Detailed proofs of theorems 1 and 2 given in [1]

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I. Background

A. Relations

We will make frequent use of the following well known relations which hold for any conformable matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$.

$$\mathrm{vec}(\mathbf{ABC}) = (\mathbf{C}^T \otimes \mathbf{A})\mathrm{vec}(\mathbf{B}), \tag{1}$$

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}, \tag{2}$$

$$\mathrm{Tr}(\mathbf{AB}) = (\mathrm{vec}(\mathbf{A}^H))^H \mathrm{vec}(\mathbf{B}). \tag{3}$$

B. General expression of the CRB

The stochastic CRB is writing through the compact expression of the FIM:

$$\mathrm{CRB}_{\mathrm{sto}}^{-1}(\boldsymbol{\alpha}) = \frac{T}{2}\left(\frac{\partial \mathbf{r}_{\tilde{y}}}{\partial \boldsymbol{\alpha}^T}\right)^H \left(\mathbf{R}_{\tilde{y}}^{-T} \otimes \mathbf{R}_{\tilde{y}}^{-1}\right)\left(\frac{\partial \mathbf{r}_{\tilde{y}}}{\partial \boldsymbol{\alpha}^T}\right), \tag{4}$$

where the vectorization of $\mathbf{R}_{\tilde{y}} = \widetilde{\mathbf{A}}\mathbf{R}_s\widetilde{\mathbf{A}}^H + \sigma_n^2\mathbf{I}$ is given from (1) by

$$\mathbf{r}_{\tilde{y}} \stackrel{\mathrm{def}}{=} \mathrm{vec}(\mathbf{R}_{\tilde{y}}) = (\widetilde{\mathbf{A}}^* \otimes \widetilde{\mathbf{A}})\mathrm{vec}(\mathbf{R}_s) + \sigma_n^2 \mathrm{vec}(\mathbf{I}).$$

To begin the proofs of the two theorems, all the first steps of [17] apply. In particular, using the partition

$$(\mathbf{R}_{\tilde{y}}^{-T/2} \otimes \mathbf{R}_{\tilde{y}}^{-1/2})\left(\frac{\partial \mathbf{r}_{\tilde{y}}}{\partial \boldsymbol{\omega}^T}|\frac{\partial \mathbf{r}_{\tilde{y}}}{\partial \boldsymbol{\rho}^T}, \frac{\partial \mathbf{r}_{\tilde{y}}}{\partial \sigma_n^2}\right) \stackrel{\mathrm{def}}{=} (\mathbf{G}|\mathbf{V}, \mathbf{u}), \tag{5}$$

we can deduce from (4)

$$\frac{2}{T}\mathrm{CRB}_{\mathrm{sto}}^{-1}(\boldsymbol{\omega}) = \mathbf{G}^H \boldsymbol{\Pi}_{\boldsymbol{\Delta}}^{\perp}\mathbf{G}, \tag{6}$$

with $\boldsymbol{\Delta} \stackrel{\mathrm{def}}{=} (\mathbf{V}, \mathbf{u})$ and $\boldsymbol{\Pi}_{\boldsymbol{\Delta}}^{\perp} \stackrel{\mathrm{def}}{=} \mathbf{I} - \boldsymbol{\Pi}_{\boldsymbol{\Delta}}$, where $\boldsymbol{\Pi}_{\boldsymbol{\Delta}}$ denotes the orthonormal projector on the columns of $\boldsymbol{\Delta}$. Following [17, rel.(14)], it has been proved that

$$\boldsymbol{\Pi}_{\boldsymbol{\Delta}}^{\perp} = \boldsymbol{\Pi}_{\mathbf{V}}^{\perp} - \frac{\boldsymbol{\Pi}_{\mathbf{V}}^{\perp}\mathbf{u}\mathbf{u}^H\boldsymbol{\Pi}_{\mathbf{V}}^{\perp}}{\mathbf{u}^H\boldsymbol{\Pi}_{\mathbf{V}}^{\perp}\mathbf{u}}, \tag{7}$$

where $\frac{d\mathbf{r}_{\tilde{y}}}{d\sigma_n^2} = \mathrm{vec}(\mathbf{I})$ implies by using (1), that

$$\mathbf{u} = (\mathbf{R}_{\tilde{y}}^{-T/2} \otimes \mathbf{R}_{\tilde{y}}^{-1/2})\mathrm{vec}(\mathbf{I}) = \mathrm{vec}(\mathbf{R}_{\tilde{y}}^{-1}). \tag{8}$$

