Asymptotic Generalized Eigenvalue Distribution of Block Multilevel Toeplitz Matrices

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***Abstract*—In many detection and estimation problems associated with processing of second-order stationary random processes, the observation data are the sum of two zero-mean second-order stationary processes: the process of interest and the noise process. In particular, the main performance criterion is the signal-to-noise ratio (SNR). After linear filtering, the optimal SNR corresponds to the maximal value of a Rayleigh quotient which can be interpreted as the largest generalized eigenvalue of the covariance matrices associated with the signal and noise processes, which are block multilevel Toeplitz structured for m-dimensional vector-valued second-order stationary p-dimensional random processes $\mathbf{x}_{i_1,i_2,\ldots,i_p} \in \mathbb{R}^m$. In this paper, an extension of Szegö's theorem to the generalized eigenvalues of Hermitian block multilevel Toeplitz matrices is given, providing information about the asymptotic distribution of those generalized eigenvalues and in particular of the optimal SNR after linear filtering. A simple proof of this theorem, under the hypothesis of absolutely summable elements is given. The proof is based on the notion of multilevel asymptotic equivalence between block multilevel matrix sequences derived from the celebrated Gray approach. Finally, a short example in wideband space–time beamforming is given to illustrate this theorem.**

***Index Terms*—Asymptotic distribution, block multilevel Toeplitz matrix, generalized eigenvalues, multidimensional second-order vector valued stationary random process, Szegö's theorem.**

I. Introduction

Multidimensional discrete random processes appear in many signal processing applications. In imaging problems, observation data are often modeled by 2-D discrete random processes. However, many examples can also be found where data are 3-D such as, for instance, hyperspectral imaging (x-spatial dimension $\times$ y-spatial dimension $\times$ wavelength; see, e.g., [1]) or interferometric synthetic aperture radar (IF-SAR) imaging (x-spatial dimension $\times$ y-spatial dimension $\times$ elevation; see, e.g., [2] and [3]) for which multidimensional processing is used (see, e.g., [4]–[6]). Furthermore, in some applications, the data can be vector-valued (e.g., in array processing). In those applications, if the random process is second-order stationary, its covariance matrix will be structured. More specifically, consider a m-dimensional vector-valued second-order stationary p-dimensional random process $\mathbf{x}_{i_1,i_2,\ldots,i_p} \in \mathbb{R}^m$ and a block of $n_1 \times n_2 \times \cdots \times n_p$ samples. If the data are grouped into a vector by stacking the data of the different dimensions in alphabetical order, the vector's covariance matrix will have an m block p-level Toeplitz structure (e.g., see [7] in the scalar case).

In many detection and estimation problems associated with processing of such p-dimensional random processes, the data are the sum of two zero-mean second-order stationary processes: the process of interest and the noise process. In this case, the main performance criterion is the signal-to-noise ratio (SNR), or the signal-to-interference-plus-noise ratio (SINR), in the presence of interference processes and background noise. After linear filtering, the optimal SNR corresponds to the maximal value of a Rayleigh quotient, which can be interpreted as the largest generalized eigenvalue of the covariance matrices associated with the signal and noise processes.

In this paper, we study the problem of the influence of the size of the samples vector along with the different dimensions, on the generalized eigenvalues of m block p-level Toeplitz matrices. More specifically, we assume that the number of samples along with the different dimensions $n_1, n_2, \ldots, n_p$ tend to infinity with p and m fixed and analyze the asymptotic generalized eigenvalue distribution of such matrices.

The problem of the asymptotic eigenvalue distribution of Toeplitz matrices was first analyzed by Grenander and Szegö, whose famous result asserts that the eigenvalues of a sequence of Hermitian Toeplitz matrices asymptotically behave like the samples of the Fourier transform of its entries [8]. However, this analysis has been performed by use of sophisticated mathematics under the general hypothesis that the Toeplitz matrix is generated by a measurable and bounded spectrum. Then, Gray has proposed a simpler proof of this result for banded Toeplitz matrices [9] (called finite-order Toeplitz matrices) and then for infinite-order Toeplitz matrices, under the assumption of absolutely summable elements, based on the asymptotic equivalence between two matrices [10]. Following this approach and under the same assumption, the Grenander and Szegö result has been later extended to the eigenvalues of block Toeplitz with Toeplitz block matrices where both the size and the number of blocks tend to infinity [11], then to the eigenvalues of block Toeplitz where only the size of blocks tends to infinity in [12]. Let us note that extension of the celebrated Grenander and Szegö result has been also extensively studied in the mathematical literature under the general assumption that the involved Toeplitz, block Toeplitz, multilevel Toeplitz matrices are generated by integrable spectra (e.g., see [14] and [15]) for the asymptotic eigenvalue and generalized eigenvalue distribution. However, the highly sophisticated mathematical tools employed in these papers are sometimes beyond the grasp of the engineering community. Consequently, the results obtained in these papers have not received appreciation they deserve in the signal processing literature. In the present work, we propose an extension of Szegö's theorem to the generalized eigenvalues of m block p-level Toeplitz matrices, under the hypothesis of absolutely summable elements, which relies on an extension of the notion of asymptotic equivalence between matrix sequences established by Gray in [9]. An example of application of this theorem to the wideband space–time beamforming has been given in [16].

This paper is organized as follows. In Section II, we give an interpretation of the generalized eigenvalues of Hermitian matrices from the point of view of SNR after linear filtering. Then, in Section III, the notation of m block p-level Toeplitz matrices and m block p-level circulant matrices are introduced and preliminary results about asymptotic equivalence between multilevel matrix sequences are given. Then, a generalized eigenvalue distribution theorem of m block p-level Toeplitz matrices which is our main result is proven in Section IV. Finally, a short example is given in Section V to illustrate this theorem.

