}}_k)\mathrm{Tr}(\widetilde{\boldsymbol{\Gamma}}^{-1}\widetilde{\boldsymbol{\Gamma}}_l) + \mathrm{E}\left(\phi^2(\tilde{\eta})\frac{\partial \tilde{\eta}}{\partial \alpha_k}\frac{\partial \tilde{\eta}}{\partial \alpha_l}\right). \tag{25}$$

It follows from (18) that $\widetilde{\boldsymbol{\Gamma}}^{-1/2}(\widetilde{\mathbf{z}} - \widetilde{\boldsymbol{\mu}}) =_d \sqrt{\mathcal{Q}}\ \widetilde{\mathbf{u}}$ and hence from (21) we get

$$\begin{aligned}
\phi^2(\tilde{\eta})\frac{\partial \tilde{\eta}}{\partial \alpha_k}\frac{\partial \tilde{\eta}}{\partial \alpha_l} \quad =_d \quad & \mathcal{Q}\phi^2(\mathcal{Q})\mathrm{Re}\left(\tilde{\boldsymbol{\mu}}_k^H\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}}\right)\mathrm{Re}\left(\tilde{\boldsymbol{\mu}}_l^H\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}}\right) \\
+ \quad & \frac{1}{2}\mathcal{Q}^{3/2}\phi^2(\mathcal{Q})\mathrm{Re}\left(\tilde{\boldsymbol{\mu}}_l^H\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}}\right)[\tilde{\mathbf{u}}^H\widetilde{\mathbf{H}}_k\tilde{\mathbf{u}}] + \frac{1}{2}\mathcal{Q}^{3/2}\phi^2(\mathcal{Q})\mathrm{Re}\left(\tilde{\boldsymbol{\mu}}_k^H\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}}\right)[\tilde{\mathbf{u}}^H\widetilde{\mathbf{H}}_l\tilde{\mathbf{u}}] \\
+ \quad & \frac{1}{4}\mathcal{Q}^2\phi^2(\mathcal{Q})[\tilde{\mathbf{u}}^H\widetilde{\mathbf{H}}_k\tilde{\mathbf{u}}][\tilde{\mathbf{u}}^H\widetilde{\mathbf{H}}_l\tilde{\mathbf{u}}].
\end{aligned} \tag{26}$$

The first term of (26) can be further simplified as

$$\mathrm{Re}\left(\tilde{\boldsymbol{\mu}}_k^H\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}}\right)\mathrm{Re}\left(\tilde{\boldsymbol{\mu}}_l^H\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}}\right)=\frac{1}{2}\mathrm{Re}\left(\tilde{\boldsymbol{\mu}}_k^H\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}}\tilde{\mathbf{u}}^H\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\boldsymbol{\mu}}_l\right)+\frac{1}{2}\mathrm{Re}\left(\tilde{\boldsymbol{\mu}}_k^T\widetilde{\boldsymbol{\Gamma}}^{-*1/2}\tilde{\mathbf{u}}^*\tilde{\mathbf{u}}^H\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\boldsymbol{\mu}}_l\right),$$

and thanks to the independence between $\mathcal{Q}$ and $\tilde{\mathbf{u}}$, the expected value of the first term of (26) is given by

$$\begin{aligned}\mathrm{E}[\mathcal{Q}\phi^2(\mathcal{Q})]\mathrm{E}\left(\mathrm{Re}\left(\tilde{\boldsymbol{\mu}}_k^H\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}}\right)\mathrm{Re}\left(\tilde{\boldsymbol{\mu}}_l^H\widetilde{\boldsymbol{\Gamma}}^{-1/2}\tilde{\mathbf{u}}\right)\right) &= \\ \frac{\mathrm{E}[\mathcal{Q}\phi^2(\mathcal{Q})]}{2M}\mathrm{Re}\left(\tilde{\boldsymbol{\mu}}_k^H\widetilde{\boldsymbol{\Gamma}}^{-1}\tilde{\boldsymbol{\mu}}_l\right)+\frac{\mathrm{E}[\mathcal{Q}\phi^2(\mathcal{Q})]}{2M}\mathrm{Re}\left(\tilde{\boldsymbol{\mu}}_k^T\widetilde{\boldsymbol{\Gamma}}^{-*}\mathbf{J}'\tilde{\boldsymbol{\mu}}_l\right) &= \frac{\mathrm{E}[\mathcal{Q}\phi^2(\mathcal{Q})]}{M}\mathrm{Re}\left(\tilde{\boldsymbol{\mu}}_k^H\widetilde{\boldsymbol{\Gamma}}^{-1}\tilde{\boldsymbol{\mu}}_l\right),\end{aligned}\tag{27}$$

