Closed-Form Expressions of the Exact Cramer–Rao Bound for Parameter Estimation of BPSK, MSK, or QPSK Waveforms

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***Abstract*—This letter addresses the stochastic Cramer–Rao bound (CRB) pertaining to the joint estimation of the carrier frequency offset, the carrier phase and the noise and signal powers of binary phase-shift keying (BPSK), minimum shift keying (MSK), and quaternary phase-shift keying (QPSK) modulated signals corrupted by additive white circular Gaussian noise. Because the associated models are governed by simple Gaussian mixture distributions, an explicit expression of the Fisher information matrix is given and an explicit expression for the stochastic CRB of these four parameters are deduced. Specialized expressions for low and high SNR are presented as well. Finally, these expressions are related to the modified CRB and our proposed analytical expressions are numerically compared with the approximate expressions previously given in the literature.**

***Index Terms*—BPSK, modified Cramer–Rao bound, minimum shift keying (MSK), quaternary phase-shift keying (QPSK), stochastic Cramer–Rao bound.**

I. Introduction

THE stochastic Cramer–Rao bound (CRB) is a well-known lower bound on the variance of any unbiased estimate and, as such, serves as a useful benchmark for practical estimators. Unfortunately, the evaluation of this CRB is mathematically quite difficult when the observed signal contains, in addition to the parameters to be estimated, random discrete data and random noise. A typical example of such a situation that has been studied by many authors (see, e.g., [1] and the references therein) is the observation of noisy linearly modulated waveforms that are a function of deterministic parameters such that the time delay, the carrier frequency offset, the carrier phase, noise and signal powers, as well as the data symbol sequence. Because the analytical computation of this CRB has been considered to be unfeasible, a modified CRB (MCRB) which is much simpler to evaluate than the exact CRB has been introduced in [2]. However, this MCRB may not be as tight as the exact CRB [3] for joint estimation of all parameters. To circumvent this difficulty, asymptotic expressions at low [4] or high [5] signal-to-noise ratio (SNR) have been investigated. Unfortunately, though, these asymptotic expressions do not apply at moderate SNR, for which only combined analytical/numerical (see, e.g., [1], [5], and [6]) approaches are available until now.

In this letter, we investigate an analytical expression of the stochastic CRB (i.e., if the information symbols are viewed as nuisance parameters and thus applicable in non-data-aided estimation) associated with the joint estimation of the carrier frequency offset, the carrier phase and the noise and signal powers of BPSK, QPSK, or MSK modulated signals corrupted by additive white circular Gaussian noise, which is valid for arbitrary SNR. This letter is organized as follows. After formulating the problem in Section II, an explicit expression of the Fisher information matrix (FIM) associated with all the deterministic parameters is given in Section III. Because the carrier frequency offset and the carrier phase parameters are decoupled from the signal noise and signal powers parameters, simple explicit expressions for the stochastic CRB of these four parameters are deduced. Specialized expressions for low and high SNR are presented as well. Finally, in Section IV, our proposed analytical expressions are numerically compared with the previously given approximate expressions.

II. Problem Formulation

Consider BPSK, QPSK, or MSK modulated signals. The received signals are bandpass filtered and after down conversion the signal to baseband, the in-phase and quadrature components are paired to obtain complex signals. We assume Nyquist shaping and ideal sample timing so that the inter-symbol interference at each symbol spaced sampling instance can be ignored. In the presence of frequency offset and carrier phase, the signals at the output of the matched filters yield the observation vector $\mathbf{y} = (y_{k_0}, \ldots, y_{k_0+K-1})$, with

$$y_k = a s_k e^{i2\pi k\nu} e^{i\phi} + n_k$$

for $k = k_0, \ldots, k_0 + K - 1$. $\{s_k\}$ is a sequence of independent identically distributed (IID) data symbols taking values ± 1 and $\pm\sqrt{2}/2 \pm i\sqrt{2}/2$ with equal probabilities for BPSK and QPSK, respectively, and for MSK are defined by $s_{k+1} = i s_k c_k$, where c_k is a sequence of independent BPSK symbols with equal probabilities, where the original value s_{k_0} remains unspecified in the set $\{+1, +i, -1, -i\}$. The deterministic unknown parameters a, ν, and ϕ represent the amplitude, the carrier frequency offset normalized to the symbol rate, and the carrier phase at $k = 0$. Finally, the sequence $\{n_k\}$ consists of IID zero-mean complex circular Gaussian noise random variables[1] of variance $\sigma^2 \stackrel{\text{def}}{=} E|n_k|^2$. The symbols s_k are assumed to be independent from n_k. If no *a priori* information is available concerning the transmitted symbols, the distribution of $\mathbf{y}$ is