II. SNR After Linear Filtering of Multidimensional Processes

In this section, we give an interpretation of the generalized eigenvalues of two Hermitian matrices. Thus, let $\mathbf{x}$ be a data vector where the vector-valued p-dimensional data are arranged in alphabetical order of dimension $m \times n_1 \times n_2 \times \cdots \times n_p$ composed by the sum of $\mathbf{s}$, the

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signal of interest and $\mathbf{n}$, the noise process. Both processes are modelled by zero-mean second-order stationary processes uncorrelated with each other with $mn_1n_2\dots n_p \times mn_1n_2\dots n_p$ Hermitian covariance matrices $\mathbf{R}_s = E(\mathbf{s}\mathbf{s}^H)$ and $\mathbf{R}_n = E(\mathbf{n}\mathbf{n}^H)$, respectively. Since the processes are assumed to be second-order stationary, the covariance matrices are m block p-level Toeplitz structured. Moreover, since $\mathbf{n}$ is composed of background noise and possibly interference uncorrelated which each other, $\mathbf{R}_n$ is not only positive semi definite but positive definite. After linear filtering with $m \times n_1 \times n_2 \times \dots \times n_p$-dimensional filter $\mathbf{w}$ of the data $\mathbf{x}$, the SNR is given by the Rayleigh quotient

$$\mathrm{SNR} = r(\mathbf{w}) = \frac{\mathbf{w}^H\mathbf{R}_s\mathbf{w}}{\mathbf{w}^H\mathbf{R}_n\mathbf{w}}. \qquad (1)$$

This ratio is closely related to the generalized eigenvalue problem. Indeed, by taking the complex gradient of (1) w.r.t. $\mathbf{w}$ and setting the result to zero, we obtain: $\mathbf{R}_s\mathbf{w} = r(\mathbf{w})\mathbf{R}_n\mathbf{w}$. This is the expression of a generalized eigenproblem. Therefore, the stationary points $\mathbf{w}$ and stationary values $r(\mathbf{w})$ of the Rayleigh quotient (1) are, respectively, the generalized eigenvectors and eigenvalues of $(\mathbf{R}_s, \mathbf{R}_n)$ denoted $\lambda_k(\mathbf{R}_s, \mathbf{R}_n)$ which are real-valued and positive. In particular, the maximum of this SNR is given by the maximum generalized eigenvalue of $(\mathbf{R}_s, \mathbf{R}_n)$. Moreover, since generally $\mathbf{R}_n$ is positive definite, these generalized eigenvalues are given by the eigenvalues[1] of $\mathbf{R}_n^{-1}\mathbf{R}_s$ that are denoted by $\lambda_k(\mathbf{R}_s, \mathbf{R}_n) = \lambda_k(\mathbf{R}_n^{-1}\mathbf{R}_s)$.

III. Notations And Preliminary Results

We first formalize the definition of the structured m block p-level Toeplitz and m block p-level circulant matrices and extend the asymptotic equivalence of matrix sequences to multilevel matrix sequences. Then, we give some preliminary lemmas necessary to prove our main result in Section IV.

Definition 1 (m Block p-Level Toeplitz Matrix): Given a multi-index $\mathbf{n}_p = (n_1, n_2, \dots, n_p)$, an m block p-level Toeplitz matrix $\mathbf{A}_{\mathbf{n}_p,m}$ is defined recursively as follows. If $p = 1$ and $m = 1$, then it is a customary Toeplitz matrix of order n_1. If $p > 1$ and $m = 1$, then $\mathbf{A}_{\mathbf{n}_p,1}$ can be partitioned into $n_1 \times n_1$ blocks

$$\mathbf{A}_{\mathbf{n}_p,1} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{A}_0^{\mathbf{n}_{p-1}} & \mathbf{A}_1^{\mathbf{n}_{p-1}} & \cdots & \mathbf{A}_{n_1-1}^{\mathbf{n}_{p-1}} \\ \mathbf{A}_{-1}^{\mathbf{n}_{p-1}} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{A}_1^{\mathbf{n}_{p-1}} \\ \mathbf{A}_{-(n_1-1)}^{\mathbf{n}_{p-1}} & \cdots & \mathbf{A}_{-1}^{\mathbf{n}_{p-1}} & \mathbf{A}_0^{\mathbf{n}_{p-1}} \end{bmatrix}$$

and each block $\mathbf{A}_{q_1}^{\mathbf{n}_{p-1}}$, $q_1 = -(n_1-1), \dots, 0, \dots, +(n_1-1)$ is a $(p-1)$-level Toeplitz matrix of multi-index $\mathbf{n}_{p-1} = (n_2, \dots, n_p)$. In other words, $\mathbf{A}_{\mathbf{n}_p,1}$ has an outermost block Toeplitz structure of order n_1, each block is itself a block Toeplitz matrix of order n_2, and so on, down to the innermost block level, made of ordinary Toeplitz matrices of order n_p

$$\begin{bmatrix} a_{\mathbf{q}_{p-1},0} & a_{\mathbf{q}_{p-1},1} & \cdots & a_{\mathbf{q}_{p-1},n_p-1} \\ a_{\mathbf{q}_{p-1},-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{\mathbf{q}_{p-1},1} \\ a_{\mathbf{q}_{p-1},-(n_p-1)} & \cdots & a_{\mathbf{q}_{p-1},-1} & a_{\mathbf{q}_{p-1},0} \end{bmatrix}$$

where $a_{\mathbf{q}_{p-1},q_p} = a_{q_1,q_2,\dots,q_p}$ with $q_i = -(n_i-1), \dots, 0, +(n_i-1)$, $i = 1, \dots, p$, is the general term of $\mathbf{A}_{\mathbf{n}_p,1}$. Finally, for p and m arbitrary, $\mathbf{A}_{\mathbf{n}_p,m}$ is an $m \times m$ block matrix constituted by p-level Toeplitz matrices previously defined. The general term of $\mathbf{A}_{\mathbf{n}_p,m}$ is denoted $a_{q_1,q_2,\dots,q_p}^{m_1,m_2}$ with $m_1, m_2 = 1, 2, \dots, m$.

The approach of this paper is to relate the generalized eigenvalues of m block p-level Toeplitz matrices to those of simpler structured associated m block p-level circulant matrices that we now formalize.

Definition 2 (m Block p-Level Circulant Matrix): Given a multi-index $\mathbf{n}_p = (n_1, n_2, \dots, n_p)$, an m block p-level circulant matrix is defined recursively as the m block p-level Toeplitz matrix previously defined, where Toeplitz blocks are replaced by circulant blocks.