using $\mathrm{E}(\tilde{\mathbf{u}}\tilde{\mathbf{u}}^H)=\frac{1}{M}\mathbf{I}$ and $\mathrm{E}(\tilde{\mathbf{u}}^*\tilde{\mathbf{u}}^H)=\frac{1}{M}\mathbf{J}$, $\widetilde{\boldsymbol{\Gamma}}^{-*1/2}\mathbf{J}'\widetilde{\boldsymbol{\Gamma}}^{-1/2}=\widetilde{\boldsymbol{\Gamma}}^{-*}\mathbf{J}'$ and $\mathbf{J}'\tilde{\boldsymbol{\mu}}_l=\tilde{\boldsymbol{\mu}}_l^*$. The expected value of the second and third terms of (26) are zero because the third-order moments of $\mathbf{u}$ are zero. Because $\mathbf{y}=_d\|\mathbf{y}\|\mathbf{u}$, where $\|\mathbf{y}\|$ and $\mathbf{u}$ are independent when $\mathbf{y}\sim\mathbb{C}\mathcal{N}_M(\mathbf{0},\mathbf{I})$, we get

$$\mathrm{E}[(\tilde{\mathbf{u}}^H\widetilde{\mathbf{H}}_k\tilde{\mathbf{u}})(\tilde{\mathbf{u}}^H\widetilde{\mathbf{H}}_l\tilde{\mathbf{u}})]=\frac{1}{\mathrm{E}(\|\mathbf{y}\|^4)}\mathrm{E}[(\tilde{\mathbf{y}}^H\widetilde{\mathbf{H}}_k\tilde{\mathbf{y}})(\tilde{\mathbf{y}}^H\widetilde{\mathbf{H}}_l\tilde{\mathbf{y}})].$$

Noting that $\widetilde{\mathbf{H}}_k$ and $\widetilde{\mathbf{H}}_l$ are structured as $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ of the Lemma 1, this lemma applies to the couples $(\widetilde{\mathbf{H}}_k,\widetilde{\mathbf{H}}_l)$ and $(\mathbf{I},\mathbf{I})$ giving $\mathrm{E}[(\tilde{\mathbf{y}}^H\widetilde{\mathbf{H}}_k\tilde{\mathbf{y}})(\tilde{\mathbf{y}}^H\widetilde{\mathbf{H}}_l\tilde{\mathbf{y}})]=\mathrm{Tr}(\widetilde{\mathbf{H}}_k)\mathrm{Tr}(\widetilde{\mathbf{H}}_l)+2\mathrm{Tr}(\widetilde{\mathbf{H}}_k\widetilde{\mathbf{H}}_l)$ and $\mathrm{E}[\|\tilde{\mathbf{y}}\|^4]=4M(M+1)$. Consequently the expected value of the last term of (26) is given by

$$\begin{aligned}\mathrm{E}\left(\frac{1}{4}\mathcal{Q}^2\phi^2(\mathcal{Q})[\tilde{\mathbf{u}}^H\widetilde{\mathbf{H}}_k\tilde{\mathbf{u}}][\tilde{\mathbf{u}}^H\widetilde{\mathbf{H}}_l\tilde{\mathbf{u}}]\right) &= \frac{\mathrm{E}(\mathcal{Q}^2\phi^2(\mathcal{Q}))}{4M(M+1)}\left(\mathrm{Tr}(\widetilde{\mathbf{H}}_k)\mathrm{Tr}(\widetilde{\mathbf{H}}_l)+2\mathrm{Tr}(\widetilde{\mathbf{H}}_k\widetilde{\mathbf{H}}_l)\right)\\ &= \frac{\mathrm{E}(\mathcal{Q}^2\phi^2(\mathcal{Q}))}{4M(M+1)}\left(\mathrm{Tr}(\widetilde{\boldsymbol{\Gamma}}_k\widetilde{\boldsymbol{\Gamma}}^{-1})\mathrm{Tr}(\widetilde{\boldsymbol{\Gamma}}_l\widetilde{\boldsymbol{\Gamma}}^{-1})+2\mathrm{Tr}(\widetilde{\boldsymbol{\Gamma}}_k\widetilde{\boldsymbol{\Gamma}}^{-1}\widetilde{\boldsymbol{\Gamma}}_l\widetilde{\boldsymbol{\Gamma}}^{-1})\right)\end{aligned}\tag{28}$$

Gathering (27) (28) in (25) concludes the proof. ■

IV. Proof of Eq. (9) of [1]

Using that [1, (4)] is a p.d.f. with $\int_0^\infty\delta_{M,g}^{-1}\mathcal{Q}_t^{M-1}g(\mathcal{Q}_t)d\mathcal{Q}_t=1$ and that $\mathrm{E}(\mathcal{Q})=\mathrm{E}(\mathcal{R}^2)<\infty$, we get

$$\mathrm{E}(\mathcal{Q}\phi(\mathcal{Q}))=\int_0^\infty\delta_{M,g}^{-1}\mathcal{Q}^Mg'(\mathcal{Q})d\mathcal{Q}=\left[\delta_{M,g}^{-1}\mathcal{Q}^Mg(\mathcal{Q})\right]_0^\infty-M\int_0^\infty\delta_{M,g}^{-1}\mathcal{Q}^{M-1}g(\mathcal{Q})d\mathcal{Q}=-M.\tag{29}$$

It follows from Cauchy-Schwarz inequality that

$$M^2=(\mathrm{E}(\mathcal{Q}\phi(\mathcal{Q})))^2\le\mathrm{E}(\mathcal{Q})\mathrm{E}(\mathcal{Q}\phi^2(\mathcal{Q}))=\mathrm{E}(\mathcal{Q})M\xi_1.\tag{30}$$