Consequently using (6) and (7), if $\mathbf{g}_k$ denotes the k-th column of $\mathbf{G}$, the (k,l) element of $\frac{2}{T}\mathrm{CRB}_{\mathrm{sto}}^{-1}(\boldsymbol{\omega})$ can be written elementwise as

$$\frac{2}{T}\left[\mathrm{CRB}_{\mathrm{sto}}^{-1}(\boldsymbol{\omega})\right]_{k,l} = \mathbf{g}_k^H \boldsymbol{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_l - \frac{\mathbf{g}_k^H\boldsymbol{\Pi}_{\mathbf{V}}^{\perp}\mathbf{u}\mathbf{u}^H\boldsymbol{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_l}{\mathbf{u}^H\boldsymbol{\Pi}_{\mathbf{V}}^{\perp}\mathbf{u}}. \tag{9}$$

To proceed, we need to determine the expressions of $\boldsymbol{\Pi}_{\mathbf{V}}^{\perp}$ associated with the two parametrizations of the real symmetric matrix $\mathbf{R}_s$. But as the steps of the proof given in [17] do not apply, we have to elaborate a little bit.

II. Proof of theorem 1

For arbitrary real symmetric matrix $\mathbf{R}_s$, let $\mathbf{r}_{s,k}$ denote the *kth* column of $\mathbf{R}_s$. We get from $\mathbf{R}_{\tilde{y}} \stackrel{\text{def}}{=} \mathrm{E}(\widetilde{\mathbf{y}}_t\widetilde{\mathbf{y}}_t^H) = \widetilde{\mathbf{A}}\mathbf{R}_s\widetilde{\mathbf{A}}^H + \sigma_n^2\mathbf{I}$,

$$\begin{aligned}\frac{d\mathbf{R}_{\tilde{y}}}{dw_k} &= \Big(\mathbf{0},.,\tilde{\mathbf{a}}_k^{'},.,\mathbf{0}\Big)\mathbf{R}_s\tilde{\mathbf{A}}^H + \tilde{\mathbf{A}}\mathbf{R}_s\begin{pmatrix}\mathbf{0}^T\\ \vdots \\ \tilde{\mathbf{a}}_k^{'H} \\ \vdots \\ \mathbf{0}^T\end{pmatrix}\\ &= \tilde{\mathbf{a}}_k^{'}\mathbf{r}_{s,k}^T\tilde{\mathbf{A}}^H + \tilde{\mathbf{A}}\mathbf{r}_{s,k}\tilde{\mathbf{a}}_k^{'H}.\end{aligned}$$

where $\tilde{\mathbf{a}}_k^{'} \stackrel{\text{def}}{=} d\tilde{\mathbf{a}}_k/d\omega_k$. Hence using (1), the *kth* column of $\mathbf{G}$ in (5) is given by

$$\mathbf{g}_k = (\mathbf{R}_{\tilde{y}}^{-T/2}\otimes\mathbf{R}_{\tilde{y}}^{-1/2})\mathrm{vec}\left(\frac{d\mathbf{R}_{\tilde{y}}}{d\omega_k}\right) = \mathrm{vec}\left(\mathbf{R}_{\tilde{y}}^{-1/2}\frac{d\mathbf{R}_{\tilde{y}}}{d\omega_k}\mathbf{R}_{\tilde{y}}^{-1/2}\right) = \mathrm{vec}(\mathbf{Z}_k^H + \mathbf{Z}_k), \tag{10}$$

where

$$\mathbf{Z}_k \stackrel{\text{def}}{=} \mathbf{R}_{\tilde{y}}^{-1/2}\tilde{\mathbf{A}}\mathbf{r}_{s,k}\tilde{\mathbf{a}}_k^{'H}\mathbf{R}_{\tilde{y}}^{-1/2}. \tag{11}$$

Next, we determine $\mathbf{V}$. The key observation to note here, is that the real-valued symmetric matrix $\mathbf{R}_s$, using [20, rel.(7.18)], can be written as

$$\mathrm{vec}(\mathbf{R}_s) = \mathbf{D}_K\boldsymbol{\rho},$$

where $\mathbf{D}_K$ is the so-called duplication matrix, and hence from (5)

$$\mathbf{V} = (\mathbf{R}_{\tilde{y}}^{-T/2}\tilde{\mathbf{A}}^*\otimes\mathbf{R}_{\tilde{y}}^{-1/2}\tilde{\mathbf{A}})\mathbf{D}_K \stackrel{\text{def}}{=} \mathbf{W}\mathbf{D}_K, \tag{12}$$