[1]Note that many papers consider the parameters a^2 and σ^2 denoted usually as the symbol energy E_s and the noise power spectral density N_0 as known. They usually suppose a unit variance for the noise and use the ratio $\epsilon \stackrel{\text{def}}{=} (E_s/N_0)^{1/2}$ as the modulation amplitude, but in practice, these two parameters are unknown as well.

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parameterized by $\boldsymbol{\theta} \stackrel{\text{def}}{=} (\nu, \phi, a, \sigma)$. We note that the MSK is modeled equivalently (see, e.g., [7]) by $s_k = i^{k-k_0} b_k s_{k_0}$, where b_k is another sequence of independent BPSK symbols $\{-1, +1\}$ with equal probabilities. Consequently, similar to the BPSK and QPSK, $(y_k)_{k=k_0,\ldots,k_0+K-1}$ are independently non-identically distributed along the following: *mixture of circular Gaussian distribution*[2]

$$p(y_k; \boldsymbol{\theta}) = \frac{1}{L\pi\sigma^2} \sum_{l=1}^{L} \exp\left(-\frac{|y_k - a s_{l,k} e^{i2\pi k\nu} e^{i\phi}|^2}{\sigma^2}\right) \quad (1)$$

with $s_{l,k} = \pm 1 (L = 2)$, $s_{l,k} = \pm\sqrt{2}/2 \pm i\sqrt{2}/2 (L = 4)$, or $s_{l,k} = i^{k-k_0} b_l s_{k_0}$ with $b_l = \pm 1 (L = 2)$ associated with BPSK, QPSK, or MSK, respectively.

III. Stochastic CRB: Analytical Results

A. General Closed-Form Expression

Using the independence of the random variables y_k, the FIM is given (elementwise) by

$$(\mathbf{I}_F)_{i,j} = -\sum_{k=k_0}^{k_0+K-1} E\left(\frac{\partial^2 \ln p(y_k; \boldsymbol{\theta})}{\partial\theta_i \partial\theta_j}\right) i, \quad j = 1, \ldots, 4 \quad (2)$$

where the PDFs (1) take the following forms:

$$p(y_k; \boldsymbol{\theta}) = \frac{1}{\pi\sigma^2} \exp\left(-\frac{|y_k|^2 + a^2}{\sigma^2}\right) c(y_k)$$

[3]where $c(y_k)$ is respectively equal to $\cosh\left(a/\sigma^2 g_1(y_k)\right)$, $\cosh\left(a/\sigma^2\sqrt{2} g_1(y_k)\right) \cosh\left(a/\sigma^2\sqrt{2} g_2(y_k)\right)$, and $\cosh\left(a/\sigma^2 g_3(y_k)\right)$ for the BPSK, QPSK, and MSK, respectively, with $g_1(y_k) \stackrel{\text{def}}{=} 2\Re(e^{i2\pi k\nu} e^{i\phi} y_k^*)$, $g_2(y_k) \stackrel{\text{def}}{=} 2\Im(e^{i2\pi k\nu} e^{i\phi} y_k^*)$, and $g_3(y_k) \stackrel{\text{def}}{=} 2\Re(i^{k-k_0} e^{i2\pi k\nu} e^{i\phi} s_{k_0} y_k^*)$. Extending the approach used in [8] for the parameters a and σ only and in [9] for the direction of arrival (DOA) parameters, the following lemma is proved in Appendix A.