Next, note that

$$\mathrm{E}(\mathcal{Q})=\int_0^\infty\delta_{M,g}^{-1}\mathcal{Q}^Mg(\mathcal{Q})d\mathcal{Q}=\delta_{M,g}^{-1}\delta_{M+1,g}\int_0^\infty\delta_{M+1,g}^{-1}\mathcal{Q}^Mg(\mathcal{Q})d\mathcal{Q}=\delta_{M,g}^{-1}\delta_{M+1,g}=M.\tag{31}$$

Plugging (31) in (30) proves Eq. (9) of [1]. ■

V. Proof of Result 4

Because $\xi_2=1$ for Gaussian distributions, we get for NC-CES distributions:

$$\mathbf{I}_{\mathrm{CES}}^{\mathrm{NC}}(\boldsymbol{\alpha}_2)-\mathbf{I}_{\mathrm{CN}}^{\mathrm{NC}}(\boldsymbol{\alpha}_2)=\frac{\xi_2-1}{2}\left(\frac{d\mathrm{vec}(\widetilde{\boldsymbol{\Gamma}})}{d\boldsymbol{\alpha}_2^T}\right)^H\left((\widetilde{\boldsymbol{\Gamma}}^{-T}\otimes\widetilde{\boldsymbol{\Gamma}}^{-1})+\frac{1}{2}\mathrm{vec}(\widetilde{\boldsymbol{\Gamma}}^{-1})\mathrm{vec}^H(\widetilde{\boldsymbol{\Gamma}}^{-1})\right)\frac{d\mathrm{vec}(\widetilde{\boldsymbol{\Gamma}})}{d\boldsymbol{\alpha}_2^T}\tag{32}$$

where $(\widetilde{\boldsymbol{\Gamma}}^{-T}\otimes\widetilde{\boldsymbol{\Gamma}}^{-1})+\frac{1}{2}\mathrm{vec}(\widetilde{\boldsymbol{\Gamma}}^{-1})\mathrm{vec}^H(\widetilde{\boldsymbol{\Gamma}}^{-1})$ is positive definite. Replacing $\widetilde{\boldsymbol{\Gamma}}$ by $\boldsymbol{\Gamma}$, the proof is identical for C-CES distributions. ■

VI. PROOF OF RESULT 5

We note first that the general expressions of the SCRB proved here is valid for arbitrary parameterization of $\mathbf{A}_\theta$ if the real-valued parameter of interest $\boldsymbol{\theta} \in \mathbb{R}^L$ is characterized by the subspace generated by the columns of the full column rank $M \times K$ matrix $\mathbf{A}_\theta$ with $K < M$. It can be applied for example to near or far-field DOA modeling with scalar or vector-sensors for an arbitrary number of parameters per source $s_{t,k}$ (with $\mathbf{s}_t \stackrel{\text{def}}{=} (s_{t,1}, .., s_{t,K})^T$ and many other modelings as the SIMO and MIMO modelings. Let us start with the circular case for which $\boldsymbol{\Omega} = \mathbf{0}$ and thus $\widetilde{\boldsymbol{\Gamma}} = \text{Diag}(\boldsymbol{\Sigma}, \boldsymbol{\Sigma}^*)$ where $\boldsymbol{\Sigma} = \mathbf{A}_\theta \mathbf{R}_s \mathbf{A}_\theta^H + \sigma_n^2 \mathbf{I}$. The SCRB form for this case can be then written through the compact expression of the general FIM given in Result 2, using (1) and (2), as follows:

$$\frac{1}{T}\text{SCRB}_{\text{CES}}^{-1}(\boldsymbol{\alpha}) = \left(\frac{d\text{vec}(\boldsymbol{\Sigma})}{d\boldsymbol{\alpha}^T}\right)^H \left(\xi_2(\boldsymbol{\Sigma}^{-T} \otimes \boldsymbol{\Sigma}^{-1}) + (\xi_2 - 1)\text{vec}(\boldsymbol{\Sigma}^{-1})\text{vec}^H(\boldsymbol{\Sigma}^{-1})\right) \left(\frac{d\text{vec}(\boldsymbol{\Sigma})}{d\boldsymbol{\alpha}^T}\right). \quad (33)$$

The SCRB of $\boldsymbol{\theta}$ alone can be deduced from (33) as follows:

$$\frac{1}{T}\text{SCRB}_{\text{CES}}^{-1}(\boldsymbol{\theta}) = \mathbf{G}^H \boldsymbol{\Pi}_{\boldsymbol{\Delta}}^{\perp} \mathbf{G}, \quad (34)$$

with $\mathbf{G} \stackrel{\text{def}}{=} \mathbf{T}_i^{1/2}(\boldsymbol{\Sigma}^{-T/2} \otimes \boldsymbol{\Sigma}^{-1/2})\frac{\partial\text{vec}(\boldsymbol{\Sigma})}{\partial\boldsymbol{\theta}^T}$ and $\boldsymbol{\Delta} \stackrel{\text{def}}{=} \mathbf{T}_i^{1/2}(\boldsymbol{\Sigma}^{-T/2} \otimes \boldsymbol{\Sigma}^{-1/2})\frac{\partial\text{vec}(\boldsymbol{\Sigma})}{\partial\boldsymbol{\alpha}_n^T}$ where

$$\mathbf{T}_i \stackrel{\text{def}}{=} \xi_2 \mathbf{I} + (\xi_2 - 1)\text{vec}(\mathbf{I})\text{vec}^T(\mathbf{I}). \quad (35)$$