and consequently:

$$\begin{aligned}\boldsymbol{\Pi}_{\mathbf{V}}^{\perp} &= \mathbf{I} - \mathbf{V}(\mathbf{V}^H\mathbf{V})^{-1}\mathbf{V}^H = \mathbf{I} - \mathbf{W}\mathbf{D}_K(\mathbf{D}_K^T\mathbf{W}^H\mathbf{W}\mathbf{D}_K)^{-1}\mathbf{D}_K^T\mathbf{W}^H\\ &= \mathbf{I} - \mathbf{W}\mathbf{D}_K(\mathbf{D}_K^T(\mathbf{U}\otimes\mathbf{U})\mathbf{D}_K)^{-1}\mathbf{D}_K^T\mathbf{W}^H,\end{aligned} \tag{13}$$

using

$$\mathbf{W}^H\mathbf{W} = (\tilde{\mathbf{A}}^T\mathbf{R}_{\tilde{y}}^{-T}\tilde{\mathbf{A}}^*)\otimes(\tilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1}\tilde{\mathbf{A}}) \stackrel{\text{def}}{=} \mathbf{U}\otimes\mathbf{U},$$

deduced from (2), where

$$\mathbf{U} \stackrel{\text{def}}{=} \tilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1}\tilde{\mathbf{A}} \tag{14}$$

is an $K\times K$ real symmetric non-singular matrix. Then it follows from [20, Theorem 7.38], and some simple algebraic manipulations using [20, Theorem 7.37, rel.(c)] and [20, Theorem 7.34, rel.(d)], that (13) becomes

$$\boldsymbol{\Pi}_{\mathbf{V}}^{\perp} = \mathbf{I} - \mathbf{W}(\mathbf{U}^{-1}\otimes\mathbf{U}^{-1})\mathbf{W}^H. \tag{15}$$

Now let us prove that $\mathbf{u}^H\boldsymbol{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_k = 0$.

Using the formula (1), we get from (11)

$$\begin{aligned}\mathbf{W}^H\mathbf{g}_k &= (\tilde{\mathbf{A}}^T\mathbf{R}_{\tilde{y}}^{-T/2}\otimes\tilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1/2})\mathrm{vec}(\mathbf{Z}_k^H + \mathbf{Z}_k)\\ &= \mathrm{vec}(\tilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1/2}\mathbf{Z}_k^H\mathbf{R}_{\tilde{y}}^{-1/2}\tilde{\mathbf{A}}) + \tilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1/2}\mathbf{Z}_k\mathbf{R}_{\tilde{y}}^{-1/2}\tilde{\mathbf{A}})\\ &= \mathrm{vec}(\mathbf{b}_k\mathbf{c}_k^T + \mathbf{c}_k\mathbf{b}_k^T) \stackrel{\text{def}}{=} \mathrm{vec}(\mathbf{H}_k),\end{aligned} \tag{16}$$

where $\mathbf{b}_k$ and $\mathbf{c}_k$ are the $K\times 1$ real-valued vectors given by

$$\mathbf{b}_k^T \stackrel{\text{def}}{=} \tilde{\mathbf{a}}_k^{'H}\mathbf{R}_{\tilde{y}}^{-1}\tilde{\mathbf{A}} \;\text{ and }\; \mathbf{c}_k \stackrel{\text{def}}{=} \mathbf{U}\mathbf{r}_{s,k}. \tag{17}$$

From (10), (13) and (16) we obtain

$$\begin{aligned}
\mathbf{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_k &= \mathbf{g}_k - \mathbf{W}(\mathbf{U}^{-1}\otimes\mathbf{U}^{-1})\mathbf{W}^H\mathbf{g}_k \\
&= \mathbf{g}_k - \mathbf{W}(\mathbf{U}^{-1}\otimes\mathbf{U}^{-1})\mathrm{vec}(\mathbf{H}_k) \\
&= \mathbf{g}_k - \mathrm{vec}(\mathbf{R}_{\tilde{y}}^{-1/2}\tilde{\mathbf{A}}\mathbf{U}^{-1}\mathbf{H}_k\mathbf{U}^{-1}\tilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1/2}) \\
&= \mathbf{g}_k - \mathrm{vec}(\mathbf{R}_{\tilde{y}}^{-1/2}\tilde{\mathbf{A}}\mathbf{U}^{-1}(\mathbf{b}_k\mathbf{c}_k^T+\mathbf{c}_k\mathbf{b}_k^T)\mathbf{U}^{-1}\tilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1/2}) \\
&= \mathbf{g}_k - \mathrm{vec}(\mathbf{R}_{\tilde{y}}^{-1/2}\tilde{\mathbf{A}}\mathbf{U}^{-1}\mathbf{b}_k\mathbf{c}_k^T\mathbf{U}^{-1}\tilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1/2} + \mathbf{R}_{\tilde{y}}^{-1/2}\tilde{\mathbf{A}}\mathbf{U}^{-1}\mathbf{c}_k\mathbf{b}_k^T\mathbf{U}^{-1}\tilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1/2}). \qquad (18)
\end{aligned}$$