The eigenvalue decomposition (EVD) of p-level circulant matrices is derived in the same way as those of circulant matrices. This EVD proven in [7, Th. 5.84] in the case of circulant matrices of level 3 extends easily to circulant matrices of arbitrary level p. More precisely, an $n_1n_2\dots n_p \times n_1n_2\dots n_p$ p-level circulant matrix $\mathbf{C}_{\mathbf{n}_p,1}$ is diagonalizable by the unitary matrix

$$\mathbf{U}_{\mathbf{n}_p} = \mathbf{U}_{n_p} \otimes \cdots \otimes \mathbf{U}_{n_1} \qquad (2)$$

where $(\mathbf{U}_{n_i})_{i=1\dots p}$ are the $n_i \times n_i$ unitary discrete Fourier transform (DFT) matrices of terms $(\mathbf{U}_{n_i})_{k,l} = 1/\sqrt{n_i}e^{-j2\pi(k-1)(l-1)/n_i}$ where the associated eigenvalues are the p-dimensional DFT of its first row

$$\lambda_{k_1,\dots,k_p} = \sum_{q_1=0}^{n_1-1}\sum_{q_2=0}^{n_2-1}\cdots\sum_{q_p=0}^{n_p-1} c_{q_1,\dots,q_p} \times e^{-j2\pi\left(q_1k_1/n_1+q_2k_2/n_2+\cdots+q_pk_p/n_p\right)}.$$

Now, we define multilevel asymptotic equivalence that extends the asymptotic equivalence introduced by Gray [9] for $p = 1$.

Definition 3 (Multilevel Asymptotic Equivalence): Let $\{\mathbf{A}_{\mathbf{n}_p,m}\}_{\mathbf{n}_p}$ and $\{\mathbf{B}_{\mathbf{n}_p,m}\}_{\mathbf{n}_p}$ be sequences of $mn_1n_2\dots n_p$th square matrices where m is fixed. These matrices are said to be multilevel asymptotically equivalent[2] and noted $\mathbf{A}_{\mathbf{n}_p,m} \sim \mathbf{B}_{\mathbf{n}_p,m}$ if the following conditions hold:

1) $\|\mathbf{A}_{\mathbf{n}_p,m}\| \le M < \infty$ and $\|\mathbf{B}_{\mathbf{n}_p,m}\| \le M < \infty$
2) $\lim_{n_1,n_2,\dots,n_p\to\infty} |\mathbf{A}_{\mathbf{n}_p,m} - \mathbf{B}_{\mathbf{n}_p,m}| = 0$

where $\|.\|$ is the spectral norm and $|.|$ is a normalized Frobenius norm defined by

$$|\mathbf{A}_{\mathbf{n}_p,m}|^2 = \frac{1}{mn_1n_2\dots n_p}\sum_{m_1=1}^{m}\sum_{m_2=1}^{m}\sum_{k_1=1}^{n_1}\sum_{l_1=1}^{n_1}\sum_{k_2=1}^{n_2}\sum_{l_2=1}^{n_1}\cdots \sum_{k_p=1}^{n_p}\sum_{l_p=1}^{n_p}\left|a_{k_1,l_1,k_2,l_2,\dots,k_p,l_p}^{m_1,m_2}\right|^2$$

where $a_{k_1,l_1,k_2,l_2,\dots,k_p,l_p}^{m_1,m_2}$ denotes the generic term of $\{\mathbf{A}_{\mathbf{n}_p,m}\}_{\mathbf{n}_p}$. Then, we prove in the Appendix the following lemma about the asymptotic eigenvalue distribution of multilevel asymptotically equivalent matrices.

Lemma 1: Let $\{\mathbf{A}_{\mathbf{n}_p,m}\}$ and $\{\mathbf{B}_{\mathbf{n}_p,m}\}$ be sequences of multilevel asymptotically equivalent $mn_1n_2\dots n_p \times mn_1n_2\dots n_p$

[1]Note that the generalized eigenvalues of $(\mathbf{R}_s, \mathbf{R}_n)$ are also given in this case by the eigenvalues of the Hermitian matrix $\mathbf{R}_n^{-H/2}\mathbf{R}_s\mathbf{R}_n^{-1/2}$ where $\mathbf{R}_n^{1/2}$ is an arbitrary square root of $\mathbf{R}_n$. But this decomposition does not facilitate the extension of Szegö's theorem.

[2]Note that this relation is symmetric and transitive, but it is not reflexive for arbitrary sequences $\{\mathbf{A}_{\mathbf{n}_p,m}\}_{\mathbf{n}_p}$ in contrast to the set of Hermitian m block p-level Toeplitz sequences where $\left\{a_{i_1,i_2,\dots,i_p}^{m_1,m_2}\right\}_{i_1,i_2,\dots,i_p}$ is absolutely summable for which this relation satisfies the reflexive property.

matrices with eigenvalues $\lambda_k(\mathbf{A}_{\mathbf{n}_p,m})$ and $\lambda_k(\mathbf{B}_{\mathbf{n}_p,m})$ for $k = 1,\ldots,mn_1n_2\ldots n_p$, respectively. Then for any positive integer s

$$\lim_{n_1,n_2,\ldots,n_p\to\infty}\frac{1}{n_1n_2\ldots n_p}\times\sum_{k=1}^{mn_1n_2\ldots n_p}(\lambda_k^s(\mathbf{A}_{\mathbf{n}_p,m})-\lambda_k^s(\mathbf{B}_{\mathbf{n}_p,m}))=0$$

and hence if either limit exists individually

$$\lim_{n_1,n_2,\ldots,n_p\to\infty}\frac{1}{n_1n_2\ldots n_p}\sum_{k=1}^{mn_1n_2\ldots n_p}\lambda_k^s(\mathbf{A}_{\mathbf{n}_p,m})\times\lim_{n_1,n_2,\ldots,n_p\to\infty}\frac{1}{n_1n_2\ldots n_p}\sum_{k=1}^{mn_1n_2\ldots n_p}\lambda_k^s(\mathbf{B}_{\mathbf{n}_p,m}).$$

From now on, we consider Hermitian m block p-level Toeplitz matrices only, for which $a^{m_1,m_2}_{-i_1,-i_2,\ldots,-i_p}=(a^{m_2,m_1}_{i_1,i_2,\ldots,i_p})^*$ is equivalent to a real-valued $m\times m$ matrix Fourier transform

$$\{\mathbf{A}(\omega_1,\omega_2,\ldots,\omega_p)\}_{m_1,m_2}=\sum_{i_1,i_2,\ldots,i_p}a^{m_1,m_2}_{i_1,i_2,\ldots,i_p}e^{-j\sum_{k=1}^{p}\omega_k i_k}$$

whose existence is guaranteed if the sequence $\{a^{m_1,m_2}_{i_1,i_2,\ldots,i_p}\}_{i_1,i_2,\ldots,i_p}$ is absolutely summable.