Let's further partition the matrix $\boldsymbol{\Delta}$ as $\boldsymbol{\Delta} = \mathbf{T}_i^{1/2}(\boldsymbol{\Sigma}^{-T/2} \otimes \boldsymbol{\Sigma}^{-1/2})\left[\frac{\partial\text{vec}(\boldsymbol{\Sigma})}{\partial\boldsymbol{\rho}^T} \mid \frac{\partial\text{vec}(\boldsymbol{\Sigma})}{\partial\sigma_n^2}\right] \stackrel{\text{def}}{=} [\mathbf{V} \mid \mathbf{u}_n]$. In the sequel, the proofs presented here follow the lines of the proof presented in [6] for circular Gaussian distributed observations. It follows from [6, rel. (14)] that

$$\boldsymbol{\Pi}_{\boldsymbol{\Delta}}^{\perp} = \boldsymbol{\Pi}_{\mathbf{V}}^{\perp} - \frac{\boldsymbol{\Pi}_{\mathbf{V}}^{\perp}\mathbf{u}_n\mathbf{u}_n^H\boldsymbol{\Pi}_{\mathbf{V}}^{\perp}}{\mathbf{u}_n^H\boldsymbol{\Pi}_{\mathbf{V}}^{\perp}\mathbf{u}_n}. \quad (36)$$

Using $\frac{\partial\text{vec}(\boldsymbol{\Sigma})}{\partial\sigma_n^2} = \text{vec}(\mathbf{I})$, we obtain

$$\mathbf{u}_n = \mathbf{T}_i^{1/2}\text{vec}(\boldsymbol{\Sigma}^{-1}). \quad (37)$$

Consequently using (34) and (36), if $\mathbf{g}_k$ denotes the *kth* column of $\mathbf{G}$, the (k,l) element of $\text{SCRB}_{\text{CES}}^{-1}(\boldsymbol{\alpha})$ can be written elementwise as

$$\frac{1}{T}\left[\text{SCRB}_{\text{CES}}^{-1}(\boldsymbol{\theta})\right]_{k,l} = \mathbf{g}_k^H\boldsymbol{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_l - \frac{\mathbf{g}_k^H\boldsymbol{\Pi}_{\mathbf{V}}^{\perp}\mathbf{u}_n\mathbf{u}_n^H\boldsymbol{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_l}{\mathbf{u}_n^H\boldsymbol{\Pi}_{\mathbf{V}}^{\perp}\mathbf{u}_n}. \quad (38)$$

Let us proceed now to determine the expression of $\mathbf{g}_k$. Letting $\mathbf{A}'_{\theta_k} \stackrel{\text{def}}{=} \frac{\partial\mathbf{A}_\theta}{\partial\theta_k}$, we get

$$\frac{\partial\boldsymbol{\Sigma}}{\partial\theta_k} = \mathbf{A}'_{\theta_k}\mathbf{R}_s\mathbf{A}_\theta^H + \mathbf{A}_\theta\mathbf{R}_s\mathbf{A}'^H_{\theta_k}, \quad (39)$$

Hence, using (1), the *kth* column of $\mathbf{G}$ in (38) is given by

$$\mathbf{g}_k = \mathbf{T}_i^{1/2}\text{vec}(\mathbf{Z}_k + \mathbf{Z}_k^H) \quad \text{where} \quad \mathbf{Z}_k \stackrel{\text{def}}{=} \boldsymbol{\Sigma}^{-1/2}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}'^H_{\theta_k}\boldsymbol{\Sigma}^{-1/2}. \quad (40)$$

Next, we determine $\mathbf{V}$ and then $\boldsymbol{\Pi}_{\mathbf{V}}^{\perp}$. Since $\mathbf{R}_s$ is a Hermitian matrix, it can be then factorized as

$$\text{vec}(\mathbf{R}_s) = \mathbf{J}\boldsymbol{\rho} \quad (41)$$

where $\mathbf{J}$ is a $K^2 \times K^2$ constant nonsingular matrix. It follows, using (1), that $\mathbf{V}$ can be be expressed as

$$\mathbf{V} = \mathbf{T}_i^{1/2}(\boldsymbol{\Sigma}^{-T/2}\mathbf{A}_\theta^* \otimes \boldsymbol{\Sigma}^{-1/2}\mathbf{A}_\theta)\mathbf{J} \stackrel{\text{def}}{=} \mathbf{T}_i^{1/2}\mathbf{W}\mathbf{J}.$$

Note from (38) that the SCRB depends on $\mathbf{V}$ only via $\mathbf{\Pi}_{\mathbf{V}}^{\perp}$, that can be expressed as

$$\mathbf{\Pi}_{\mathbf{V}}^{\perp} = \mathbf{I} - \mathbf{V}(\mathbf{V}^H\mathbf{V})^{-1}\mathbf{V}^H = \mathbf{I} - \mathbf{T}_i^{1/2}\mathbf{W}(\mathbf{W}^H\mathbf{T}_i\mathbf{W})^{-1}\mathbf{W}^H\mathbf{T}_i^{1/2}. \tag{42}$$

