Direct Derivation of the Stochastic CRB of DOA Estimation for Rectilinear Sources

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***Abstract*—Several direction of arrival (DOA) estimation algorithms have been proposed to exploit the structure of rectilinear or strictly second-order noncircular signals. But until now, only the compact closed-form expressions of the corresponding deterministic Cramér–Rao bound (DCRB) have been derived because it is much easier to derive than the stochastic CRB (SCRB). As this latter bound is asymptotically achievable by the maximum likelihood estimator, while the DCRB is unattainable, it is important to have a compact closed-form expression for this SCRB to assess the performance of DOA estimation algorithms for rectilinear signals. The aim of this letter is to derive this expression directly from the Slepian–Bangs formula including in particular the case of prior knowledge of uncorrelated or coherent sources. Some properties of these SCRBs are proved and numerical illustrations are given.**

***Index Terms*—Circular, deterministic Cramér–Rao bound (DCRB), direction of arrival (DOA), noncircular, rectilinear, Slepian–Sangs formula, stochastic Cramér–Rao bound (SCRB), strictly noncircular.**

I. Introduction

VARIOUS direction of arrival (DOA) estimation algorithms, such as MUSIC [1], [2], root-MUSIC [3], standard ESPRIT [4], and unitary ESPRIT [5], [6] have been adapted to exploit the structure of rectilinearity or strictly second-order noncircularity of signals, which include commonly used digital modulation schemes, such as BPSK and ASK. These algorithms are known to achieve a higher estimation accuracy and can resolve up to twice as many sources compared to the traditional DOA algorithms. To assess the performance of these algorithms, it is necessary to derive the stochastic Cramér–Rao bound (SCRB) for rectilinear sources. Nonetheless, only the SCRB for arbitrary noncircular sources [7], [8] and the deterministic CRB (DCRB) for rectilinear sources [9]–[11] are available, among many other bounds (e.g., [12] and references therein). But, the first bound does not take into account the prior knowledge of rectilinearity and the second bound, although providing valuable engineering insight is unattainable.

As generally the exploitation of prior knowledge usually reduces the estimation error, this letter derives closed-form expressions of the SCRB for arbitrary rectilinear sources and for the specific prior knowledge of uncorrelated, and fully correlated (referred to coherent) sources. Note that explicit expressions of circular and noncircular SCRBs for DOA parameter alone have been derived by two different methods for arbitrary sources. The first one consists of computing the asymptotic covariance matrix of the concentrated maximum likelihood estimator [13], which is asymptotically efficient and the other one is obtained directly from the Slepian–Bangs formula [14], [15]. Similarly to [16], we present here a direct derivation of the different rectilinear SCRBs from the extended Slepian–Bangs formula [7] for noncircular Gaussian distributions. Finally, some properties of these SCRBs are proved and numerical illustrations are given.

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II. Data Model and Problem Formulation

Consider K zero-mean narrow-band signals $(x_{t,k})_{k=1,\ldots,K}$ impinging on an arbitrary array of M sensors. These signals are supposed rectilinear (also called strictly second-order noncircular), i.e., described by the following model

$$x_{t,k} = s_{t,k} e^{i\phi_k} \quad \text{with} \quad s_{t,k} \ \text{real-valued} \tag{1}$$

where the phases ϕ_k associated with different propagation delays are assumed fixed, but unknown during the array observation. The array output at time t is modeled as

$$\mathbf{y}_t = \mathbf{A}_\theta \boldsymbol{\Delta}_\phi \mathbf{s}_t + \mathbf{n}_t, \quad t = 1, \ldots, T \tag{2}$$

where $(\mathbf{y}_t)_{t=1,\ldots,T}$ are independent. $\mathbf{A}_\theta \stackrel{\text{def}}{=} [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K)]$ denotes the conventional steering matrix, $\boldsymbol{\Delta}_\phi \stackrel{\text{def}}{=} \text{Diag}(e^{i\phi_1}, \ldots, e^{i\phi_K})$ and $\mathbf{s}_t \stackrel{\text{def}}{=} (s_{t,1}, \ldots, s_{t,K})^T$. $\mathbf{n}_t$ is the additive noise, which is assumed zero-mean circular complex Gaussian, spatially uncorrelated with $\text{E}(\mathbf{n}_t \mathbf{n}_t^H) = \sigma_n^2 \mathbf{I}$ and independent from $s_{t,k}$. $(s_{t,k})_{k=1,\ldots,K,t=1,..T}$ are either real-valued deterministic unknown parameters (in the so-called conditional or deterministic model), or zero-mean real-valued Gaussian distributed with covariance $\text{E}(\mathbf{s}_t \mathbf{s}_t^T) = \mathbf{R}_s$ (in the so-called unconditional or stochastic model).

To derive the CRB from the Slepian–Bangs formula, we have to carefully specify the parameters of the Gaussian distribution of $(\mathbf{y}_t)_{t=1,\ldots,T}$. Under the deterministic assumption, $\mathbf{y}_t$ are circularly Gaussian distributed with mean $(\mathbf{A}_\theta \boldsymbol{\Delta}_\phi \mathbf{s}_t)_{t=1,\ldots,T}$ and covariance $\sigma_n^2 \mathbf{I}$, which are parameterized by the real-valued parameter

$$\boldsymbol{\alpha} = (\boldsymbol{\theta}^T, \boldsymbol{\phi}^T, \boldsymbol{\rho}^T, \sigma_n^2)^T \tag{3}$$

where $\boldsymbol{\theta} \stackrel{\text{def}}{=} (\theta_1, \ldots, \theta_K)^T$, $\boldsymbol{\phi} \stackrel{\text{def}}{=} (\phi_1, \ldots, \phi_K)^T$ and $\boldsymbol{\rho} \stackrel{\text{def}}{=} (\mathbf{s}_1^T, \ldots, \mathbf{s}_T^T)^T$. On the other hand, under the stochastic assumption, $\mathbf{y}_t$ is noncircularly Gaussian distributed with zero-mean, covariance