To construct a sequence $\{\mathbf{C}_{\mathbf{n}_p,m}(a)\}_{\mathbf{n}_p}$ of m block p-level circulant matrices that are multilevel asymptotically equivalent to $\{\mathbf{A}_{\mathbf{n}_p,m}\}_{\mathbf{n}_p}$, we begin by defining for each couple (m_1,m_2) where $m_1,m_2=1,\ldots,m$, the sequence [see (3) shown at the bottom of the page] and $\{\mathbf{C}^{(m_1,m_2)}_{\mathbf{n}_p}(a)\}_{\mathbf{n}_p}$, the sequence of p-level circulant matrices[3] induced by $\left\{c^{(m_1,m_2)}_{q_1,q_2,\ldots,q_p}(a)\right\}_{q_1,q_2,\ldots,q_p}$. We note that $\mathbf{C}^{(m_1,m_2)}_{\mathbf{n}_p}(a)$ may be written more compactly as

$$\mathbf{C}^{(m_1,m_2)}_{\mathbf{n}_p}(a)=\mathbf{U}^H_{\mathbf{n}_p}\boldsymbol{\Delta}^{m_1,m_2}_{\mathbf{n}_p}(a)\mathbf{U}_{\mathbf{n}_p}\tag{4}$$

where $\mathbf{U}_{\mathbf{n}_p}$ is defined as in (2) and $\boldsymbol{\Delta}^{m_1,m_2}_{\mathbf{n}_p}(a)$ is the $n_1n_2\ldots n_p\times n_1n_2\ldots n_p$ diagonal matrix of elements $\{\mathbf{A}(2\pi i_1/n_1,2\pi i_2/n_2,\ldots,2\pi i_p/n_p)\}_{m_1,m_2}$, arranged in alphabetical order. Finally, the sequence $\{\mathbf{C}_{\mathbf{n}_p,m}(a)\}_{\mathbf{n}_p}$ of m block p-level Circulant matrices is built from the $m\times m$ blocks $\mathbf{C}^{\mathbf{n}_p,(m_1,m_2)}(a)$. Consequently[4]

$$\mathbf{C}_{\mathbf{n}_p,m}(a)=\mathbf{U}^H_{\mathbf{n}_p,m}\boldsymbol{\Delta}_{\mathbf{n}_p,m}(a)\mathbf{U}_{\mathbf{n}_p,m}\tag{5}$$

where $\mathbf{U}_{\mathbf{n}_p,m}\stackrel{\text{def}}{=}\mathbf{I}_m\otimes\mathbf{U}_{\mathbf{n}_p}$ and where $\boldsymbol{\Delta}_{\mathbf{n}_p,m}(a)$ is the m block matrix whose block (m_1,m_2) is $\boldsymbol{\Delta}^{m_1,m_2}_{\mathbf{n}_p}(a)$. We are now ready to state the following lemma.

[3]The p-level circulant matrix structure is easily shown by noticing that the sequence defined by (3) is $(n_1,n_2,\ldots,n_p)$-periodic.

[4]Let us note that contrary to the original Toeplitz case, the matrices $\mathbf{C}_{\mathbf{n}_p,m}(a)$ are no longer circulant, nor is $\boldsymbol{\Delta}_{\mathbf{n}_p,m}(a)$ diagonal.

Lemma 2: Let $\{a^{m_1,m_2}_{q_1,q_2,\ldots,q_p}\}_{q_1,q_2,\ldots,q_p}$ be Hermitian, absolutely summable sequences with Fourier transform $\{\mathbf{A}(\omega_1,\omega_2,\ldots,\omega_p)\}_{m_1,m_2}$. Let $c^{(m_1,m_2)}_{q_1,q_2,\ldots,q_p}(a)$ be defined by (3). Then, the induced sequences of matrices $\{\mathbf{A}_{\mathbf{n}_p,m}\}_{\mathbf{n}_p}$ and $\{\mathbf{C}_{\mathbf{n}_p,m}(a)\}_{\mathbf{n}_p}$ are multilevel asymptotically equivalent.

Proof: This lemma is proven in [11, Lemma 1] for Hermitian Toeplitz block Toeplitz matrices, i.e., for $p=2$, $m=1$ and in [12, Lemma 3] for Hermitian block Toeplitz matrices, i.e., for $p=1$ and arbitrary m. The extension to arbitrary p and m is straightforward. ■

IV. Block Multilevel Toeplitz Matrices Generalized Eigenvalue Distribution Theorem

The aim of this section is to extend Szegö's theorem to the case of the generalized eigenvalues of Hermitian block multilevel Toeplitz matrices, under the assumption that the elements generating the matrices are absolutely summable. To prove this extension, three preliminary lemmas are necessary. More precisely, we first prove in Lemma 3 that the eigenvalues of Hermitian block multilevel Toeplitz matrices generated from absolutely summable sequences are bounded by the minimum and maximum eigenvalues of the matrix-valued multidimensional Fourier transforms of the sequences. Then, this lemma is used for the proof of the multilevel asymptotic equivalence between the inverse of a positive definite Hermitian block multilevel Toeplitz matrix and the inverse of its multilevel asymptotically equivalent block multilevel circulant matrix, given by Lemma 4. Furthermore, Lemma 5 shows that the product of the inverse of a positive definite Hermitian block multilevel Toeplitz matrix by a Hermitian block multilevel Toeplitz matrix is multilevel asymptotically equivalent to the product of the inverse of the Hermitian block multilevel circulant matrix by a Hermitian block multilevel circulant matrix, both derived from (3). Finally, using this multilevel asymptotic equivalence and Lemma 1, we straightforwardly obtain extended Theorem 1.

Lemma 3: Let $\{a^{m_1,m_2}_{q_1,q_2,\ldots,q_p}\}_{q_1,q_2,\ldots,q_p}$ be Hermitian, absolutely summable sequences with Fourier transforms $\{\mathbf{A}(\omega_1,\omega_2,\ldots,\omega_p)\}_{m_1,m_2}$. Then, for all eigenvalues $\lambda(\mathbf{A}_{\mathbf{n}_p,m})$ of the induced sequences of matrices $\{\mathbf{A}_{\mathbf{n}_p,m}\}_{\mathbf{n}_p}$, we have

$$\begin{aligned}m_a=\min_{\lambda,\omega_1,\omega_2,\ldots,\omega_p}\lambda\left(\mathbf{A}(\omega_1,\omega_2,\ldots,\omega_p)\right)&\le\lambda(\mathbf{A}_{\mathbf{n}_p,m})\\&\le\max_{\lambda,\omega_1,\omega_2,\ldots,\omega_p}\lambda\left(\mathbf{A}(\omega_1,\omega_2,\ldots,\omega_p)\right)=M_a.\end{aligned}$$

Proof: This lemma is proven in the first step of the proof of [11, Lemma 1] for Toeplitz block Toeplitz matrices, i.e., for $p=2$, $m=1$ and in [13] for Hermitian block Toeplitz matrices, i.e., for $p=1$, and arbitrary m. The extension to arbitrary p and m is straightforward.