After some algebraic manducation, using (1) and (2), we obtain

$$\mathbf{W}^H\mathbf{T}_i\mathbf{W} = \xi_2(\mathbf{U}^* \otimes \mathbf{U}) + (\xi_2 - 1)\mathrm{vec}(\mathbf{U})\mathrm{vec}^H(\mathbf{U}),$$

where $\mathbf{U} \stackrel{\text{def}}{=} \mathbf{A}_\theta^H\mathbf{\Sigma}^{-1}\mathbf{A}_\theta$ is a $K \times K$ Hermitian nonsingular matrix. It follows from matrix inverse lemma (given by (7)), that its inverse can be expressed as

$$(\mathbf{W}^H\mathbf{T}_i\mathbf{W})^{-1} = \frac{1}{\xi_2}(\mathbf{U}^{-*} \otimes \mathbf{U}^{-1}) - \eta\mathrm{vec}(\mathbf{U}^{-1})\mathrm{vec}^H(\mathbf{U}^{-1})$$

where $\eta \stackrel{\text{def}}{=} \frac{\xi_2-1}{\xi_2^2(1+\frac{\xi_2-1}{\xi_2}\mathrm{vec}^H(\tilde{\mathbf{U}})(\tilde{\mathbf{U}}^{-*}\otimes\tilde{\mathbf{U}}^{-1})\mathrm{vec}(\tilde{\mathbf{U}}))}$ can be simplified, using (4), as $\eta \stackrel{\text{def}}{=} \frac{\xi_2-1}{\xi_2^2(1+\frac{\xi_2-1}{\xi_2}K)}$. Thus, using (1) and (2), we obtain

$$\mathbf{W}(\mathbf{W}^H\mathbf{T}_i\mathbf{W})^{-1}\mathbf{W}^H = \frac{1}{\xi_2}(\mathbf{H}_1^* \otimes \mathbf{H}_1) - \eta\mathrm{vec}(\mathbf{H}_1)\mathrm{vec}^H(\mathbf{H}_1) \stackrel{\text{def}}{=} \mathcal{B}, \tag{43}$$

where $\mathbf{H}_1 \stackrel{\text{def}}{=} \mathbf{\Sigma}^{-1/2}\mathbf{A}_\theta\mathbf{U}^{-1}\mathbf{A}_\theta^H\mathbf{\Sigma}^{-1/2}$. Therefore, (42) becomes

$$\mathbf{\Pi}_{\mathbf{V}}^{\perp} = \mathbf{I} - \mathbf{T}_i^{1/2}\mathcal{B}\mathbf{T}_i^{1/2}. \tag{44}$$

Now let us show that $\mathbf{u}_n^H\mathbf{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_k = 0$. It follows from (37) and (40), using (44), that

$$\mathbf{u}_n^H\mathbf{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_k = \mathrm{vec}^H(\mathbf{\Sigma}^{-1})\mathbf{T}_i\mathrm{vec}(\mathbf{Z}_k + \mathbf{Z}_k^H) - \mathrm{vec}^H(\mathbf{\Sigma}^{-1})\mathbf{T}_i\mathcal{B}\mathbf{T}_i\mathrm{vec}(\mathbf{Z}_k + \mathbf{Z}_k^H). \tag{45}$$

It follows, after some algebraic manipulation, using (1), (3) and (43) that

$$\begin{aligned}\mathbf{T}_i\mathcal{B}\mathbf{T}_i &= \xi_2(\mathbf{H}_1^* \otimes \mathbf{H}_1) - \xi_2^2\eta\mathrm{vec}(\mathbf{H}_1)\mathrm{vec}^H(\mathbf{H}_1)\\ &+ (\xi_2 - 1)(1 - K\eta\xi_2)\left(\mathrm{vec}(\mathbf{I})\mathrm{vec}^H(\mathbf{H}_1) + \mathrm{vec}(\mathbf{H}_1)\mathrm{vec}^T(\mathbf{I})\right)\\ &+ \frac{(\xi_2 - 1)^2K}{\xi_2}(1 - K\eta\xi_2)\mathrm{vec}(\mathbf{I})\mathrm{vec}^T(\mathbf{I}),\end{aligned} \tag{46}$$

using $\mathbf{H}_1^2 = \mathbf{H}_1$ and $\mathrm{Tr}(\mathbf{H}_1) = K$. Using the definition (35) for $\mathbf{T}_i$ and (3), the first term of (45) can be expressed as