$$\mathbf{R}_y \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{y}_t\mathbf{y}_t^H) = \mathbf{A}_\theta\boldsymbol{\Delta}_\phi\mathbf{R}_s\boldsymbol{\Delta}_\phi^*\mathbf{A}_\theta^H + \sigma_n^2\mathbf{I} \quad (4)$$

and complementary covariance

$$\mathbf{C}_y \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{y}_t\mathbf{y}_t^T) = \mathbf{A}_\theta\boldsymbol{\Delta}_\phi\mathbf{R}_s\boldsymbol{\Delta}_\phi\mathbf{A}_\theta^T \quad (5)$$

which is also generally parameterized by (3), but, where $\boldsymbol{\rho}$ is now the $K(K+1)/2$ vector made from $[\mathbf{R}_s]_{i,j}$ for $1 \leq i \leq j \leq K$. In the particular case, where prior knowledge of uncorrelated or full coherent sources $s_{t,k}$ are incorporated, the parameter $\boldsymbol{\rho}$ reduces to $\boldsymbol{\rho} = (\sigma_1^2, .., \sigma_k^2, .., \sigma_K^2)^T$ where $\sigma_k^2 \stackrel{\text{def}}{=} \mathrm{E}(s_{t,k}^2) = [\mathbf{R}_s]_{k,k}$ and to $\boldsymbol{\rho} = \mathbf{c} = (c_1, \ldots, c_K)^T$ where $\mathbf{R}_s = \mathbf{c}\mathbf{c}^T$, respectively.

Under the stochastic assumption, $(\mathbf{y}_t)_{t=1,..,T}$ are independent and noncircular Gaussian distributed and therefore, the Fisher information matrix (FIM) for the parameter $\boldsymbol{\alpha}$ is given (elementwise) by [7]

$$\mathrm{FIM}_{i,j} = \frac{T}{2}\mathrm{Tr}\left[\frac{\partial \mathbf{R}_{\tilde{y}}}{\partial \alpha_i}\mathbf{R}_{\tilde{y}}^{-1}\frac{\partial \mathbf{R}_{\tilde{y}}}{\partial \alpha_j}\mathbf{R}_{\tilde{y}}^{-1}\right] \quad (6)$$

where $\mathbf{R}_{\tilde{y}}$ is the covariance of the extended signal $\widetilde{\mathbf{y}}_t \stackrel{\text{def}}{=} [\mathbf{y}_t^T, \mathbf{y}_t^H]^T$ given by

$$\mathbf{R}_{\tilde{y}} \stackrel{\text{def}}{=} \mathrm{E}(\widetilde{\mathbf{y}}_t\widetilde{\mathbf{y}}_t^H) = \widetilde{\mathbf{A}}\mathbf{R}_s\widetilde{\mathbf{A}}^H + \sigma_n^2\mathbf{I} \quad (7)$$

where $\widetilde{\mathbf{A}} \stackrel{\text{def}}{=} [\begin{smallmatrix}\mathbf{A}_\theta\boldsymbol{\Delta}_\phi \\ \mathbf{A}_\theta^*\boldsymbol{\Delta}_\phi^*\end{smallmatrix}] = [\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_K]$ with $\tilde{\mathbf{a}}_k \stackrel{\text{def}}{=} [\mathbf{a}^T(\theta_k)e^{i\phi_k}, \mathbf{a}^H(\theta_k)e^{-i\phi_k}]^T$.

The purpose of the next section is to directly derive the SCRB of the parameter $\boldsymbol{\theta}$ alone from the FIM (6). Noting that the parameters θ_k and ϕ_k are nonlinearly related in the extended steering vector $\tilde{\mathbf{a}}_k$, closed-form expressions of the SCRB of the couple $\boldsymbol{\omega} \stackrel{\text{def}}{=} (\boldsymbol{\theta}^T, \boldsymbol{\phi}^T)^T$ are first derived through its inverse $[\mathrm{CRB}_{\mathrm{sto}}(\boldsymbol{\omega})]^{-1} \stackrel{\text{def}}{=} [\begin{smallmatrix}\mathbf{I}_{\boldsymbol{\theta},\boldsymbol{\theta}} & \mathbf{I}_{\boldsymbol{\theta},\boldsymbol{\phi}} \\ \mathbf{I}_{\boldsymbol{\theta},\boldsymbol{\phi}}^T & \mathbf{I}_{\boldsymbol{\phi},\boldsymbol{\phi}}\end{smallmatrix}]$. Thus, the SCRB of $\boldsymbol{\theta}$ alone is deduced by

$$\mathrm{CRB}_{\mathrm{sto}}(\boldsymbol{\theta}) = (\mathbf{I}_{\boldsymbol{\theta},\boldsymbol{\theta}} - \mathbf{I}_{\boldsymbol{\theta},\boldsymbol{\phi}}\mathbf{I}_{\boldsymbol{\phi},\boldsymbol{\phi}}^{-1}\mathbf{I}_{\boldsymbol{\theta},\boldsymbol{\phi}}^T)^{-1}. \quad (8)$$