Lemma 1: The parameter $\boldsymbol{\theta} = (\nu, \phi, a, \sigma)$ is partitioned in two decoupled parameters (ν, ϕ) and (a, σ) in the FIM associated with the BPSK, QPSK, and MSK, as follows:

$$\mathbf{I}_{\text{BPSK}} = \mathbf{I}_{\text{MSK}} = \begin{bmatrix} \mathbf{I}_B^{(1)} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_B^{(2)} \end{bmatrix}$$
$$\mathbf{I}_{\text{QPSK}} = \begin{bmatrix} \mathbf{I}_Q^{(1)} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_Q^{(2)} \end{bmatrix}$$

with

$$\mathbf{I}_B^{(1)} = 2\rho^2(1 - f_1(\rho)) \times \begin{bmatrix} (2\pi)^2 \sum_{k=k_0}^{k_0+K-1} k^2 & 2\pi \sum_{k=k_0}^{k_0+K-1} k \\ 2\Pi \sum_{k=k_0}^{k_0+K-1} k & K \end{bmatrix}$$

$$\mathbf{I}_B^{(2)} = 2K\frac{\rho}{a^2} \begin{bmatrix} 1 - f_2(\rho) & 2\sqrt{\rho} f_2(\rho) \\ 2\sqrt{\rho} f_2(\rho) & 2(1 - 2\rho f_2(\rho)) \end{bmatrix}$$
$$\mathbf{I}_Q^{(1)} = 2\rho^2\left(1 - (1 + \rho) f_1\left(\frac{\rho}{2}\right)\right) \times \begin{bmatrix} (2\pi)^2 \sum_{k=k_0}^{k_0+K-1} k^2 & 2\pi \sum_{k=k_0}^{k_0+K-1} k \\ 2\pi \sum_{k=k_0}^{k_0+K-1} k & K \end{bmatrix}$$
$$\mathbf{I}_Q^{(2)} = 2K\frac{\rho}{a^2} \begin{bmatrix} 1 - f_2(\frac{\rho}{2}) & 2\sqrt{\rho} f_2(\frac{\rho}{2}) \\ 2\sqrt{\rho} f_2(\frac{\rho}{2}) & 2(1 - 2\rho f_2(\frac{\rho}{2})) \end{bmatrix}$$

where ρ is the SNR a^2/σ^2 and f_1 and f_2 are the following decreasing functions of ρ:

$$f_1(\rho) \stackrel{\text{def}}{=} \frac{2e^{-\rho}}{\sqrt{2\pi}} \int_0^{+\infty} \frac{e^{-u^2/2}}{\cosh(u\sqrt{2\rho})} du$$
$$f_2(\rho) \stackrel{\text{def}}{=} \frac{2e^{-\rho}}{\sqrt{2\pi}} \int_0^{+\infty} \frac{u^2 e^{-u^2/2}}{\cosh(u\sqrt{2\rho})} du.$$

The determinants of $\mathbf{I}_B^{(1)}$ and $\mathbf{I}_Q^{(1)}$ do not depend on the time k_0 at which the first sample is taken, and consequently, the CRB for the frequency does not depend on it either, but the CRB for the phase does. The minimum value for this CRB is attained for $k_0 = -(K - 1)/2$. This particular choice of k_0 renders $\mathbf{I}_B^{(1)}$ and $\mathbf{I}_Q^{(1)}$ diagonal, and we obtain in this case the following result, where the MCRB are straightforwardly derived from [2]:

$$\text{MCRB}(\theta_i) = \frac{1}{E_{\mathbf{y},\mathbf{s}}\left(\frac{\partial^2 \ln p(\frac{\mathbf{y}}{\mathbf{s};\boldsymbol{\theta}})}{\partial\theta_i^2}\right)}, \quad i = 1, \ldots, 4.$$

1) Result 1: The CRB for the joint estimation of the parameters (ν, ϕ, a, σ) associated with the BPSK and MSK are given by

$$\text{CRB}(\nu) = \frac{6}{(2\pi)^2 K(K^2 - 1)\rho(1 - f_1(\rho))} = \text{MCRB}(\nu)\left(\frac{1}{1 - f_1(\rho)}\right) \quad (3)$$

$$\text{CRB}(\phi) = \frac{1}{2K\rho(1 - f_1(\rho))} = \text{MCRB}(\phi)\left(\frac{1}{1 - f_1(\rho)}\right) \quad (4)$$

$$\text{CRB}(a) = \frac{a^2(1 - 2\rho f_2(\rho))}{2K\rho(1 - f_2(\rho) - 2\rho f_2(\rho))} = \text{MCRB}(a)\left(\frac{1 - 2\rho f_2(\rho)}{1 - f_2(\rho) - 2\rho f_2(\rho)}\right) \quad (5)$$

$$\text{CRB}(\sigma) = \frac{a^2(1 - f_2(\rho))}{4K\rho(1 - f_2(\rho) - 2\rho f_2(\rho))} = \text{MCRB}(\sigma)\left(\frac{1 - 2\rho f_2(\rho)}{1 - f_2(\rho) - 2\rho f_2(\rho)}\right). \quad (6)$$

The CRBs associated with the QPSK are obtained by replacing $f_1(\rho)$ and $f_2(\rho)$ by, respectively, $(1 + \rho) f_1(\rho/2)$ and $f_2(\rho/2)$ in (3)–(6).