$$\begin{aligned}\mathrm{vec}^H(\mathbf{\Sigma}^{-1})\mathbf{T}_i\mathrm{vec}(\mathbf{Z}_k + \mathbf{Z}_k^H) &= \xi_2\mathrm{Tr}(\mathbf{\Sigma}^{-1}(\mathbf{Z}_k + \mathbf{Z}_k^H)) + (\xi_2 - 1)\mathrm{Tr}(\mathbf{\Sigma}^{-1})\mathrm{Tr}(\mathbf{Z}_k + \mathbf{Z}_k^H)\\ &= 2\xi_2\mathrm{Re}(\mathrm{Tr}(\mathbf{\Sigma}^{-2}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}_{\theta_k}^{\prime H})) + 2(\xi_2 - 1)\mathrm{Tr}(\mathbf{\Sigma}^{-1})\mathrm{Re}(\mathrm{Tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}_{\theta_k}^{\prime H}))\end{aligned} \tag{47}$$

using $\mathrm{Tr}(\mathbf{\Sigma}^{-1}(\mathbf{Z}_k + \mathbf{Z}_k^H)) = 2\mathrm{Re}(\mathrm{Tr}(\mathbf{\Sigma}^{-2}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}_{\theta_k}^{\prime H}))$ and $\mathrm{Tr}(\mathbf{Z}_k + \mathbf{Z}_k^H) = 2\mathrm{Re}(\mathrm{Tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}_{\theta_k}^{\prime H}))$. After simple algebraic manipulations, using (46), (1) and (3), and that $\mathrm{Tr}(\mathbf{Z}_k + \mathbf{Z}_k^H) = \mathrm{Tr}((\mathbf{Z}_k + \mathbf{Z}_k^H)\mathbf{H}_1) = \mathrm{Tr}(\mathbf{H}_1(\mathbf{Z}_k + \mathbf{Z}_k^H)\mathbf{H}_1) = 2\mathrm{Re}(\mathrm{Tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}_{\theta_k}^{\prime H}))$ and $\mathrm{Tr}(\mathbf{\Sigma}^{-1}\mathbf{H}_1^2) = \mathrm{Tr}(\mathbf{\Sigma}^{-1}\mathbf{H}_1)$, the second term of (45) can be simplified as

$$\begin{aligned}&\mathrm{vec}^H(\mathbf{\Sigma}^{-1})\mathbf{T}_i\mathcal{B}\mathbf{T}_i\mathrm{vec}(\mathbf{Z}_k + \mathbf{Z}_k^H)\\ &= \xi_2\mathrm{Tr}(\mathbf{\Sigma}^{-1}\mathbf{H}_1(\mathbf{Z}_k + \mathbf{Z}_k^H)\mathbf{H}_1) + (\xi_2 - 1)\mathrm{Tr}(\mathbf{\Sigma}^{-1})\mathrm{Tr}(\mathbf{Z}_k + \mathbf{Z}_k^H)\\ &= 2\xi_2\mathrm{Re}(\mathrm{Tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{U}^{-1}\mathbf{A}_\theta^H\mathbf{\Sigma}^{-2}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}_{\theta_k}^{\prime H})) + 2(\xi_2 - 1)\mathrm{Tr}(\mathbf{\Sigma}^{-1})\mathrm{Re}(\mathrm{Tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}_{\theta_k}^{\prime H}))\\ &= 2\xi_2\mathrm{Re}(\mathrm{Tr}(\mathbf{\Sigma}^{-2}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}_{\theta_k}^{\prime H})) + 2(\xi_2 - 1)\mathrm{Tr}(\mathbf{\Sigma}^{-1})\mathrm{Re}(\mathrm{Tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}_{\theta_k}^{\prime H})),\end{aligned} \tag{48}$$

where the first term in the last line is obtained using $\mathbf{A}_\theta\mathbf{U}^{-1}\mathbf{A}_\theta^H\mathbf{\Sigma}^{-2}\mathbf{A}_\theta = \mathbf{\Sigma}^{-1}\mathbf{A}_\theta$. It follows, therefore, from (45), (47) and (48) that

$$\mathbf{u}_n^H\mathbf{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_k = 0.$$

This identity together with (40) and (44) allows us to rewrite the individual elements of (38) as

$$\begin{aligned}\frac{1}{T}\left[\mathrm{SCRB}^{-1}_{\mathrm{CES}}(\boldsymbol{\theta})\right]_{k,l} &= \mathbf{g}_k^H \mathbf{\Pi}^{\perp}_{\mathbf{V}} \mathbf{g}_l \\ &= \mathrm{vec}^H(\mathbf{Z}_k+\mathbf{Z}_k^H)\mathbf{T}_i \mathrm{vec}(\mathbf{Z}_l+\mathbf{Z}_l^H) - \mathrm{vec}^H(\mathbf{Z}_k+\mathbf{Z}_k^H)\mathbf{T}_i\mathcal{B}\mathbf{T}_i\mathrm{vec}(\mathbf{Z}_l+\mathbf{Z}_l^H). \quad (49)\end{aligned}$$