III. Derivation of the Different CRB

Writing the FIM (6) in compact matrix from as

$$\mathrm{FIM} = \frac{T}{2}\left(\frac{\partial \mathbf{r}_{\tilde{y}}}{\partial \boldsymbol{\alpha}^T}\right)^H(\mathbf{R}_{\tilde{y}}^{-T} \otimes \mathbf{R}_{\tilde{y}}^{-1})\left(\frac{\partial \mathbf{r}_{\tilde{y}}}{\partial \boldsymbol{\alpha}^T}\right) \quad (9)$$

where $\mathbf{r}_{\tilde{y}} \stackrel{\text{def}}{=} \mathrm{vec}(\mathbf{R}_{\tilde{y}}) = (\widetilde{\mathbf{A}}^* \otimes \widetilde{\mathbf{A}})\mathrm{vec}(\mathbf{R}_s) + \sigma_n^2\mathrm{vec}(\mathbf{I})$ (with $\otimes$ is the Kronecker product), all the first steps of the proof given in [16] apply. In particular, using the partition $(\mathbf{R}_{\tilde{y}}^{-T/2} \otimes \mathbf{R}_{\tilde{y}}^{-1/2})(\frac{\partial \mathbf{r}_{\tilde{y}}}{\partial \boldsymbol{\omega}^T} | \frac{\partial \mathbf{r}_{\tilde{y}}}{\partial \boldsymbol{\rho}^T}, \frac{\partial \mathbf{r}_{\tilde{y}}}{\partial \sigma_n^2}) \stackrel{\text{def}}{=} (\mathbf{G}|\mathbf{V}\ \mathbf{u})$, we can deduce from (9)

$$\frac{2}{T}[\mathrm{CRB}_{\mathrm{sto}}(\boldsymbol{\omega})]^{-1} = \mathbf{G}^H\boldsymbol{\Pi}_{\boldsymbol{\Delta}}^\perp\mathbf{G} \quad (10)$$

with $\boldsymbol{\Delta} \stackrel{\text{def}}{=} (\mathbf{V}\ \mathbf{u})$ and $\boldsymbol{\Pi}_{\boldsymbol{\Delta}}^\perp \stackrel{\text{def}}{=} \mathbf{I} - \boldsymbol{\Pi}_{\boldsymbol{\Delta}}$, where $\boldsymbol{\Pi}_{\boldsymbol{\Delta}}$ denotes the orthonormal projector on the columns of $\boldsymbol{\Delta}$. Starting from (10), where $\boldsymbol{\Pi}_{\boldsymbol{\Delta}}^\perp$ is given by [16, rel. (14)], the main steps of the proof of the following theorem are given in the appendix.

Theorem 1: The SCRB under the general rectilinear assumption is given by for $K < 2M$

$$\mathrm{CRB}_{\mathrm{sto}}^{\mathrm{rec}_1}(\boldsymbol{\omega}) = \frac{\sigma_n^2}{T}\left((\widetilde{\mathbf{D}}_\omega^H\boldsymbol{\Pi}_{\widetilde{\mathbf{A}}}^\perp\widetilde{\mathbf{D}}_\omega) \odot \left(\begin{pmatrix}1 & 1 \\ 1 & 1\end{pmatrix} \otimes \widetilde{\mathbf{H}}\right)\right)^{-1} \quad (11)$$

with $\boldsymbol{\Pi}_{\widetilde{\mathbf{A}}}^\perp \stackrel{\text{def}}{=} \mathbf{I} - \boldsymbol{\Pi}_{\widetilde{\mathbf{A}}}$, where $\boldsymbol{\Pi}_{\widetilde{\mathbf{A}}}$ denotes the orthonormal projector on the columns of $\widetilde{\mathbf{A}}$, $\widetilde{\mathbf{D}}_\omega \stackrel{\text{def}}{=} [\widetilde{\mathbf{D}}_\theta, \widetilde{\mathbf{D}}_\phi]$ with $\widetilde{\mathbf{D}}_\theta \stackrel{\text{def}}{=} [\frac{\partial \tilde{\mathbf{a}}_1}{\partial \theta_1}, \ldots, \frac{\partial \tilde{\mathbf{a}}_K}{\partial \theta_K}]$, $\widetilde{\mathbf{D}}_\phi \stackrel{\text{def}}{=} [\frac{\partial \tilde{\mathbf{a}}_1}{\partial \phi_1}, \ldots, \frac{\partial \tilde{\mathbf{a}}_K}{\partial \phi_K}]$, $\widetilde{\mathbf{H}} \stackrel{\text{def}}{=} \mathbf{R}_s\widetilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1}\widetilde{\mathbf{A}}\mathbf{R}_s$, and $\odot$ denotes the element by element matrix product. Applying (8), the SCRB on $\boldsymbol{\theta}$ alone is given by

$$\mathrm{CRB}_{\mathrm{sto}}^{\mathrm{rec}_1}(\boldsymbol{\theta}) = \frac{\sigma_n^2}{T}\left(\left[\left(\widetilde{\mathbf{D}}_\theta^H\boldsymbol{\Pi}_{\widetilde{\mathbf{A}}}^\perp\widetilde{\mathbf{D}}_\theta\right) \odot \widetilde{\mathbf{H}}\right] - \left[\left(\widetilde{\mathbf{D}}_\theta^H\boldsymbol{\Pi}_{\widetilde{\mathbf{A}}}^\perp\widetilde{\mathbf{D}}_\phi\right) \odot \widetilde{\mathbf{H}}\right]\left[\left(\widetilde{\mathbf{D}}_\phi^H\boldsymbol{\Pi}_{\widetilde{\mathbf{A}}}^\perp\widetilde{\mathbf{D}}_\phi\right) \odot \widetilde{\mathbf{H}}\right]^{-1}\left[\left(\widetilde{\mathbf{D}}_\phi^H\boldsymbol{\Pi}_{\widetilde{\mathbf{A}}}^\perp\widetilde{\mathbf{D}}_\theta\right) \odot \widetilde{\mathbf{H}}\right]\right)^{-1}. \quad (12)$$

Including the prior knowledge that the K rectilinear sources are coherent (where $\mathbf{R}_s = \mathbf{c}\mathbf{c}^T$), which appears in specular multipath propagation, the main steps of the proof of the following theorem are given in the appendix.