Note that the proof of the decoupling that is not trivial [see (11) and (12)] has not appeared in the literature despite the first expressions $\text{CRB}(\nu)$ and (ϕ) coincide with the expressions given in [10] for $\text{CRB}(\nu)$ with (ϕ, a, σ) known and $\text{CRB}(\phi)$

[2]Usually for such a mixture, explicit closed-form expressions of the CRB are not available.

[3]Note that this productform does not extend to arbitrary QAM (see, e.g., [6, rel. (41)] for the 16QAM).

with (ν, a, σ) known, for the BPSK only. The first expressions $\mathrm{CRB}(\nu)$ and $\mathrm{CRB}(\phi)$ do not appear in [10] for MSK and QPSK including for (ϕ, a, σ) or (ν, a, σ) known. Using the definition of f_1 and f_2, these asymptotic CRBs coincide with the MCRB for high SNR. This extends a property proved in [3] for a scalar parameter only.

B. Low-SNR Expression

For low SNR, $f_1(\rho)$ and $f_2(\rho)$ approach 1. We resort to a Taylor series expansion of these functions obtained by expanding $e^{-\rho}$ and $\cosh(u\sqrt{2\rho})$ around $\rho = 0$. Then, using the values $(2n)!/2^n n!$ of the moments of order $2n$ of zero-mean unit variance Gaussian random variables, we obtain, after tedious but straightforward algebraic manipulations

$$f_1(\rho) = 1 - 2\rho + 4\rho^2 - \frac{40}{3}\rho^3 + \frac{208}{3}\rho^4 + o(\rho^4),$$
$$f_2(\rho) = 1 - 4\rho + 16\rho^2 - \frac{256}{3}\rho^3 + \frac{12544}{21}\rho^4 + o(\rho^4).$$

Inserting these expansions in Result 1 allows us to prove the following result.[4]

1) Result 2: The CRB for the joint estimation of the parameters (ν, ϕ, a, σ) associated with the BPSK, MSK, and QPSK are given for low SNR by

$$\begin{aligned}\mathrm{CRB}(\nu) &= \frac{6}{(2\pi)^2 K(K^2-1)}\frac{L!}{L^2\rho^L}(1 + L\rho + o(\rho))\\ &= \mathrm{MCRB}(\nu)\frac{L!}{L^2\rho^{L-1}}(1 + L\rho + o(\rho)) \qquad (7)\end{aligned}$$

$$\begin{aligned}\mathrm{CRB}(\phi) &= \frac{1}{2K}\frac{L!}{L^2\rho^L}(1 + L\rho + o(\rho))\\ &= \mathrm{MCRB}(\phi)\frac{L!}{L^2\rho^{L-1}}(1 + L\rho + o(\rho)) \qquad (8)\end{aligned}$$

$$\begin{aligned}\mathrm{CRB}(a) &= \frac{a^2}{K\alpha_L\rho^L}(1 + L\rho + o(\rho))\\ &= \mathrm{MCRB}(a)\frac{2}{\alpha_L\rho^{L-1}}(1 + L\rho + o(\rho)) \qquad (9)\end{aligned}$$

$$\begin{aligned}\mathrm{CRB}(\sigma) &= \frac{a^2}{K\beta_L\rho^{L-1}}(1 + \gamma_L\rho^{3-L/2} + o(\rho^{3-L/2}))\\ &= \mathrm{MCRB}(\sigma)\frac{4}{\beta_L\rho^{L-2}}(1 + \gamma_L\rho^{3-L/2} + o(\rho^{3-L/2})) \qquad (10)\end{aligned}$$

with $L = 2$ for BPSK and MSK and $L = 4$ for QPSK, and $\alpha_2 = 4$, $\alpha_4 = 16/3$, $\beta_2 = 2$, $\beta_4 = 16/3$, $\gamma_2 = -16/3$, and $\gamma_4 = 4$.