Lemma 4: Let $\mathbf{B}_{\mathbf{n}_p,m}$ be a positive definite Hermitian, m block p-level Toeplitz matrix generated from the absolutely summable sequences $\{b^{m_1,m_2}_{q_1,q_2,\ldots,q_p}\}_{q_1,q_2,\ldots,q_p}$ with Fourier transforms $\{\mathbf{B}(\omega_1,\omega_2,\ldots,\omega_p)\}_{m_1,m_2}$ and the associated multilevel asymptotically equivalent m block p-level circulant matrix $\mathbf{C}_{\mathbf{n}_p,m}(b)$ as defined in Lemma 2. If $\min_{\lambda,\omega_1,\omega_2,\ldots,\omega_p}\lambda\left(\mathbf{B}(\omega_1,\omega_2,\ldots,\omega_p)\right)=m_b>0$, then

$$(\mathbf{B}_{\mathbf{n}_p,m})^{-1}\sim(\mathbf{C}_{\mathbf{n}_p,m}(b))^{-1}.$$

Proof: Using Lemma 3, the proof is given in the Appendix.

$$\{c^{(m_1,m_2)}_{q_1,q_2,\ldots,q_p}(a)\}_{q_1,q_2,\ldots,q_p}\stackrel{\text{def}}{=}\frac{1}{n_1n_2\ldots n_p}\sum_{i_1=0}^{n_1-1}\sum_{i_2=0}^{n_2-1}\cdots\sum_{i_p=0}^{n_p-1}\left\{\mathbf{A}\left(2\pi\frac{i_1}{n_1},2\pi\frac{i_2}{n_2},\ldots,2\pi\frac{i_p}{n_p}\right)\right\}_{m_1,m_2}e^{j2\pi\sum_{k=1}^{p}q_ki_k/n_k}\tag{3}$$

Lemma 5: With the assumptions of Lemma 4, if $\mathbf{A}_{\mathbf{n}_p,m}$ is a Hermitian block multilevel Toeplitz matrix generated by absolutely summable sequences $\{a^{m_1,m_2}_{q_1,q_2,\ldots,q_p}\}_{q_1,q_2,\ldots,q_p}$, the associated block multilevel circulant matrices $\mathbf{C}_{\mathbf{n}_p,m}(a)$ and $\mathbf{C}_{\mathbf{n}_p,m}(b)$ issued from (3) satisfy

$$(\mathbf{B}_{\mathbf{n}_p,m})^{-1}\mathbf{A}_{\mathbf{n}_p,m} \sim (\mathbf{C}_{\mathbf{n}_p,m}(b))^{-1}\mathbf{C}_{\mathbf{n}_p,m}(a).$$

Proof: The proof is given in the Appendix.

We now introduce the interval

$$I_\omega = \left[\min_{\lambda,\omega_1,\omega_2,\ldots,\omega_p} \lambda\left(\mathbf{B}^{-1}(\omega_1,\omega_2,\ldots,\omega_p)\mathbf{A}(\omega_1,\omega_2,\ldots,\omega_p)\right); \max_{\lambda,\omega_1,\omega_2,\ldots,\omega_p} \lambda\left(\mathbf{B}^{-1}(\omega_1,\omega_2,\ldots,\omega_p)\mathbf{A}(\omega_1,\omega_2,\ldots,\omega_p)\right)\right]$$

and give a theorem about the asymptotic distribution of the generalized eigenvalues of Hermitian block multilevel Toeplitz matrices which is proven in the Appendix.

Theorem 1: Let $\mathbf{A}_{\mathbf{n}_p,m}$ and $\mathbf{B}_{\mathbf{n}_p,m}$ be two Hermitian m block p-level Toeplitz matrices, such that $\mathbf{B}_{\mathbf{n}_p,m}$ is positive definite, generated by absolutely summable sequences $\{a^{m_1,m_2}_{q_1,q_2,\ldots,q_p}\}_{q_1,q_2,\ldots,q_p}$ and $\{b^{m_1,m_2}_{q_1,q_2,\ldots,q_p}\}_{q_1,q_2,\ldots,q_p}$, respectively, with $\min_{\lambda,\omega_1,\omega_2,\ldots,\omega_p}\lambda(\mathbf{B}(\omega_1,\omega_2,\ldots,\omega_p)) = m_b > 0$. Then if the eigenvalues of $(\mathbf{B}_{\mathbf{n}_p,m})^{-1}\mathbf{A}_{\mathbf{n}_p,m}$ lie in I_ω, for all continuous functions F on I_ω

$$\lim_{n_1,n_2,\ldots,n_p\to\infty} \frac{1}{n_1 n_2 \ldots n_p} \times \sum_{k=1}^{mn_1n_2\ldots n_p} F(\lambda_k(\mathbf{A}_{\mathbf{n}_p,m},\mathbf{B}_{\mathbf{n}_p,m})) = \frac{1}{(2\pi)^p}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\sum_{k=1}^{m} F(\lambda_k(\mathbf{A}(\omega_1,\ldots,\omega_p), \mathbf{B}(\omega_1,\ldots,\omega_p)))d\omega_1\ldots d\omega_p.$$

As shown in [9] and [10], and combined with the fact that, for all $n_1, n_2, \ldots, n_p$, the eigenvalues of $(\mathbf{B}_{\mathbf{n}_p,m})^{-1}\mathbf{A}_{\mathbf{n}_p,m}$ lie in I_ω, Theorem 1 leads to the following corollary.

Corollary 1: For any positive integer l, the smallest and the largest l generalized eigenvalues of $(\mathbf{A}_{\mathbf{n}_p,m}, \mathbf{B}_{\mathbf{n}_p,m})$ are convergent in $n_1, n_2, \ldots, n_p$ and

$$\lim_{n_1,n_2,\ldots,n_p\to\infty} \lambda_{n_1n_2\ldots n_p-l+1}(\mathbf{A}_{\mathbf{n}_p,m},\mathbf{B}_{\mathbf{n}_p,m}) = \min_{\omega_1,\omega_2,\ldots,\omega_p} \lambda_1\left(\mathbf{A}(\omega_1,\omega_2,\ldots,\omega_p),\mathbf{B}(\omega_1,\omega_2,\ldots,\omega_p)\right)$$

$$\lim_{n_1,n_2,\ldots,n_p\to\infty} \lambda_l(\mathbf{A}_{\mathbf{n}_p,m},\mathbf{B}_{\mathbf{n}_p,m}) = \max_{\omega_1,\omega_2,\ldots,\omega_p} \lambda_m\left(\mathbf{A}(\omega_1,\omega_2,\ldots,\omega_p),\mathbf{B}(\omega_1,\omega_2,\ldots,\omega_p)\right)$$

where the eigenvalues are ranked in decreasing order.