Theorem 2: The SCRB under the prior knowledge of fully coherent rectilinear sources is given by for $K < 2M$

$$\mathrm{CRB}_{\mathrm{sto}}^{\mathrm{rec}_2}(\boldsymbol{\omega}) = \frac{\sigma_n^2}{T\kappa_c}\left(\left(\widetilde{\mathbf{D}}_\omega^H\boldsymbol{\Pi}_{\widetilde{\mathbf{A}}}^\perp\widetilde{\mathbf{D}}_\omega\right) \odot \left(\begin{pmatrix}1 & 1 \\ 1 & 1\end{pmatrix} \otimes \mathbf{R}_s\right)\right)^{-1} \quad (13)$$

where $\kappa_c \stackrel{\text{def}}{=} \mathbf{c}^T\widetilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1}\widetilde{\mathbf{A}}\mathbf{c}$.

Including now the prior knowledge that the K rectilinear sources are uncorrelated, the following theorem (presented in [9]) is proved in the appendix

Theorem 3: The SCRB under the prior knowledge of uncorrelated rectilinear sources is given by $K < 2M$

$$\mathrm{CRB}_{\mathrm{sto}}^{\mathrm{rec}_3}(\boldsymbol{\omega}) = \frac{2}{T}\left(\widetilde{\boldsymbol{\Delta}}_\sigma\overline{\mathbf{D}}_\omega^H\widetilde{\mathbf{B}}(\widetilde{\mathbf{B}}^H\overline{\mathbf{G}}\widetilde{\mathbf{B}})^{-1}\widetilde{\mathbf{B}}^H\overline{\mathbf{D}}_\omega\widetilde{\boldsymbol{\Delta}}_\sigma\right)^{-1} \quad (14)$$

where $\overline{\mathbf{G}} \stackrel{\text{def}}{=} \mathbf{R}_{\tilde{y}}^T \otimes \mathbf{R}_{\tilde{y}} + \frac{\sigma_n^4}{2M-K}\mathrm{vec}(\boldsymbol{\Pi}_{\widetilde{\mathbf{A}}})\mathrm{vec}^H(\boldsymbol{\Pi}_{\widetilde{\mathbf{A}}})$, $\widetilde{\boldsymbol{\Delta}}_\sigma \stackrel{\text{def}}{=} \mathrm{Diag}(\sigma_1^2, \ldots, \sigma_K^2, \sigma_1^2, \ldots, \sigma_K^2)$, $\overline{\mathbf{D}}_\omega \stackrel{\text{def}}{=} (\widetilde{\mathbf{A}}^*|\widetilde{\mathbf{A}}^*) \circ (\widetilde{\mathbf{D}}_\theta|\widetilde{\mathbf{D}}_\phi) + (\widetilde{\mathbf{D}}_\theta^*|\widetilde{\mathbf{D}}_\phi^*) \circ (\widetilde{\mathbf{A}}\,|\,\widetilde{\mathbf{A}})$, and $\widetilde{\mathbf{B}}$ is any $(2M)^2 \times ((2M)^2 - K)$ matrix whose columns span the null space of $\widetilde{\mathbf{A}}^* \circ \widetilde{\mathbf{A}}$, where $\circ$ is the columnwise Kronecker product.

Finally, to make comparisons with the DCRB, we recall its closed-form expression derived in [9] and [10] and then in [11] under more general rectilinear models for $K < 2M$

$$\mathrm{CRB}_{\mathrm{det}}^{\mathrm{rec}}(\boldsymbol{\omega}) = \frac{\sigma_n^2}{T}\left(\left(\widetilde{\mathbf{D}}_\omega^H\boldsymbol{\Pi}_{\widetilde{\mathbf{A}}}^\perp\widetilde{\mathbf{D}}_\omega\right) \odot \left(\begin{pmatrix}1 & 1 \\ 1 & 1\end{pmatrix} \otimes \mathbf{R}_{s,T}\right)\right)^{-1} \quad (15)$$

where $\mathbf{R}_{s,T} \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^T \mathbf{s}_t\mathbf{s}_t^T$. We also recall the closed-form expression of the SCRB derived under the assumption of arbitrary noncircular sources in [7], applied to rectilinear sources for $K < M$

$$\mathrm{CRB}^{\mathrm{nc}}_{\mathrm{sto}}(\boldsymbol{\theta}) = \frac{\sigma_n^2}{2T}\left(\mathrm{Re}\left[\left(\mathbf{D}_\theta^H \mathbf{\Pi}^{\perp}_{\mathbf{A}_\theta} \mathbf{D}_\theta\right) \odot \left(\mathbf{\Delta}^*_\phi \widetilde{\mathbf{H}} \mathbf{\Delta}_\phi\right)\right]\right)^{-1} \tag{16}$$

where $\mathbf{D}_\theta \stackrel{\mathrm{def}}{=} \left[\frac{\partial \mathbf{a}_1}{\partial \theta_1}, \ldots, \frac{\partial \mathbf{a}_K}{\partial \theta_K}\right]$.

IV. Analytical and Numerical Comparisons

Considering the comparison of the previously introduced closed-form expressions of the CRB, the main steps of the proof of the following theorem are given in the appendix.