After simple algebraic manipulations, using the definition (35) for $\mathbf{T}_i$, (1) and (3), the first term in (49) can be simplified as

$$\begin{aligned}\mathrm{vec}^H(\mathbf{Z}_k+\mathbf{Z}_k^H)\mathbf{T}_i\mathrm{vec}(\mathbf{Z}_l+\mathbf{Z}_l^H) &= \xi_2\mathrm{Tr}((\mathbf{Z}_k+\mathbf{Z}_k^H)(\mathbf{Z}_l+\mathbf{Z}_l^H)) + (\xi_2-1)\mathrm{Tr}(\mathbf{Z}_k+\mathbf{Z}_k^H)\mathrm{Tr}(\mathbf{Z}_l+\mathbf{Z}_l^H) \\ &= 2\xi_2\Big[\mathrm{Re}(\mathrm{Tr}((\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}'^H_{\theta_l})(\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}'^H_{\theta_k}))) \\ &+ \mathrm{Re}(\mathrm{Tr}((\mathbf{\Sigma}^{-1}\mathbf{A}'_{\theta_l}\mathbf{R}_s\mathbf{A}_\theta)(\mathbf{\Sigma}^{-1}\mathbf{R}_s\mathbf{A}'^H_{\theta_k})))\Big] \\ &+ 4(\xi_2-1)\mathrm{Re}(\mathrm{Tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}'^H_{\theta_k}))\mathrm{Re}(\mathrm{Tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}'^H_{\theta_l})) \quad (50)\end{aligned}$$

Similarly, after some algebraic manipulations, using (46), (1) and (4), the second term in (49) can be simplified as

$$\begin{aligned}\mathrm{vec}^H(\mathbf{Z}_k+\mathbf{Z}_k^H)\mathbf{T}_i\mathcal{B}\mathbf{T}_i\mathrm{vec}(\mathbf{Z}_l+\mathbf{Z}_l^H) &= 2\xi_2\Big[\mathrm{Tr}(\mathrm{Re}((\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}'^H_{\theta_l})(\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}'^H_{\theta_k}))) \\ &+ \mathrm{Tr}(\mathrm{Re}((\mathbf{\Sigma}^{-1}\mathbf{A}\mathbf{U}^{-1}\mathbf{A}^H\mathbf{\Sigma}^{-1}\mathbf{A}'_{\theta_l}\mathbf{R}_s\mathbf{A}^H_\theta)(\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}'^H_{\theta_k})))\Big] \\ &+ 4(\xi_2-1)\mathrm{Tr}(\mathrm{Re}(\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}'^H_{\theta_k}))\mathrm{Tr}(\mathrm{Re}(\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}'^H_{\theta_l})). \quad (51)\end{aligned}$$

It follows then from (50) and (51) that (49) can be simplified as

$$\begin{aligned}\frac{1}{T}\left[\mathrm{SCRB}^{-1}_{\mathrm{CES}}(\boldsymbol{\theta})\right]_{k,l} &= 2\xi_2\mathrm{Re}\left(\mathrm{Tr}\left[(\mathbf{\Sigma}^{-1}-\mathbf{\Sigma}^{-1}\mathbf{A}\mathbf{U}^{-1}\mathbf{A}^H\mathbf{\Sigma}^{-1})(\mathbf{A}'_{\theta_l}\mathbf{R}_s\mathbf{A}^H_\theta\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}'^H_{\theta_k})\right]\right) \\ &= \frac{2\xi_2}{\sigma_n^2}\mathrm{Re}\left(\mathrm{Tr}\left[(\mathbf{\Pi}^{\perp}_{\mathbf{A}_\theta})(\mathbf{A}'_{\theta_l}\mathbf{R}_s\mathbf{A}^H_\theta\mathbf{\Sigma}^{-1}\mathbf{A}_\theta\mathbf{R}_s\mathbf{A}'^H_{\theta_k})\right]\right) \\ &= \frac{2\xi_2}{\sigma_n^2}\mathrm{Re}\left(\mathrm{Tr}\left[\mathbf{\Pi}^{\perp}_{\mathbf{A}_\theta}\mathbf{A}'_{\theta_l}\mathbf{H}\mathbf{A}'^H_{\theta_k}\right]\right), \quad (52)\end{aligned}$$

where the second equality is obtained using $\mathbf{\Sigma}^{-1}-\mathbf{\Sigma}^{-1}\mathbf{A}\mathbf{U}^{-1}\mathbf{A}^H\mathbf{\Sigma}^{-1} = \frac{1}{\sigma_n^2}\mathbf{\Pi}^{\perp}_{\mathbf{A}}$ thanks to $\mathbf{A}\mathbf{U}^{-1}\mathbf{A}^H\mathbf{\Sigma}^{-1} = \mathbf{A}(\mathbf{A}^H\mathbf{A})^{-1}\mathbf{A}^H$. Using (4), we can write (52) in matrix form as is shown in Result 5.

In the noncircular case, the proof follows the similar above steps by replacing $\mathbf{T}_i$ by $\widetilde{\mathbf{T}}_i \stackrel{\mathrm{def}}{=} \frac{\xi_2}{2}\mathbf{I} + \frac{\xi_2-1}{4}\mathrm{vec}(\mathbf{I})\mathrm{vec}^T(\mathbf{I})$, and $\mathbf{\Sigma}$ by $\widetilde{\mathbf{\Gamma}}$ where (39) is replaced by $\frac{\partial\widetilde{\mathbf{\Gamma}}}{\partial\theta_k} = \tilde{\mathbf{A}}'_{\theta_k}\mathbf{R}_{\tilde{s}}\tilde{\mathbf{A}}^H_\theta + \tilde{\mathbf{A}}_\theta\mathbf{R}_{\tilde{s}}\tilde{\mathbf{A}}'^H_{\theta_k}$ with $\tilde{\mathbf{A}}_\theta \stackrel{\mathrm{def}}{=} \mathrm{Diag}(\mathbf{A}_\theta, \mathbf{A}^*_\theta)$ and $\tilde{\mathbf{A}}'_{\theta_k} \stackrel{\mathrm{def}}{=} \frac{\partial\tilde{\mathbf{A}}_\theta}{\partial\theta_k}$.