V. Illustration

An application of this corollary to the asymptotic optimal detection for wideband space-time beamforming has been given in [16]. In this case, the optimal space-time SINR is given by the maximal generalized eigenvalue of the signal and interference plus noise covariance matrices. Since both matrices are Hermitian block-Toeplitz structured, corollary 1 applies (with $p = 1, n_1$ corresponding to the number of taps and m corresponding to the number of sensors). Using this corollary, closed-form expressions for the asymptotic optimal SINR performance bound has been proven when the number of taps increase to ∞, where the number of sensors is fixed.

In particular if $\mathcal{S}_s(f)$, $\mathcal{S}_j(f)$, σ_n^2, B, and $\boldsymbol{\phi}(\theta, f_0+f)$ denote, respectively, the power spectral densities of the signal of interest and of J interferences, the power of the spatially white background noise, the bandwidth of the signals around the center frequency f_0 and the steering vectors associated with de direction of arrival θ and the frequency f_0+f, the asymptotic SINR for the optimal space-time beamforming sampled at the Shannon rate is given by

$$\max_{f\in I_f}\{\mathcal{S}_s(f)\boldsymbol{\phi}(\theta_s, f_0+f)^H \mathbf{R}_n^{-1}(f)\boldsymbol{\phi}(\theta_s, f_0+f)\} \quad (6)$$

where $\mathbf{R}_n(f) \stackrel{\text{def}}{=} \sum_{j=1}^{J}\mathcal{S}_j(f)\boldsymbol{\phi}(\theta_j, f_0+f)\boldsymbol{\phi}(\theta_j, f_0+f)^H + \sigma_n^2/B\mathbf{I}$ and $I_f = [-B/2; B/2]$.

Expression (6) has been analyzed for particular scenarios and compared to the optimal SINR $\sigma_s^2\boldsymbol{\phi}(\theta_s, f_0)^H\mathbf{R}_n^{-1}\boldsymbol{\phi}(\theta_s, f_0)$ given in narrowband spatial beamforming.

VI. Conclusion

In this paper, we have given an extension of Szegö's theorem to the generalized eigenvalues of Hermitian block multilevel Toeplitz matrices that has been proven using Gray's machinery under the hypothesis of absolutely summable elements. Among the applications of this theorem, one of its corollary allows one in particular to derive asymptotic bounds of the SNR performance. A short example in wideband space-time beamforming has been given to illustrate this theorem.

Appendix

Proof of Lemma 1: Let $\mathbf{D}_{\mathbf{n}_p,m} \stackrel{\text{def}}{=} \mathbf{A}_{\mathbf{n}_p,m} - \mathbf{B}_{\mathbf{n}_p,m} = \{d_{k,j}\}$. Applying the Cauchy–Schwarz inequality to $\mathrm{Tr}(\mathbf{D}_{\mathbf{n}_p,m})$ yields

$$\left|\mathrm{Tr}(\mathbf{D}_{\mathbf{n}_p,m})\right|^2 = \left|\sum_{k=1}^{mn_1n_2\ldots n_p} d_{k,k}\right|^2 \le mn_1n_2\ldots n_p \sum_{k=1}^{mn_1n_2\ldots n_P} |d_{k,k}|^2 \le (mn_1n_2\ldots n_p)^2\left|\mathbf{D}_{\mathbf{n}_p,m}\right|^2$$

and we deduce $\lim_{n_1,n_2,\ldots,n_p\to\infty} 1/n_1n_2\ldots n_p\mathrm{Tr}(\mathbf{D}_{\mathbf{n}_p,m}) = 0$ since the matrices $\mathbf{A}_{\mathbf{n}_p,m}$ and $\mathbf{B}_{\mathbf{n}_p,m}$ are multilevel asymptotically equivalent. Finally, noticing that

$$\sum_{k=1}^{mn_1n_2\ldots n_p}\left(\lambda_k(\mathbf{A}_{\mathbf{n}_p,m}) - \lambda_k(\mathbf{B}_{\mathbf{n}_p,m})\right) = \mathrm{Tr}(\mathbf{D}_{\mathbf{n}_p,m})$$

we obtain

$$\lim_{n_1,n_2,\ldots,n_p\to\infty}\frac{1}{n_1n_2\ldots n_p} \times \sum_{k=1}^{n_1n_2\ldots n_p}(\lambda_k(\mathbf{A}_{\mathbf{n}_p,m}) - \lambda_k(\mathbf{B}_{\mathbf{n}_p,m})) = 0.$$

Then, following the steps of the proof given in [10, Th. 2] completes the proof for arbitrary integer s. ■

Proof of Lemma 4: First, from Lemma 2, $\lambda(\mathbf{B}_{\mathbf{n}_p,m}) \geq m_b > 0$, so $\mathbf{B}_{\mathbf{n}_p,m}$ is nonsingular and since $\mathbf{B}_{\mathbf{n}_p,m}$ is Hermitian $\lambda((\mathbf{B}_{\mathbf{n}_p,m})^{-1}) = \lambda^{-1}(\mathbf{B}_{\mathbf{n}_p,m})$ and $\|(\mathbf{B}_{\mathbf{n}_p,m})^{-1}\| = \max_\lambda \lambda^{-1}(\mathbf{B}_{\mathbf{n}_p,m}) \leq 1/m_b$.

Considering the block diagonal matrix $\mathbf{\Delta}_{\mathbf{n}_p,m}(b)$ associated with $\mathbf{B}_{\mathbf{n}_p,m}$ given by Lemma 1, we have with partitioning the $(mn_1n_2\ldots,n_p)$th vector $\mathbf{w}$ with intuitive notations [see the equation at the bottom of the page].

Consequently, $\lambda(\mathbf{C}_{\mathbf{n}_p,m}(b)) \geq m_b > 0$, $\mathbf{C}_{\mathbf{n}_p,m}(b)$ is nonsingular and satisfies $\|\mathbf{C}_{\mathbf{n}_p,m}(b)^{-1}\| \leq 1/m_b$ as well and Condition 1 of the multilevel asymptotic equivalence is satisfied.