We note that (7) and (8) for BPSK and QPSK are refinements of the expressions of $\mathrm{CRB}(\nu)$ and $\mathrm{CRB}(\phi)$ given in [4].

C. High-SNR Expression

For high SNR, the MCRB approaches the CRB at the same rate that $f_1(\rho)$ and $f_2(\rho)$ approach 0. Because we prove in Appendix B that these functions are bounded above by $e^{-\rho}/\sqrt{\pi\rho}$ and more precisely that $f_1(\rho)/e^{-\rho}\ln(2)/\sqrt{\pi\rho}$ tends to 1 when ρ tends to ∞, the CRB are practically equal to the MCRB for moderate SNR. For example: $\rho = 2$ (3 dB) and $\rho = 4$ (6 dB)] give, respectively, the upper bound 0.05 and 0.005 for $f_1(\rho)$ and $f_2(\rho)$, and consequently, the ratios CRB/MCRB is around one from these values of SNR.

[4]Of course, these bounds are going to infinity as the SNR decreases to zero; consequently, for the parameters ν and ϕ with finite support, these results are useful for not too low SNR only (typically in the range $[-5$ dB, 0 dB$]$).

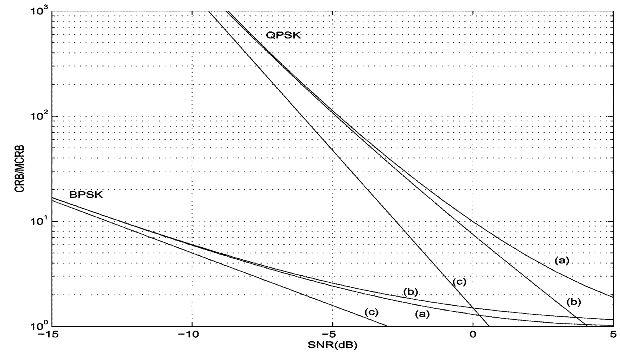

Fig. 1. Ratio $\mathrm{CRB}(\nu)/\mathrm{MCRB}(\nu) = \mathrm{CRB}(\phi)/\mathrm{MCRB}(\phi)$ at low SNR. (a) Exact value given by (3) and (4). (b) Approximate value given by (7) and (8). (c) Approximate value given in [4].

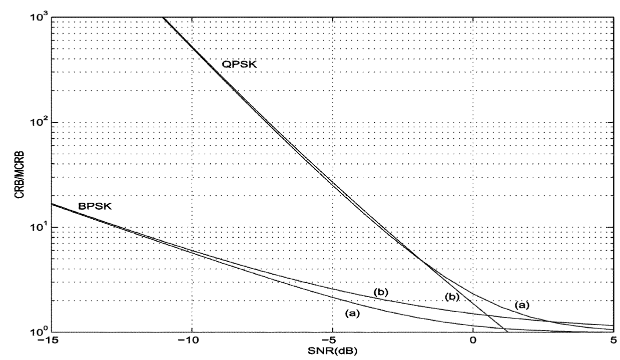

Fig. 2. Ratio CRB(a)/MCRB(a) at low SNR. (a) Exact value given by (5). (b) Approximate value given by (9).

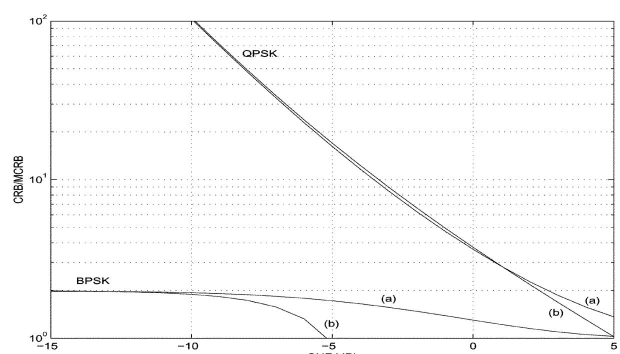

Fig. 3. Ratio $\mathrm{CRB}(\sigma)/\mathrm{MCRB}(\sigma)$ at low SNR. (a) Exact value given by (6). (b) Approximate value given by (10).

IV. Numerical Results

The analytical result 1 is numerically compared with the approximations given in result 2 and to the approximations given in [4] for $\mathrm{CRB}(\nu)$ and $\mathrm{CRB}(\phi)$ of BPSK and QPSK at low SNR.