Theorem 4: Under the general rectilinear assumption, the DCRB (for $T \to \infty$, i.e., replacing $\mathbf{R}_{s,T}$ by $\mathbf{R}_s$) and SCRB have the relationships

$$\mathrm{CRB}^{\mathrm{rec}}_{\mathrm{det}}(\boldsymbol{\theta}) \leq \mathrm{CRB}^{\mathrm{rec}_1}_{\mathrm{sto}}(\boldsymbol{\theta}) \leq \mathrm{CRB}^{\mathrm{nc}}_{\mathrm{sto}}(\boldsymbol{\theta}). \tag{17}$$

Note that for a finite value of T, we cannot be sure that $\mathrm{CRB}^{\mathrm{rec}}_{\mathrm{det}}(\boldsymbol{\theta}) \leq \mathrm{CRB}^{\mathrm{rec}_1}_{\mathrm{sto}}(\boldsymbol{\theta})$. In fact for low T and high SNRs, the inequality reverses.

If we consider now the exploitation of prior knowledge, the following theorem is proved in the appendix.

Theorem 5: Under the prior knowledge that the sources are rectilinear uncorrelated, $\mathrm{CRB}^{\mathrm{rec}_3}_{\mathrm{sto}}(\boldsymbol{\theta})$ is reduced w.r.t. $\mathrm{CRB}^{\mathrm{rec}_1}_{\mathrm{sto}}(\boldsymbol{\theta})$. In contrast, the exploitation of fully coherency of the sources does not reduce $\mathrm{CRB}^{\mathrm{rec}_1}_{\mathrm{sto}}(\boldsymbol{\theta})$

$$\mathrm{CRB}^{\mathrm{rec}_3}_{\mathrm{sto}}(\boldsymbol{\theta}) \leq \mathrm{CRB}^{\mathrm{rec}_1}_{\mathrm{sto}}(\boldsymbol{\theta}),\ \mathrm{CRB}^{\mathrm{rec}_2}_{\mathrm{sto}}(\boldsymbol{\theta}) = \mathrm{CRB}^{\mathrm{rec}_1}_{\mathrm{sto}}(\boldsymbol{\theta}). \tag{18}$$

Note that similar properties have been proved for circular sources in [17] for uncorrelated sources and in [18] for fully coherent sources.

Finally, in the case of a single rectilinear source, we have proved after tedious algebraic manipulations that the SCRBs of θ_1 alone deduced from (11), (13), and (14) reduce to

$$\mathrm{CRB}^{\mathrm{rec}}_{\mathrm{sto}}(\theta_1) = \frac{1}{2T\mathbf{a}_1'^H \mathbf{\Pi}^{\perp}_{\mathbf{a}_1} \mathbf{a}_1'} \frac{\sigma_n^2}{\sigma_s^2}\left(1 + \frac{\sigma_n^2}{2\sigma_s^2 \|\mathbf{a}_1\|^2}\right) \tag{19}$$

where $\mathbf{a}_1 \stackrel{\mathrm{def}}{=} \mathbf{a}(\theta_1)$ and $\mathbf{a}_1' \stackrel{\mathrm{def}}{=} d\mathbf{a}(\theta_1)/d\theta_1$. Comparing (19) to the SCRB derived in [7] under the general noncircular assumption, we see that $\mathrm{CRB}^{\mathrm{rec}}_{\mathrm{sto}}(\theta_1) = \mathrm{CRB}^{\mathrm{nc}}_{\mathrm{sto}}(\theta_1)$, i.e., the SCRB is not reduced by exploiting the rectilinear prior knowledge.

To illustrate, the difference between the different CRBs, we consider now the case of two equal-power rectilinear sources of signal-to-noise ratio $10\log_{10}(\sigma_s^2/\sigma_n^2) = 10$ dB and correlation ρ, impinging on an ULA of $M = 6$ sensors with half-wavelength spacing. Figs. 1–3 exhibit different ratios of CRB. Fig. 1 shows that there are significant gaps between the DSCB and the SCRB for closely spaced and strongly correlated rectilinear sources. More generally, extensive numerical comparisons have shown that this gap increases for low SNR, low phase and DOA separations and high source correlation, but this gap will always vanish for high SNR. This proves that the conclusions based on the DCRB may be very optimistic for not too high SNR. Fig. 2 highlights that the exploitation of the prior of rectilinearity greatly reduces the estimation error for closely spaced and uncorrelated sources. Finally, Fig. 3 proves that the joint exploitation of the prior of uncorrelatedness and rectilinearity

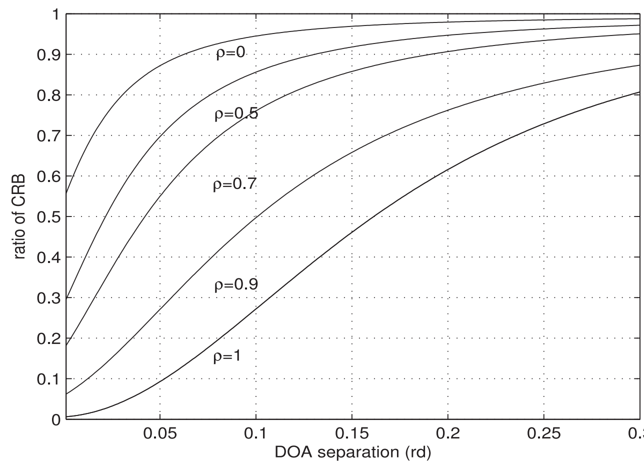

Fig. 1. Ratio $\mathrm{CRB}^{\mathrm{rec}}_{\mathrm{det}}(\theta_1)/\mathrm{CRB}^{\mathrm{rec}_1}_{\mathrm{sto}}(\theta_1)$ for $\phi_1 - \phi_2 = 0.1$ rd.