VII. Proof of Result 6

The proof of this result follows similar steps as the proof of Result 5 based on [7, th. 1] by replacing $\mathbf{\Sigma}$ by $\widetilde{\mathbf{\Gamma}} = \tilde{\mathbf{A}}_\omega\mathbf{R}_r\tilde{\mathbf{A}}^H_\omega + \sigma_n^2\mathbf{I}$, $\mathbf{A}_\theta$ by $\widetilde{\mathbf{A}}_\omega = \begin{pmatrix}\mathbf{A}_\theta\mathbf{\Delta}_\phi \\ \mathbf{A}^*_\theta\mathbf{\Delta}^*_\phi\end{pmatrix}$ where $\boldsymbol{\omega} \stackrel{\mathrm{def}}{=} (\boldsymbol{\theta}^T, \boldsymbol{\phi}^T)^T$ with $\boldsymbol{\phi} \stackrel{\mathrm{def}}{=} (\phi_1, ..., \phi_K)^T$, and also by pointing out that $\mathbf{R}_r \in \mathbb{R}^{K\times K}$ is symmetric which lead us to replace $\mathbf{J}$ in (41) by $\mathbf{D}_\rho$ defined in [7, th. 1] to get $\mathrm{vec}(\mathbf{R}_r) = \mathbf{D}_\rho\boldsymbol{\rho}$. Thus, $\mathbf{V}$ becomes $\mathbf{V} = \widetilde{\mathbf{T}}_i^{1/2}\mathbf{W}\mathbf{D}_\rho$ with $\mathbf{W} = (\widetilde{\mathbf{\Gamma}}^{-T/2}\widetilde{\mathbf{A}}^*_\omega \otimes \widetilde{\mathbf{\Gamma}}^{-1/2}\widetilde{\mathbf{A}}_\omega)$. Hence $\mathbf{\Pi}^{\perp}_{\mathbf{V}}$ in [7, th. 1] takes here the following key form expression: $\mathbf{\Pi}^{\perp}_{\mathbf{V}} = \mathbf{I} - \widetilde{\mathbf{T}}_i^{1/2}\mathcal{B}\widetilde{\mathbf{T}}_i^{1/2}$ with $\mathcal{B} = \frac{2}{\xi_2}\mathbf{W}(\mathbf{U}^{-1}\otimes\mathbf{U}^{-1})\mathbf{N}_K\mathbf{W}^H - \tilde{\eta}\mathrm{vec}(\mathbf{H}_1)\mathrm{vec}^H(\mathbf{H}_1)$ where $\mathbf{U} \stackrel{\mathrm{def}}{=} \widetilde{\mathbf{A}}^H_\omega\widetilde{\mathbf{\Gamma}}^{-1}\widetilde{\mathbf{A}}_\omega$, $\mathbf{N}_K$ is defined in [7, th. 1] and $\tilde{\eta} \stackrel{\mathrm{def}}{=} \frac{\xi_2-1}{\xi_2^2(1+\frac{\xi_2-1}{2\xi_2}K)}$. The rest of the proof follows the same lines of arguments as that of the proof of Result 5.

REFERENCES

[1] H. Abeida and J.P. Delmas, ”Slepian-Bangs formula and Cramér Rao bound for circular and non-circular complex elliptical symmetric distributions,” accepted to *IEEE Signal Process. Lett.*, August 2019.

[2] H. Abeida and J.P. Delmas, ”MUSIC-like estimation of direction of arrival for noncircular sources,” *IEEE Trans. Signal Process.*, vol. 54, no. 7, pp. 2678-2690, 2006.

[3] E. Ollila and V. Koivunen, ”Generalized complex elliptical distributions,” *in Proc. SAM Workshop*, pp. 460-464, July 2004.

[4] E. Ollila, D. Tyler, V. Koivunen and H. Poor, ”Complex elliptically symmetric distributions: Survey, new results and applications,” *IEEE Trans. Signal Process.*, vol. 60, no. 11, pp. 5597-5625, Nov. 2012.

[5] O. Besson and Y. I. Abramovich, ”On the Fisher information matrix for multivariate elliptically contoured distributions,” *IEEE Signal Process. Lett.*, vol. 20, no. 11, pp. 1130-1133, Nov. 2013.

[6] P. Stoica, A. G. Larsson, and A. B. Gershman, “The stochastic CRB for array processing: A textbook derivation,” *IEEE Signal Process. Lett.*, vol. 8, no. 5, pp. 148-150, May 2001.

[7] H. Abeida and J.P. Delmas, ”Direct derivation of the stochastic CRB of DOA estimation for rectilinear sources,” *IEEE Signal Process. Lett.*, vol. 24, no. 10, pp. 1522-1526, 2017.