To simplify the expression (18), we need the following equality (19)

$$\begin{aligned}
\tilde{\mathbf{A}}\mathbf{U}^{-1}\tilde{\mathbf{A}}^H &= \tilde{\mathbf{A}}(\tilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1}\tilde{\mathbf{A}})^{-1}\tilde{\mathbf{A}}^H \\
&= \tilde{\mathbf{A}}(\tilde{\mathbf{A}}^H\tilde{\mathbf{A}})^{-1}(\tilde{\mathbf{A}}^H\tilde{\mathbf{A}}\mathbf{R}_s+\sigma_n^2\mathbf{I})\tilde{\mathbf{A}}^H \\
&= \tilde{\mathbf{A}}\mathbf{R}_s\tilde{\mathbf{A}}^H + \sigma_n^2\mathbf{\Pi}_{\tilde{\mathbf{A}}} \\
&= \mathbf{\Pi}_{\tilde{\mathbf{A}}}\mathbf{R}_{\tilde{y}}, \qquad (19)
\end{aligned}$$

where $\mathbf{\Pi}_{\tilde{\mathbf{A}}} \stackrel{\mathrm{def}}{=} \tilde{\mathbf{A}}(\tilde{\mathbf{A}}^H\tilde{\mathbf{A}})^{-1}\tilde{\mathbf{A}}^H$. Using $\mathbf{b}_k^T\mathbf{U}^{-1}\tilde{\mathbf{A}}^H = \tilde{\mathbf{a}}_k^{\prime H}\mathbf{\Pi}_{\tilde{\mathbf{A}}}$ and $\mathbf{U}^{-1}\mathbf{c}_k = \mathbf{r}_{s,k}$ deduced from (19) and (17), (18) can be simplified as

$$\begin{aligned}
\mathbf{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_k &= \mathbf{g}_k - \mathrm{vec}(\mathbf{R}_{\tilde{y}}^{-1/2}\tilde{\mathbf{A}}\mathbf{r}_{s,k}\tilde{\mathbf{a}}_k^{\prime H}\mathbf{\Pi}_{\tilde{\mathbf{A}}}\mathbf{R}_{\tilde{y}}^{-1/2} + \mathbf{R}_{\tilde{y}}^{-1/2}\mathbf{\Pi}_{\tilde{\mathbf{A}}}\tilde{\mathbf{a}}_k^{\prime}\mathbf{r}_{s,k}^T\tilde{\mathbf{A}}^H\mathbf{R}_{\tilde{y}}^{-1/2}) \\
&= \mathbf{g}_k - \mathrm{vec}(\mathbf{Y}_k+\mathbf{Y}_k^H) = \mathrm{vec}(\mathbf{Z}_k-\mathbf{Y}_k+\mathbf{Z}_k^H-\mathbf{Y}_k^H) \qquad (20)
\end{aligned}$$

where $\mathbf{Y}_k \stackrel{\mathrm{def}}{=} \mathbf{R}_{\tilde{y}}^{-1/2}\tilde{\mathbf{A}}\mathbf{r}_{s,k}\tilde{\mathbf{a}}_k^{\prime H}\mathbf{\Pi}_{\tilde{\mathbf{A}}}\mathbf{R}_{\tilde{y}}^{-1/2}$. From (20) and (8) together with the identity (3), we get

$$\begin{aligned}
\mathbf{u}^H\mathbf{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_k &= (\mathrm{vec}(\mathbf{R}_{\tilde{y}}^{-1}))^H\mathrm{vec}(\mathbf{Z}_k-\mathbf{Y}_k+\mathbf{Z}_k^H-\mathbf{Y}_k^H) \\
&= \mathrm{Tr}(\mathbf{R}_{\tilde{y}}^{-1}(\mathbf{Z}_k-\mathbf{Y}_k+\mathbf{Z}_k^H-\mathbf{Y}_k^H)) \\
&= \mathrm{Tr}(\mathbf{R}_{\tilde{y}}^{-1}(\mathbf{Z}_k-\mathbf{Y}_k)) + \mathrm{Tr}((\mathbf{Z}_k^H-\mathbf{Y}_k^H)\mathbf{R}_{\tilde{y}}^{-1}) \\
&\stackrel{\mathrm{def}}{=} \mathrm{Tr}(\mathbf{F}_k) + \mathrm{Tr}(\mathbf{F}_k^H). \qquad (21)
\end{aligned}$$