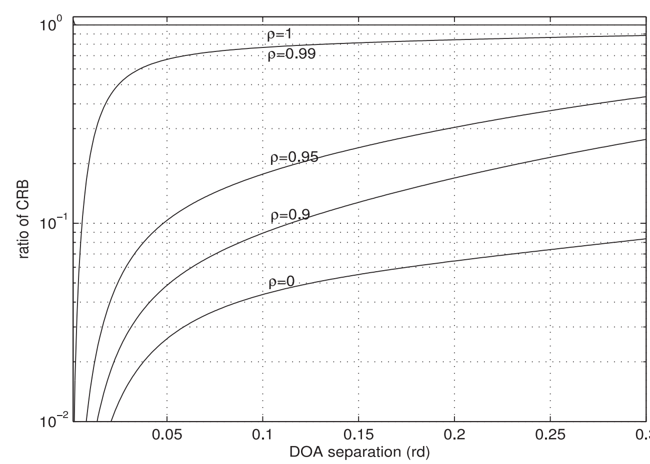

Fig. 2. Ratio $\mathrm{CRB}^{\mathrm{rec}_1}_{\mathrm{sto}}(\theta_1)/\mathrm{CRB}^{\mathrm{nc}}_{\mathrm{sto}}(\theta_1)$ for $\phi_1 - \phi_2 = 0.1$ rd.

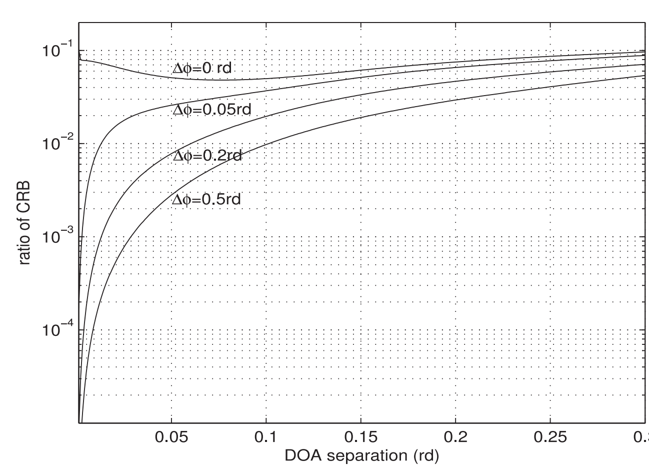

Fig. 3. Ratio $\mathrm{CRB}^{\mathrm{rec}_3}_{\mathrm{sto}}(\theta_1)/\mathrm{CRB}^{\mathrm{nc}}_{\mathrm{sto}}(\theta_1)$ for $\rho = 0$.

greatly reduces the estimation error for closely spaced sources with different phases.

V. Conclusion

Closed-form expressions of the SCRB of DOA estimation for rectilinear sources have been directly derived form the extended Slepian–Bangs formula, including the case of prior knowledge of uncorrelated or coherent rectilinear sources. Analytical and numerical comparisons with the DCRB and the SCRB for noncircular sources have shown in particular that the DCRB may be very optimistic.

Appendix

Proof of Theorem 1: (Detailed proofs of Theorem 1 are available at [23]) Since $\mathbf{R}_s$ is a $(K \times K)$ real symmetric matrix, it then follows from [19, rel.(7.18)] that $\mathrm{vec}(\mathbf{R}_s) = \mathbf{D}_K \boldsymbol{\rho}$, where $\mathbf{D}_K$ is a so-called duplication matrix, and hence [16, rel.(19)] becomes

$$\mathbf{V} = \left(\mathbf{R}_{\tilde{y}}^{-T/2}\tilde{\mathbf{A}}^{*} \otimes \mathbf{R}_{\tilde{y}}^{-1/2}\tilde{\mathbf{A}}\right)\mathbf{D}_K \stackrel{\text{def}}{=} \mathbf{W}\mathbf{D}_K.$$

Then, it follows from [19, Th. 7.38], and some simple algebraic manipulations using [19, Th. 7.34, rel.(b)] that [16, rel.(20)] becomes

$$\mathbf{\Pi}_{\mathbf{V}}^{\perp} = \mathbf{I} - \mathbf{W}(\mathbf{U} \otimes \mathbf{U})\mathbf{N}_K\mathbf{W}^H$$

where $\mathbf{U} \stackrel{\text{def}}{=} \tilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1}\tilde{\mathbf{A}}$ and $\mathbf{N}_K$ is an $(K \times K)$ matrix defined in [19, Th. 7.34]. By evaluating the derivatives in $\mathbf{G}$ and $\mathbf{u}$, and through some further algebra, one finds

$$\mathbf{u}^H\mathbf{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_k = 0$$

where $\mathbf{g}_k$ is the kth column of $\mathbf{G}$ given by $\mathbf{g}_k = \mathrm{vec}(\mathbf{Z}_k + \mathbf{Z}_k^H)$ and where $\mathbf{Z}_k \stackrel{\text{def}}{=} \mathbf{R}_{\tilde{y}}^{-1/2}\tilde{\mathbf{A}}\mathbf{r}_{s,k}\tilde{\mathbf{a}}_k'^H\mathbf{R}_{\tilde{y}}^{-1/2}$, $\tilde{\mathbf{a}}_k' \stackrel{\text{def}}{=} d\tilde{\mathbf{a}}_k/dw_k$ and $\mathbf{r}_{s,k}$ is the kth column of $\mathbf{R}_s$. This identity allows us to rewrite the individual elements of (10) as

$$\frac{2}{T}\left[\mathrm{CRB}_{\mathrm{sto}}^{-1}(\boldsymbol{\omega})\right]_{k,l} = \mathbf{g}_k^H\mathbf{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_l.$$

By further calculations we get

$$\left[\mathrm{CRB}_{\mathrm{sto}}^{-1}(\boldsymbol{\omega})\right]_{k,l} = \frac{T}{\sigma_n^2}\mathrm{Re}\left(\left(\tilde{\mathbf{a}}_k'^H\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\tilde{\mathbf{a}}_l'\right)\left(\mathbf{r}_{s,k}^T\tilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1}\tilde{\mathbf{A}}\mathbf{r}_{s,l}\right)\right). \tag{20}$$