In these conditions, we see in Figs. 1–3 a good agreement between the numerical values derived from results 1 and 2 in a large range of low SNR. Furthermore, we note that the ratio CRB/MCRB is unbounded except for the noise power of BPSK and MSK for which it tends to 2 when the SNR tends to 0.

APPENDIX A
PROOF OF LEMMA 1

To evaluate the FIM (2) for the BPSK, the partial derivatives are straightforwardly derived, in particular

$$\begin{aligned}\frac{\partial^2 \ln p(y_k;\boldsymbol{\theta})}{\partial \phi^2} &= \frac{a^2 g_2^2(y_k)}{\sigma^4}\frac{1}{\cosh^2\left(\frac{a g_1(y_k)}{\sigma^2}\right)} \\ &\quad - \frac{a g_1(y_k)}{\sigma^2}\tanh\left(\frac{a g_1(y_k)}{\sigma^2}\right) \\ \frac{\partial^2 \ln p(y_k;\boldsymbol{\theta})}{\partial a \partial \phi} &= -\frac{a g_1(y_k) g_2(y_k)}{\sigma^4}\frac{1}{\cosh^2\left(\frac{a g_1(y_k)}{\sigma^2}\right)} \\ &\quad - \frac{g_2(y_k)}{\sigma^2}\tanh\left(\frac{a g_1(y_k)}{\sigma^2}\right) \qquad (11) \\ \frac{\partial^2 \ln p(y_k;\boldsymbol{\theta})}{\partial a \partial \nu} &= (2\pi k)\frac{\partial^2 \ln p(y_k;\boldsymbol{\theta})}{\partial a \partial \phi} \\ \frac{\partial^2 \ln p(y_k;\boldsymbol{\theta})}{\partial \sigma \partial \phi} &= \frac{2a^2 g_1(y_k) g_2(y_k)}{\sigma^5}\frac{1}{\cosh^2\left(\frac{a g_1(y_k)}{\sigma^2}\right)} \\ &\quad + \frac{2a g_2(y_k)}{\sigma^3}\tanh\left(\frac{a g_1(y_k)}{\sigma^2}\right) \qquad (12) \\ \frac{\partial^2 \ln p(y_k;\boldsymbol{\theta})}{\partial \sigma \partial \nu} &= (2\pi k)\frac{\partial^2 \ln p(y_k;\boldsymbol{\theta})}{\partial \sigma \partial \phi}.\end{aligned}$$

Using the regularity condition $\partial/\partial\theta_i \int p(y_k;\boldsymbol{\theta})dy_k = \int \partial p(y_k;\boldsymbol{\theta})/\partial\theta_i dy_k$ which is fulfilled for finite mixtures of Gaussian distributions, the following property holds: $E\left(\partial \ln p(y_k;\boldsymbol{\theta})/\partial a\right) = 0$. With $\partial \ln p(y_k;\boldsymbol{\theta})/\partial a = -2a/\sigma^2 + g_1(y_k)/\sigma^2 \tanh\left(a g_1(y_k)/\sigma^2\right)$, we obtain

$$E\left(g_1(y_k)\tanh\left(\frac{a g_1(y_k)}{\sigma^2}\right)\right) = 2a.$$

This identity enables us to straightforwardly derive the terms of $\mathbf{I}_B^{(2)}$ using the definition of the function $f_2(\rho) = E\left(g_1^2(y_k)/2\sigma^2 1/\cosh^2\left(a g_1(y_k)\sigma^2\right)\right)$, where the random variable $g_1(y_k)$ is equally weighted mixed Gaussian [$\mathcal{N}(-2a;2\sigma^2)$ and $\mathcal{N}(+2a;2\sigma^2)$].