Let us now prove that

$$\mathrm{Tr}(\mathbf{F}_k) = 0.$$

After replacing $\mathbf{Z}_k$ and $\mathbf{Y}_k$ by their expression, we obtain

$$\mathbf{Z}_k - \mathbf{Y}_k = \mathbf{R}_{\tilde{y}}^{-1/2}\tilde{\mathbf{A}}\mathbf{r}_{s,k}\tilde{\mathbf{a}}_k^{\prime H}\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\mathbf{R}_{\tilde{y}}^{-1/2}. \qquad (22)$$

Thus

$$\mathrm{Tr}(\mathbf{F}_k) = \mathrm{Tr}(\mathbf{r}_{s,k}\tilde{\mathbf{a}}_k^{\prime H}\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\mathbf{R}_{\tilde{y}}^{-2}\tilde{\mathbf{A}}). \qquad (23)$$

Since

$$\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\mathbf{R}_{\tilde{y}} = \sigma_n^2\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp} \text{ or equivalently } \mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\mathbf{R}_{\tilde{y}}^{-1} = \frac{1}{\sigma_n^2}\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}, \qquad (24)$$

we get

$$\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\mathbf{R}_{\tilde{y}}^{-2}\tilde{\mathbf{A}} = \left(\frac{1}{\sigma_n^2}\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\right)\left(\mathbf{R}_{\tilde{y}}^{-1}\tilde{\mathbf{A}}\right) = \frac{1}{\sigma_n^4}\mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}\tilde{\mathbf{A}} = \mathbf{O}, \qquad (25)$$

and thus from (23), we get $\mathrm{Tr}(\mathbf{F}_k)=0$. It follows then from (21) that $\mathbf{u}^H\mathbf{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_k = 0$. ■

This identity, together with (10) and (20) allows us to simplify (9) as

$$\begin{aligned}
\frac{2}{T}\left[\mathrm{CRB}_{\mathrm{sto}}^{-1}(\boldsymbol{\omega})\right]_{k,l} &= \mathbf{g}_k^H\mathbf{\Pi}_{\mathbf{V}}^{\perp}\mathbf{g}_l \\
&= (\mathrm{vec}(\mathbf{Z}_k^H+\mathbf{Z}_k))^H\mathrm{vec}(\mathbf{Z}_l-\mathbf{Y}_l+\mathbf{Z}_l^H-\mathbf{Y}_l^H) \\
&= 2\mathrm{Re}(\mathrm{Tr}((\mathbf{Z}_k+\mathbf{Z}_k^H)(\mathbf{Z}_l-\mathbf{Y}_l))). \qquad (26)
\end{aligned}$$

Note from (11) and (22) that

$$\mathrm{Tr}(\mathbf{Z}_k^H(\mathbf{Z}_l-\mathbf{Y}_l)) = (\tilde{\mathbf{a}}_l'^H \mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp} \mathbf{R}_{\tilde{y}}^{-1} \tilde{\mathbf{a}}_k'^H)(\mathbf{r}_{s,k}^T \tilde{\mathbf{A}}^H \mathbf{R}_{\tilde{y}}^{-1} \tilde{\mathbf{A}} \mathbf{r}_{s,l}).$$

Using (24), we get

$$\mathrm{Tr}(\mathbf{Z}_k^H(\mathbf{Z}_l-\mathbf{Y}_l)) = \frac{1}{\sigma_n^2}(\tilde{\mathbf{a}}_l'^H \mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp} \tilde{\mathbf{a}}_k'^H)(\mathbf{r}_{s,k}^T \tilde{\mathbf{A}}^H \mathbf{R}_{\tilde{y}}^{-1} \tilde{\mathbf{A}} \mathbf{r}_{s,l}), \tag{27}$$