Finally, using

$$\begin{aligned}|(\mathbf{B}_{\mathbf{n}_p,m})^{-1} - \mathbf{C}_{\mathbf{n}_p,m}(b)^{-1}| &= |\mathbf{C}_{\mathbf{n}_p,m}(b)^{-1}\mathbf{C}_{\mathbf{n}_p,m}(b)(\mathbf{B}_{\mathbf{n}_p,m})^{-1} \\ &\quad -\mathbf{C}_{\mathbf{n}_p,m}(b)^{-1}\mathbf{B}_{\mathbf{n}_p,m}(\mathbf{B}_{\mathbf{n}_p,m})^{-1}|\end{aligned}$$

with [10, Lemma 3], we obtain $|(\mathbf{B}_{\mathbf{n}_p,m})^{-1} - \mathbf{C}_{\mathbf{n}_p,m}(b)^{-1}| \leq \|\mathbf{C}_{\mathbf{n}_p,m}(b)^{-1}\| \|(\mathbf{B}_{\mathbf{n}_p,m})^{-1}\| |\mathbf{C}_{\mathbf{n}_p,m}(b) - \mathbf{B}_{\mathbf{n}_p,m}|$, proving that $\lim_{n\to\infty} |(\mathbf{B}_{\mathbf{n}_p,m})^{-1} - \mathbf{C}_{\mathbf{n}_p,m}(b)^{-1}| = 0$ that satisfies Condition 2 of the multilevel asymptotic equivalence. ■

Proof of Lemma 5: Because for the spectral norm $\|(\mathbf{B}_{\mathbf{n}_p,m})^{-1}\mathbf{A}_{\mathbf{n}_p,m}\| \leq \|(\mathbf{B}_{\mathbf{n}_p,m})^{-1}\| \|\mathbf{A}_{\mathbf{n}_p,m}\|$ and $\|\mathbf{C}_{\mathbf{n}_p,m}(b)^{-1}\mathbf{C}_{\mathbf{n}_p,m}(a)\| \leq \|\mathbf{C}_{\mathbf{n}_p,m}(b)^{-1}\| \|\mathbf{C}_{\mathbf{n}_p,m}(a)\|$, Condition 1 of the multilevel asymptotic equivalence is satisfied from Lemma 4. Then, from [10, Lemma 3]

$$\begin{aligned}&\|(\mathbf{B}_{\mathbf{n}_p,m})^{-1}\mathbf{A}_{\mathbf{n}_p,m} - \mathbf{C}_{\mathbf{n}_p,m}(b)^{-1}\mathbf{C}_{\mathbf{n}_p,m}(a)\| \\ &= \|(\mathbf{B}_{\mathbf{n}_p,m})^{-1}\mathbf{A}_{\mathbf{n}_p,m} - (\mathbf{B}_{\mathbf{n}_p,m})^{-1}\mathbf{C}_{\mathbf{n}_p,m}(a) \\ &\quad + (\mathbf{B}_{\mathbf{n}_p,m})^{-1}\mathbf{C}_{\mathbf{n}_p,m}(a) - \mathbf{C}_{\mathbf{n}_p,m}(b)^{-1}\mathbf{C}_{\mathbf{n}_p,m}(a)\| \\ &\leq \|(\mathbf{B}_{\mathbf{n}_p,m})^{-1}\| |\mathbf{A}_{\mathbf{n}_p,m} - \mathbf{C}_{\mathbf{n}_p,m}(a)| \\ &\quad + \|\mathbf{C}_{\mathbf{n}_p,m}(a)\| |(\mathbf{B}_{\mathbf{n}_p,m})^{-1} - \mathbf{C}_{\mathbf{n}_p,m}(b)^{-1}|\end{aligned}$$

and using Lemma 3 and 4, Condition 2 of the multilevel asymptotic equivalence is satisfied. ■

Proof of Theorem 1: Using Lemma 5, $(\mathbf{B}_{\mathbf{n}_p,m})^{-1}\mathbf{A}_{\mathbf{n}_p,m}$ is multilevel asymptotically equivalent to $(\mathbf{C}_{\mathbf{n}_p,m}(b))^{-1}\mathbf{C}_{\mathbf{n}_p,m}(a)$. Then using Lemma 1, we have

$$\begin{aligned}\lim_{n_1,n_2,\ldots,n_p\to\infty} &\frac{1}{n_1n_2\ldots n_p} \\ \times \sum_{k=1}^{mn_1n_2\ldots n_p} &[\lambda_k^s((\mathbf{B}_{\mathbf{n}_p,m})^{-1}\mathbf{A}_{\mathbf{n}_p,m}) - \lambda_k^s((\mathbf{C}_{\mathbf{n}_p,m}(b))^{-1} \\ &\mathbf{C}_{\mathbf{n}_p,m}(a))] = 0.\end{aligned} \tag{7}$$

Using the similarity of $\mathbf{C}_{\mathbf{n}_p,m}(a)$ and $\mathbf{C}_{\mathbf{n}_p,m}(b)$ to respectively $\mathbf{\Delta}_{\mathbf{n}_p,m}(a)$ and $\mathbf{\Delta}_{\mathbf{n}_p,m}(b)$ with the same unitary matrix $\mathbf{U}_{\mathbf{n}_p,m}$ given by (5), we have

$$\begin{aligned}&\sum_{k=1}^{mn_1n_2\ldots n_p} \lambda_k^s((\mathbf{C}_{\mathbf{n}_p,m}(b))^{-1}\mathbf{C}_{\mathbf{n}_p,m}(a)) \\ &\qquad = \sum_{k=1}^{mn_1n_2\ldots n_p} \lambda_k^s(\mathbf{\Delta}_{\mathbf{n}_p,m}^{-1}(b)\mathbf{\Delta}_{\mathbf{n}_p,m}(a)).\end{aligned} \tag{8}$$