Finally, we can write (20) in matrix form as in (11). ■

Proof of Theorem 2: (Detailed proofs of Theorem 2 are available at [23]) We follow the steps similar to those in the proof of Theorem 1. Since $\mathbf{R}_s = \mathbf{c}\mathbf{c}^T$ and its derivative w.r.t. $\mathbf{c}$ is given by $\mathbf{D}_c \stackrel{\text{def}}{=} \mathbf{c} \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{c} = 2\mathbf{N}_K(\mathbf{c} \otimes \mathbf{I})$, it follows then that $\mathbf{V}$ has the form $\mathbf{V} = \mathbf{W}\mathbf{D}_c$. After some algebraic manipulation using [19, Th. 7.34, rel.(d)], it follows that

$$\mathbf{\Pi}_{\mathbf{V}}^{\perp} \stackrel{\text{def}}{=} \mathbf{I} - \mathbf{V}(\mathbf{V}^H\mathbf{V})^{-1}\mathbf{V}^H = \mathbf{I} - \mathbf{V}_1\bar{\mathbf{V}}^{-1}\mathbf{V}_1^H$$

with $\mathbf{V}_1 \stackrel{\text{def}}{=} \mathbf{W}\mathbf{N}_K(\mathbf{c} \otimes \mathbf{I})$ and $\bar{\mathbf{V}} \stackrel{\text{def}}{=} \frac{1}{2}(\kappa_c\mathbf{U} + \mathbf{u}_c\mathbf{u}_c^T)$ where $\mathbf{u}_c \stackrel{\text{def}}{=} \mathbf{U}\mathbf{c}$ and $\kappa_c \stackrel{\text{def}}{=} \mathbf{c}^T\mathbf{U}\mathbf{c}$. Thanks to the matrix inversion lemma, we have $\bar{\mathbf{V}}^{-1} = \frac{2}{\kappa_c}(\mathbf{U}^{-1} - \frac{1}{2\kappa_c}\mathbf{c}\mathbf{c}^T)$. Through some further algebra using $\mathbf{u} = \mathrm{vec}(\mathbf{R}_{\tilde{y}}^{-1})$ and $\mathbf{g}_k = \mathrm{vec}(\mathbf{Z}_k + \mathbf{Z}_k^H)$ where $\mathbf{Z}_k \stackrel{\text{def}}{=} \mathbf{R}_{\tilde{y}}^{-1/2}\tilde{\mathbf{A}}\mathbf{c}\tilde{\mathbf{a}}_k'^H\mathbf{R}_{\tilde{y}}^{-1/2}$, one finds that $\mathbf{u}^H\mathbf{V}_1 = \mathbf{c}^T\bar{\mathbf{U}}$ where $\bar{\mathbf{U}} \stackrel{\text{def}}{=} \tilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-2}\tilde{\mathbf{A}}$, $\mathbf{u}^H\mathbf{g}_k = 2c_k\tilde{\mathbf{a}}_k'^H\mathbf{R}_{\tilde{y}}^{-2}\tilde{\mathbf{A}}\mathbf{c}$, and $\mathbf{V}_1^H\mathbf{g}_k = c_k(\nu_c^{(k)}\mathbf{u}_c + \kappa_c\tilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1}\tilde{\mathbf{a}}_k')$ where $\nu_c^{(k)} \stackrel{\text{def}}{=} \tilde{\mathbf{a}}_k'^H\mathbf{R}_{\tilde{y}}^{-1}\tilde{\mathbf{A}}\mathbf{c}$. By further calculations we arrive at $\mathbf{u}^H\mathbf{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_k = 0$. This identity allows us to rewrite the individual elements of (10) as

$$\frac{2}{T}\left[\mathrm{CRB}_{\mathrm{sto}}^{-1}(\boldsymbol{\omega})\right]_{k,l} = \mathbf{g}_k^H\mathbf{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_l = \frac{2\kappa_c T}{\sigma_n^2}c_k c_l\left(\tilde{\mathbf{a}}_k'^H\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\tilde{\mathbf{a}}_l'\right)$$

which can also be written in the matrix form (13). ■

Proof of Theorem 3: Noting that (7) becomes $\mathbf{R}_{\tilde{y}} = \sum_{k=1}^{K}\sigma_k^2\tilde{\mathbf{a}}_k\tilde{\mathbf{a}}_k^H + \sigma_n^2\mathbf{I}$, all the steps of the proof given in [17, Appendix A] apply to the parameter $\boldsymbol{\omega}$ with the FIM (6) associated with the noncircular zero-mean Gaussian distribution of $\mathbf{y}_t$. ■

Proof of Theorem 4: Using $\mathbf{R}_s \geq \mathbf{R}_s\widetilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1}\widetilde{\mathbf{A}}\mathbf{R}_s$ thanks to [20, rel. B.6.37] and noting that $\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$ is positive semidefinite, we have $\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right) \otimes (\mathbf{R}_s - \widetilde{\mathbf{H}}) \geq 0$ from [19, Th. 7.10]. Then, noting that $\widetilde{\mathbf{D}}_\omega^H\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\widetilde{\mathbf{D}}_\omega$ is positive definite, we have $(\widetilde{\mathbf{D}}_\omega^H\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\widetilde{\mathbf{D}}_\omega) \odot (\left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right) \otimes (\mathbf{R}_s - \widetilde{\mathbf{H}})) \geq 0$ thanks to [20, rel. R.19, p. 358], and $\mathrm{CRB}_{\mathrm{det}}^{\mathrm{rec}}(\boldsymbol{\theta}) \leq \mathrm{CRB}_{\mathrm{sto}}^{\mathrm{rec}_1}(\boldsymbol{\theta})$ directly follows. ■