and

$$\begin{aligned}\mathrm{Tr}(\mathbf{Z}_k(\mathbf{Z}_l-\mathbf{Y}_l)) &= (\tilde{\mathbf{a}}_k'^H \mathbf{R}_{\tilde{y}}^{-1} \tilde{\mathbf{A}} \mathbf{r}_{s,l})(\tilde{\mathbf{a}}_l'^H \mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp} \mathbf{R}_{\tilde{y}}^{-1} \tilde{\mathbf{A}} \mathbf{r}_{s,k}) \\ &= \frac{1}{\sigma_n^2}(\tilde{\mathbf{a}}_k'^H \mathbf{R}_{\tilde{y}}^{-1} \tilde{\mathbf{A}} \mathbf{r}_{s,l})(\tilde{\mathbf{a}}_l'^H \mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp} \tilde{\mathbf{A}} \mathbf{r}_{s,k}) = 0.\end{aligned} \tag{28}$$

It follows then from (27) and (28) that (26) can be simplified as

$$\left[\mathrm{CRB}_{\mathrm{sto}}^{-1}(\boldsymbol{\omega})\right]_{k,l} = \frac{T}{\sigma_n^2}\mathrm{Re}\Big((\tilde{\mathbf{a}}_k'^H \mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp} \tilde{\mathbf{a}}_l')(\mathbf{r}_{s,k}^T \tilde{\mathbf{A}}^H \mathbf{R}_{\tilde{y}}^{-1} \tilde{\mathbf{A}} \mathbf{r}_{s,l})\Big). \tag{29}$$

Finally, writing (29) in matrix form, theorem 1 is proved.

III. Proof of theorem 2

For coherent sources for which $\mathbf{R}_s = \mathbf{c}\mathbf{c}^T$ and $\boldsymbol{\rho} = \mathbf{c}$, we follow the steps similar to those in the proof of theorem 1. First, we note that the k-th columns of $\mathbf{G}$ are still given by (10), but with now

$$\mathbf{Z}_k = c_k \mathbf{R}_{\tilde{y}}^{-1/2} \tilde{\mathbf{A}} \mathbf{c} \tilde{\mathbf{a}}_k'^H \mathbf{R}_{\tilde{y}}^{-1/2}. \tag{30}$$

Second, $\mathrm{vec}(\mathbf{R}_s) = \mathbf{c} \otimes \mathbf{c}$ implies that

$$\frac{\partial \mathrm{vec}(\mathbf{R}_s)}{\partial \mathbf{c}^T} = \mathbf{c}\otimes\mathbf{I} + \mathbf{I}\otimes\mathbf{c} = 2\mathbf{N}_K(\mathbf{c}\otimes\mathbf{I}), \tag{31}$$

where $\mathbf{N}_K$ is the $K \times K$ matrix defined in [20, Theorem 7.34]. Consequently (12) becomes

$$\mathbf{V} = 2\mathbf{W}\mathbf{N}_K(\mathbf{c}\otimes\mathbf{I}), \tag{32}$$

which gives after some algebraic manipulation using [20, Theorem 7.34, rel.(d)]:

$$\mathbf{\Pi}_{\mathbf{V}}^{\perp} \stackrel{\mathrm{def}}{=} \mathbf{I} - \mathbf{V}(\mathbf{V}^H\mathbf{V})^{-1}\mathbf{V}^H = \mathbf{I} - \mathbf{V}_1 \bar{\mathbf{V}}^{-1} \mathbf{V}_1^H, \tag{33}$$

with $\mathbf{V}_1 \stackrel{\mathrm{def}}{=} \mathbf{W}\mathbf{N}_K(\mathbf{c}\otimes\mathbf{I})$ and $\bar{\mathbf{V}} \stackrel{\mathrm{def}}{=} (\mathbf{c}^T\otimes\mathbf{I})\mathbf{N}_K(\mathbf{U}\otimes\mathbf{U})\mathbf{N}_K(\mathbf{c}\otimes\mathbf{I})$ where $\mathbf{U}$ is defined by (14). $\bar{\mathbf{V}}$ can be simplified as

$$\begin{aligned}\bar{\mathbf{V}} &= (\mathbf{c}^T\otimes\mathbf{I})\mathbf{N}_K(\mathbf{U}\otimes\mathbf{U})(\mathbf{c}\otimes\mathbf{I}) \\ &= (\mathbf{c}^T\otimes\mathbf{I})\mathbf{N}_K(\mathbf{U}\mathbf{c}\otimes\mathbf{U}) \\ &= \frac{1}{2}\left(\kappa_c\mathbf{U} + \mathbf{U}\mathbf{c}\mathbf{c}^T\mathbf{U}^T\right),\end{aligned} \tag{34}$$

where the first equality follows from [20, Theorem 7.35, rel.(a)] and the third equality follows from [20, Theorem 7.31, rel.(d)] using the definition of $\mathbf{N}_K$ [20, Theorem 7.34] and $\kappa_c \stackrel{\mathrm{def}}{=} \mathbf{c}^T\mathbf{U}\mathbf{c}$. The inverse $\bar{\mathbf{V}}^{-1}$ is deduced from the matrix inversion lemma applied to (34)

$$\bar{\mathbf{V}}^{-1} = \frac{2}{\kappa_c}\left(\mathbf{U}^{-1} - \frac{1}{2\kappa_c}\mathbf{c}\mathbf{c}^T\right). \tag{35}$$