To evaluate $\mathbf{I}_B^{(1)}$, we note that $g_1(y_k) = 2a s_k + \left(e^{i2\pi k\nu}e^{i\phi}n_k^* + e^{-i2\pi k\nu}e^{-i\phi}n_k\right)$ and $g_2(y_k) = -i\left(e^{i2\pi k\nu}e^{i\phi}n_k^* - e^{-i2\pi k\nu}e^{-i\phi}n_k\right)$. Because s_k and n_k are independent and the two Gaussian random variables $e^{i2\pi k\nu}e^{i\phi}n_k^* + e^{-i2\pi k\nu}e^{-i\phi}n_k$ and $e^{i2\pi k\nu}e^{i\phi}n_k^* - e^{-i2\pi k\nu}e^{-i\phi}n_k$ are uncorrelated and therefore independent, the three random variables s_k, $e^{i2\pi k\nu}e^{i\phi}n_k^* + e^{-i2\pi k\nu}e^{-i\phi}n_k$, and $e^{i2\pi k\nu}e^{i\phi}n_k^* - e^{-i2\pi k\nu}e^{-i\phi}n_k$ are collectively independent and thus $g_1(y_k)$ and $g_2(y_k)$ are independent. Using the definition of the function $f_1(\rho) = E\left(1/\cosh^2\left(a g_1(y_k)/\sigma^2\right)\right)$, the terms of $\mathbf{I}_B^{(1)}$ are derived.

Because $g_1(y_k)$ and $g_2(y_k)$ are independent and zero mean, $E(\partial^2 \ln p(y_k;\boldsymbol{\theta})/\partial a\partial\phi) = E(\partial^2 \ln p(y_k;\boldsymbol{\theta})/\partial\sigma\partial\phi) = E(\partial^2 \ln p(y_k;\boldsymbol{\theta})/\partial a\partial\nu) = E(\partial^2 \ln p(y_k;\boldsymbol{\theta})/\partial\sigma\partial\nu) = 0$. This implies that the parameters (ν,ϕ) and (a,σ) are decoupled in the FIM.

For the MSK, the derivations follow the same lines, replacing $g_1(y_k)$ by $g_3(y_k)$.

Finally, for the QPSK, evaluating the partial derivatives $\partial^2 \ln p(y_k;\boldsymbol{\theta})/\partial\theta_i\partial\theta_j$ and taking their expectation are derived in the same way, provided the log-likelihoods associated with $g_1(y_k)$ and $g_2(y_k)$ are gathered, and the hypothesis of independence of $\Re(s_k)$ and $\Im(s_k)$ is taken into account.

APPENDIX B
BOUNDS ON $f_1(\rho)$ AND $f_2(\rho)$

For high SNR, using the inequality $1/\cosh(u\sqrt{2\rho}) < 2e^{-u\sqrt{2\rho}}$, we obtain, after simple algebraic manipulations

$$\begin{aligned} f_1(\rho) &< 4Q(\sqrt{2\rho}) \text{ and } f_2(\rho) \\ &< 4\left((2\rho+1)Q(\sqrt{2\rho}) - \frac{\sqrt{2\rho}}{\sqrt{2\pi}}e^{-\rho}\right)\end{aligned}$$

where $Q(x)$ is the error function $\int_x^{+\infty} 1/\sqrt{2\pi}e^{-u^2/2}du$ classically bounded above by $1/x\sqrt{2\pi}e^{-x^2/2}$. Applying this upper bounds in (6.13) gives: $f_1(\rho) < e^{-\rho}/\sqrt{\pi\rho}$ and $f_2(\rho) < e^{-\rho}/\sqrt{\pi\rho}$. To specify the upper bound of $f_1(\rho)$, we use the following expansion:

$$\begin{aligned}\frac{1}{\cosh(u\sqrt{2\rho})} &= 2e^{-u\sqrt{2\rho}}(1+e^{-u\sqrt{2\rho}})^{-1} \\ &= 2\sum_{k=0}^{+\infty}(-1)^k e^{-(k+1)u\sqrt{2\rho}}.\end{aligned}$$

Inserting this into $f_1(\rho)$, we obtain after simple algebraic manipulations the following alternating expansion:

$$f_1(\rho) = 4\sum_{k=0}^{+\infty}(-1)^k e^{\rho[(k+1)^2-1]}Q[(k+1)\sqrt{2\rho}].$$

Using the standard bounds $1/x\sqrt{2\pi}(1-1/x^2)e^{-x^2/2} \leq Q(x) \leq 1/x\sqrt{2\pi}e^{-x^2/2}$ and $\ln(2) = -\sum_{k=1}^{\infty}(-1)^k/k$ in (6.14) proves, after simple algebraic manipulations, that $f_1(\rho)/e^{-\rho}\ln(2)/\sqrt{\pi\rho}$ tends to 1 when ρ tends to ∞.

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