Writing the matrices $\mathbf{\Delta}_{\mathbf{n}_p,m}(a)$ and $\mathbf{\Delta}_{\mathbf{n}_p,m}(b)$ defined after (5) in the form

$$\mathbf{\Delta}_{\mathbf{n}_p,m}(a) = \sum_{k_1=0}^{n_1-1}\cdots\sum_{k_p=0}^{n_p-1} \mathbf{A}\left(\frac{2\pi k_1}{n_1},\ldots,\frac{2\pi k_p}{n_p}\right)\otimes \mathbf{E}_{k_1,\ldots,k_p}$$

$$\mathbf{\Delta}_{\mathbf{n}_p,m}(b) = \sum_{k_1=0}^{n_1-1}\cdots\sum_{k_p=0}^{n_p-1} \mathbf{B}\left(\frac{2\pi k_1}{n_1},\ldots,\frac{2\pi k_p}{n_p}\right)\otimes \mathbf{E}_{k_1,\ldots,k_p}$$

with $\mathbf{E}_{k_1,\ldots,k_p}$ the sparse $n_1n_2..n_p \times n_1n_2..n_p$ matrix whose elements are zero except the unit term at the alphabetical position $(k_1,\ldots,k_p)$, we straightforwardly obtain

$$\mathbf{\Delta}_{\mathbf{n}_p,m}^{-1}(b) = \sum_{k_1=0}^{n_1-1}\cdots\sum_{k_p=0}^{n_p-1} \mathbf{B}^{-1}\left(\frac{2\pi k_1}{n_1},\ldots,\frac{2\pi k_p}{n_p}\right)\otimes \mathbf{E}_{k_1,\ldots,k_p}$$

and

$$\begin{aligned}\left(\mathbf{\Delta}_{\mathbf{n}_p,m}^{-1}(b)\mathbf{\Delta}_{\mathbf{n}_p,m}(a)\right)^s &= \sum_{k_1=0}^{n_1-1}\cdots\sum_{k_p=0}^{n_p-1}\Bigg[\mathbf{B}^{-1}\left(\frac{2\pi k_1}{n_1},\ldots,\frac{2\pi k_p}{n_p}\right) \\ &\qquad \mathbf{A}\left(\frac{2\pi k_1}{n_1},\ldots,\frac{2\pi k_p}{n_p}\right)\Bigg]^s \otimes \mathbf{E}_{k_1,\ldots,k_p}.\end{aligned}$$

Consequently

$$\begin{aligned}&\sum_{k=1}^{mn_1n_2\ldots n_p} \lambda_k^s(\mathbf{\Delta}_{\mathbf{n}_p,m}^{-1}(b)\mathbf{\Delta}_{\mathbf{n}_p,m}(a)) \\ &= \sum_{k_1=0}^{n_1-1}\cdots\sum_{k_p=0}^{n_p-1} \mathrm{Tr}\Bigg(\Bigg[\mathbf{B}^{-1}\left(\frac{2\pi k_1}{n_1},\ldots,\frac{2\pi k_p}{n_p}\right) \\ &\qquad \mathbf{A}\left(\frac{2\pi k_1}{n_1},\ldots,\frac{2\pi k_p}{n_p}\right)\Bigg]^s\Bigg).\end{aligned}$$

$$\begin{aligned}\mathbf{w}^H\mathbf{\Delta}_{\mathbf{n}_p,m}(b)\mathbf{w} &= \sum_{m_1=1}^{m}\sum_{m_2=1}^{m} \mathbf{w}_{m_1}^H \mathbf{\Delta}_{\mathbf{n}_p}^{m_1,m_2}(b)\mathbf{w}_{m_2} \\ &= \sum_{m_1=1}^{m}\sum_{m_2=1}^{m}\sum_{q_1=0}^{n_1-1}\cdots\sum_{q_p=0}^{n_p-1} w_{m_1}^{q_1,\ldots,q_2*}\left\{\mathbf{B}\left(2\pi\frac{q_1}{n_1},\ldots,2\pi\frac{q_p}{n_p}\right)\right\}_{m_1,m_2} w_{m_2}^{q_1,\ldots,q_2} \\ &= \sum_{q_1=0}^{n_1-1}\cdots\sum_{q_p=0}^{n_p-1}(\mathbf{w}^{q_1,\ldots,q_p})^H\mathbf{B}\left(2\pi\frac{q_1}{n_1},\ldots,2\pi\frac{q_p}{n_p}\right)\mathbf{w}^{q_1,\ldots,q_p} \\ &\geq \min_\lambda \lambda(\mathbf{B}(\omega_1,\ldots,\omega_p))\mathbf{w}^H\mathbf{w}.\end{aligned}$$

$$\lim_{n_1,n_2,\ldots,n_p\to\infty} \frac{1}{n_1 n_2 \ldots n_p} \sum_{k=1}^{m n_1 n_2 \ldots n_p} [\lambda_k^s((\mathbf{B}_{\mathbf{n}_p,m})^{-1}\mathbf{A}_{\mathbf{n}_p,m}) \\ = \frac{1}{(2\pi)^p}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \mathrm{Tr}\left(\left[\mathbf{B}^{-1}(\omega_1,\ldots,\omega_p)\mathbf{A}(\omega_1,\ldots,\omega_p)\right]^s\right) d\omega_1\ldots d\omega_p \\ = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sum_{k=1}^{m}\lambda_k^s\left(\mathbf{B}^{-1}(\omega_1,\ldots,\omega_p)\mathbf{A}(\omega_1,\ldots,\omega_p)\right) d\omega_1\ldots d\omega_p.$$

Using the definition of the Riemann integral where the continuity of the Fourier transforms $\mathbf{A}(\omega_1,\ldots,\omega_p)$ and $\mathbf{B}(\omega_1,\ldots,\omega_p)$ guarantees the existence, (8) gives

$$\lim_{n_1,\ldots n_p\to\infty} \frac{1}{n_1\ldots n_p} \sum_{k=1}^{m n_1 n_2 \ldots n_p} \lambda_k^s(\mathbf{C}_{\mathbf{n}_p,m}(b)^{-1}\mathbf{C}_{\mathbf{n}_p,m}(a)) \\ = \frac{1}{(2\pi)^p}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \mathrm{Tr}\left(\left[\mathbf{B}^{-1}(\omega_1,\ldots,\omega_p) \mathbf{A}(\omega_1,\ldots,\omega_p)\right]^s\right) d\omega_1\ldots d\omega_p.$$

Consequently, the limit of the first term of (7) exists and [see the equation at the top of the page].

Hence, for any polynomial P, we have

$$\lim_{n_1,n_2,\ldots,n_p\to\infty} \frac{1}{n_1 n_2 \ldots n_p} \sum_{k=1}^{m n_1 n_2 \ldots n_p} P(\lambda_k(\mathbf{A}_{\mathbf{n}_p,m},\mathbf{B}_{\mathbf{n}_p,m})) \\ = \frac{1}{(2\pi)^p}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\sum_{k=1}^{m} P(\lambda_k(\mathbf{A}(\omega_1,\ldots,\omega_p), \mathbf{B}(\omega_1,\ldots,\omega_p)))d\omega_1\ldots d\omega_p.$$

Invoking the Stone-Weierstrass approximation theorem (recalled in [9, Th. 2.2]), this relation extends to all functions F continuous on I_ω. ■

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