Now let us prove that $\mathbf{u}^H \mathbf{\Pi}_{\mathbf{V}}^{\perp} \mathbf{g}_k = 0$.

Using $\bar{\mathbf{U}} \stackrel{\text{def}}{=} \tilde{\mathbf{A}}^H \mathbf{R}_{\tilde{y}}^{-2} \tilde{\mathbf{A}}$ as a real-valued symmetric matrix and the identity (1), we get

$$\begin{aligned} \mathbf{u}^H \mathbf{V}_1 &= (\text{vec}(\mathbf{R}_{\tilde{y}}^{-1}))^H (\mathbf{R}_{\tilde{y}}^{-T/2} \tilde{\mathbf{A}}^* \otimes \mathbf{R}_{\tilde{y}}^{-1/2} \tilde{\mathbf{A}}) \mathbf{N}_K (\mathbf{c} \otimes \mathbf{I}) \\ &= (\text{vec}(\bar{\mathbf{U}}))^T (\mathbf{c} \otimes \mathbf{I}) \\ &= \mathbf{c}^T \bar{\mathbf{U}}, \end{aligned} \tag{36}$$

where the second equality follows from [20, Theorem 7.34, rel.(c)] and the third equality uses (1). Furthermore:

$$\begin{aligned} \mathbf{V}_1^H \mathbf{g}_k &= (\mathbf{c}^T \otimes \mathbf{I}) \mathbf{N}_K^T \mathbf{W}^H \mathbf{g}_k \\ &= c_k (\mathbf{c}^T \otimes \mathbf{I}) \mathbf{N}_K^T \text{vec}(\mathbf{b}_k \mathbf{c}^T \mathbf{U}^T + \mathbf{U} \mathbf{c} \mathbf{b}_k^T) \\ &= c_k (\mathbf{c}^T \otimes \mathbf{I}) \text{vec}(\mathbf{b}_k \mathbf{c}^T \mathbf{U}^T + \mathbf{U} \mathbf{c} \mathbf{b}_k^T) \\ &= c_k \left(\kappa_c \mathbf{b}_k + (\mathbf{b}_k^T \mathbf{c}) \mathbf{U} \mathbf{c} \right), \end{aligned} \tag{37}$$

where the second equality follows from $\mathbf{W}^H \mathbf{g}_k = c_k \text{vec}(\mathbf{b}_k \mathbf{c}_k^T + \mathbf{c}_k \mathbf{b}_k^T)$ deduced from (16) with $\mathbf{c}_k$ defined in (17) is now given be $c_k \mathbf{U} \mathbf{c}$, and the third equality follows from [20, Theorem 7.34, rel.(c)] and the property that $\mathbf{b}_k \mathbf{c}^T \mathbf{U}^T + \mathbf{U} \mathbf{c} \mathbf{b}_k^T$ is a real-valued symmetric matrix. In similar way, we have

$$\mathbf{u}^H \mathbf{g}_k = 2 c_k \tilde{\mathbf{a}}_k'^H \mathbf{R}_{\tilde{y}}^{-2} \tilde{\mathbf{A}} \mathbf{c}. \tag{38}$$

From (35) and (37), we get

$$\bar{\mathbf{V}}^{-1} \mathbf{V}_1^H \mathbf{g}_k = 2 c_k \mathbf{U}^{-1} \mathbf{b}_k. \tag{39}$$