Consider the parametrization $\boldsymbol{\alpha}_{\mathrm{nc}} = (\boldsymbol{\theta}^T, \boldsymbol{\rho}_{\mathrm{nc}}^T, \sigma_n^2)^T$ associated with the assumption of arbitrary noncircular sources in [7], where $\boldsymbol{\rho}_{\mathrm{nc}}$ is the $2K^2 + K$ vector made from $[\mathrm{Re}(\mathbf{R}_x)]_{i,j}$, $[\mathrm{Im}(\mathbf{R}_x)]_{i,j}$, $[\mathrm{Re}(\mathbf{C}_x)]_{i,j}$ and $[\mathrm{Re}(\mathbf{C}_x)]_{i,j}$ for $1 \leq i < j \leq K$, and $[\mathbf{R}_x]_{i,i}$, $[\mathrm{Re}(\mathbf{C}_x)]_{i,i}$ and $[\mathrm{Im}(\mathbf{C}_x)]_{i,i}$ for $1 \leq i \leq K$, where $\mathbf{R}_x \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{x}_t\mathbf{x}_t^H)$ and $\mathbf{C}_x \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{x}_t\mathbf{x}_t^T)$ with $\mathbf{x}_t \stackrel{\text{def}}{=} (x_{t,1}, \ldots, x_{t,K})^T$. Consider now the one to one mapping between $\boldsymbol{\alpha}_{\mathrm{nc}}$ and $\boldsymbol{\alpha}_{\mathrm{nc}}' = (\boldsymbol{\theta}^T, \boldsymbol{\phi}^T, \boldsymbol{\rho}_1^T, \boldsymbol{\rho}'^T, \sigma_n^2)^T$ where $\boldsymbol{\phi} = (\phi_1, .., \phi_k, .., \phi_K)^T$ is defined by $2\phi_k \stackrel{\text{def}}{=} \angle[\mathbf{C}_x]_{k,k}/[\mathbf{R}_x]_{k,k}$, $\boldsymbol{\rho}_1$ is the vector made from $[\mathrm{Re}(\mathbf{R}_s)]_{i,j}$ for $1 \leq i \leq j \leq K$, and $\boldsymbol{\rho}'$ is the vector gathering $[\mathrm{Im}(\mathbf{R}_s)]_{i,j}$, $[\mathrm{Im}(\mathbf{C}_s)]_{i,j}$ for $1 \leq i < j \leq K$ and $[\mathrm{Re}(\mathbf{C}_s)]_{i,j}$ for $1 \leq i \leq j \leq K$.

Let FIM_1, $\mathrm{FIM}_{\mathrm{nc}}$, and $\mathrm{FIM}_{\mathrm{nc}}'$ be the FIM associated with the stochastic parametrizations (3), $\boldsymbol{\alpha}_{\mathrm{nc}}$ and $\boldsymbol{\alpha}_{\mathrm{nc}}'$, respectively. From the one to one mapping $\boldsymbol{\alpha}_{\mathrm{nc}} \leftrightarrow \boldsymbol{\alpha}_{\mathrm{nc}}'$, we get $[\mathrm{FIM}_{\mathrm{nc}}']_\theta = [\mathrm{FIM}_{\mathrm{nc}}]_\theta$ (where $[.]_\theta$ denotes the submatrix determined by the first K rows and columns). Furthermore $\mathrm{FIM}_1 = [\mathrm{FIM'}_{\mathrm{nc}}]_1$ where $[.]_1$ denotes the principal submatrix determined by the rows and columns induced by the parametrization of FIM_1. Consequently, taking the inverse, we get $[\mathrm{FIM}_1]^{-1} \leq [\mathrm{FIM'}_{\mathrm{nc}}^{-1}]_1$ [22, th. 7.7.8]. Then, taking the principal submatrix determined by the first K rows and columns, we get $\mathrm{CRB}_{\mathrm{sto}}^{\mathrm{rec}_1}(\boldsymbol{\theta}) \leq \mathrm{CRB}_{\mathrm{sto}}^{\mathrm{nc}}(\boldsymbol{\theta})$. ■

Proof of Theorem 5: Let FIM_3 be the FIM associated with the parametrizations (3) for which $\boldsymbol{\rho} = ([\mathbf{R}_s]_{k,k;1 \leq k \leq K})^T$. For rectilinear uncorrelated sources, $\mathrm{FIM}_3 = [\mathrm{FIM}_1]_3$, where $[.]_3$ denotes the principal submatrix determined by the rows and columns induced by the parametrization of FIM_3. Taking the inverse, we get $[\mathrm{FIM}_3]^{-1} \leq [\mathrm{FIM}_1^{-1}]_3$ [22, Th. 7.7.8]. Then, taking the principal submatrix determined by the first K rows and columns, we get $\mathrm{CRB}_{\mathrm{sto}}^{\mathrm{rec}_3}(\boldsymbol{\theta}) \leq \mathrm{CRB}_{\mathrm{sto}}^{\mathrm{rec}_1}(\boldsymbol{\theta})$. ■

Finally, by noting that for coherent rectilinear sources, $\widetilde{\mathbf{H}} = \kappa_c\mathbf{c}\mathbf{c}^T$ in (11) gives $\mathrm{CRB}_{\mathrm{sto}}^{\mathrm{rec}_2}(\boldsymbol{\omega}) = \mathrm{CRB}_{\mathrm{sto}}^{\mathrm{rec}_1}(\boldsymbol{\omega})$, It follows then that $\mathrm{CRB}_{\mathrm{sto}}^{\mathrm{rec}_2}(\boldsymbol{\theta}) = \mathrm{CRB}_{\mathrm{sto}}^{\mathrm{rec}_1}(\boldsymbol{\theta})$. ■

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