It follows from (33), (39), (36) and (38) that

$$\begin{aligned} \mathbf{u}^H \mathbf{\Pi}_{\mathbf{V}}^{\perp} \mathbf{g}_k &= \mathbf{u}^H \mathbf{g}_k - \mathbf{u}^H \mathbf{V}_1 \bar{\mathbf{V}}^{-1} \mathbf{V}_1^H \mathbf{g}_k \\ &= 2 c_k \tilde{\mathbf{a}}_k'^H \mathbf{R}_{\tilde{y}}^{-2} \tilde{\mathbf{A}} \mathbf{c} - 2 c_k \mathbf{c}^T \bar{\mathbf{U}} \mathbf{U}^{-1} \mathbf{b}_k \\ &= 2 c_k \tilde{\mathbf{a}}_k'^H \mathbf{R}_{\tilde{y}}^{-2} \tilde{\mathbf{A}} \mathbf{c} - 2 c_k \mathbf{c}^T \tilde{\mathbf{A}}^H \mathbf{R}_{\tilde{y}}^{-2} \tilde{\mathbf{a}}_k' = 0, \end{aligned}$$

where the third equality follows from the identity $\bar{\mathbf{U}} \mathbf{U}^{-1} \tilde{\mathbf{A}}^H = \tilde{\mathbf{A}}^H \mathbf{R}_{\tilde{y}}^{-1}$ obtained using (19) and (25) which is equivalent to $\mathbf{R}_{\tilde{y}}^{-2} \tilde{\mathbf{A}} = \mathbf{\Pi}_{\tilde{\mathbf{A}}} \mathbf{R}_{\tilde{y}}^{-2} \tilde{\mathbf{A}}$. ∎

It follows that the elements of (6) reduce to

$$\frac{2}{T} \left[\text{CRB}_{\text{sto}}^{-1}(\boldsymbol{\omega}) \right]_{k,l} = \mathbf{g}_k^H \mathbf{\Pi}_{\mathbf{V}}^{\perp} \mathbf{g}_l = \mathbf{g}_k^H \mathbf{g}_l - \mathbf{g}_k^H \mathbf{V}_1 \bar{\mathbf{V}}^{-1} \mathbf{V}_1^H \mathbf{g}_l, \tag{40}$$

where we get

$$\begin{aligned} \mathbf{g}_k^H \mathbf{g}_l &= \text{vec}(\mathbf{Z}_k^H + \mathbf{Z}_k)^H \text{vec}(\mathbf{Z}_l^H + \mathbf{Z}_l) \\ &= \text{Tr}[(\mathbf{Z}_k^H + \mathbf{Z}_k)^H (\mathbf{Z}_l^H + \mathbf{Z}_l)] \\ &= 2 c_k c_l \left(\kappa_c \tilde{\mathbf{a}}_k'^H \mathbf{R}_{\tilde{y}}^{-1} \tilde{\mathbf{a}}_l' + (\mathbf{b}_k^T \mathbf{c})(\mathbf{b}_l^T \mathbf{c}) \right), \end{aligned} \tag{41}$$

where the first equality is deduced from the definition (10) of $\mathbf{g}_k$ associated with (30), the second equality follows from the identity (3), and the third equality follows from the definition (17) of $\mathbf{b}_k$ and the property that $\tilde{\mathbf{a}}_k'^H \mathbf{R}_{\tilde{y}}^{-1} \tilde{\mathbf{a}}_l'$ is real-valued. On the other hand, we get

$$\begin{aligned} \mathbf{g}_k^H \mathbf{V}_1 \bar{\mathbf{V}}^{-1} \mathbf{V}_1^H \mathbf{g}_l &= 2 c_k c_l \left(\mathbf{b}_k^T \mathbf{U}^{-1} \right) \left(\kappa_c \mathbf{b}_l + (\mathbf{b}_l^T \mathbf{c}) \mathbf{U} \mathbf{c} \right) \\ &= 2 c_k c_l \left(\kappa_c \tilde{\mathbf{a}}_k'^H \mathbf{R}_{\tilde{y}}^{-1} \mathbf{\Pi}_{\tilde{\mathbf{A}}} \tilde{\mathbf{a}}_l' + (\mathbf{b}_k^T \mathbf{c})(\mathbf{b}_l^T \mathbf{c}) \right), \end{aligned} \tag{42}$$

where the first equality follows from (37) and (39) and the second equality is deduced from (19). Plugging (41) and (42) into (40), we get:

$$\frac{2}{T} \left[\text{CRB}_{\text{sto}}^{-1}(\boldsymbol{\omega}) \right]_{k,l} = \frac{2 \kappa_c}{\sigma_n^2} c_k c_l (\tilde{\mathbf{a}}_k'^H \mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp} \tilde{\mathbf{a}}_l'),$$

using $\mathbf{R}_{\tilde{y}}^{-1} \mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp} = \frac{1}{\sigma_n^2} \mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp}$. Finally, writing (40) in matrix form, theorem 2 is proved.

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