Cramer–Rao Bounds of DOA Estimates for BPSK and QPSK Modulated Signals

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***Abstract*—This paper focuses on the stochastic Cramer–Rao bound (CRB) of direction of arrival (DOA) estimates for binary phase-shift keying (BPSK) and quaternary phase-shift keying (QPSK) modulated signals corrupted by additive circular complex Gaussian noise. Explicit expressions of the CRB for the DOA parameter alone in the case of a single signal waveform are given. These CRBs are compared, on the one hand, with those obtained with different *a priori* knowledge and, on the other hand, with CRBs under the noncircular and circular complex Gaussian distribution and with different deterministic CRBs. It is shown in particular that the CRBs under the noncircular [respectively, circular] complex Gaussian distribution are tight upper bounds on the CRBs under the BPSK [respectively, QPSK] distribution at very low and very high signal-to-noise ratios (SNRs) only. Finally, these results and comparisons are extended to the case of two independent BPSK or QPSK distributed sources where an explicit expression of the CRB for the DOA parameters alone is given for large SNR.**

***Index Terms*—Binary phase-shift keying (BPSK) and quaternary phase-shift keying (QPSK) signals, direction of arrival (DOA) estimation, Fisher information matrix, stochastic Cramer–Rao bound.**

I. Introduction

DETERMINISTIC and stochastic Cramer–Rao bounds (CRBs) play an important role in direction of arrival (DOA) estimation because they serve as a benchmark for the performance of actual estimators (see, e.g., [1]). Moreover, the stochastic CRB can be achieved asymptotically (in the number of measurements) by the stochastic maximum likelihood (ML) method. Unfortunately, this stochastic CRB appears to be prohibitive to compute for non-Gaussian processes including discrete signal waveforms. And to the best of our knowledge, no contribution has dealt with stochastic CRB for discrete signal waveforms in DOA estimation yet, despite some recent works on stochastic CRB for noncircular signals (e.g., [2] and [3]). To cope with this difficulty, a method sometimes used is to assume that the signals are arbitrary deterministic sequences while the noise is circular complex Gaussian, so that the distribution is still Gaussian and the associated deterministic CRB is easily deduced (see, e.g., [4, (2.13)]). But the corresponding deterministic (or conditional) ML method does not achieve this deterministic CRB because the deterministic likelihood function does not meet the required regularity conditions. Consequently, this deterministic CRB is only a nonattainable lower bound on the variance of any unbiased DOA estimator. To deal with non-Gaussian processes, another solution is to suppose that the signals are Gaussian but not necessarily complex circular. In that case, the associated CRB is under rather general conditions (see, e.g., [5, p. 293]) the largest CRB among the class of arbitrary distributions with given covariance matrices. This approach was used in [6] for noncircular complex signal waveforms such as discrete signals. But the associated CRB is only an upper bound on the true stochastic CRB. Faced with the drawbacks of the two aforementioned approximations, we need an explicit expression of the stochastic CRB under non-Gaussian distributions.

In this paper, we derive explicit expressions of the stochastic CRB for the DOA parameter alone in the case of binary phase-shift keying (BPSK) and quaternary phase-shift keying (QPSK) signal waveforms observed in additive circular complex Gaussian noise. More specifically, the main contribution of this paper is devoted to the case of a single BPSK and QPSK signal waveform. Because the distribution of these models are simple mixed Gaussian, an explicit expression of the Fisher information matrix (FIM) is derived using well-known properties of the Gaussian distribution. An explicit expression of the stochastic CRB for the DOA parameter alone is deduced. We note that apart from the DOA applications, the recent papers (e.g., [7] and [8]) that deal with stochastic CRBs for estimating the carrier phase and frequency of BPSK and QPSK waveforms do not give analytic solutions (see, e.g., [7, (9) and (16)] and [8, (16)]). Our CRBs are compared with those obtained with different *a priori* knowledge and are confronted with the noncircular and circular complex Gaussian CRBs and with different deterministic CRBs presented in [9]. It is shown in particular that the CRBs under the noncircular (respectively, circular) complex Gaussian distribution are tight upper bounds on the CRBs under the BPSK (respectively, QPSK) distribution at very low and very high SNRs only. Finally, the case of two independent BPSK distributed sources is dealt with. Due to the computational complexity, an explicit expression of the DOA parameters alone is given for large SNR only. Furthermore, numerical comparisons of the stochastic CRBs associated with BPSK and noncircular Gaussian distributed sources are given for all SNRs. In particular, they show that the CRB under the noncircular Gaussian distribution is a very loose upper bound on the CRB under the BPSK distribution. And the difference between these CRBs is more prominent for small DOA and phase separation.

The following notations are used throughout this paper. Matrices and vectors are represented by bold upper case and bold

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lower case characters, respectively. Vectors are by default in column orientation, while T, H, $*$, $\Re$, $\Im$, and i stand for transpose, conjugate transpose, conjugate, real and imaginary part, and $\sqrt{-1}$, respectively. $\otimes$ is the standard Kronecker product of matrices. The symbol 1_A denotes the indicator function of the condition A, which assumes the value 1 if this condition is satisfied and 0 otherwise, and the symbol $\mathcal{N}(m;v)$ denotes the univariate Gaussian distribution with mean m and variance v.

II. Data Model

Consider a BPSK or QPSK modulated signal impinging on an arbitrary array of M sensors. We assume that the array is perfectly calibrated for which the steering vector is a kwown function of the source's DOA. The received signals are bandpass filtered, and after downshifting the sensor signal to baseband, the in-phase and quadrature components are paired to obtain complex signals. We assume Nyquist shaping and ideal sample timing so that the intersymbol interference at each symbol spaced sampling instance can be ignored. In the absence of frequency offset but with possible phase offset, the signals at the output of the matched filter can be represented as

$$\mathbf{y}_t = s_t\mathbf{a}_1 + \mathbf{n}_t \qquad t = 1,\ldots,T$$

where $\mathbf{a}_1$ is the steering vector parametrized by the scalar DOA parameter θ_1. We suppose $\|\mathbf{a}_1\|^2 = M$. $s_t = \sigma_1 e^{i\phi_1}\epsilon_t$ where $(\epsilon_t)_{t=1,\ldots,T}$ are independent identically distributed (i.i.d.) random symbols taking values ± 1 (respectively, $\pm\sqrt{2}/2 \pm i\sqrt{2}/2$) with equal probabilities for BPSK (respectively, QPSK) modulations, where ϕ_1 and σ_1 are considered as unknown parameters. The symbols ϵ_t are assumed to be independent from $\mathbf{n}_t$. $(\mathbf{n}_t)_{t=1,\ldots,T}$ are i.i.d. M-variate zero-mean complex circular Gaussian random vectors with $\mathrm{E}(\mathbf{n}_t\mathbf{n}_t^H) = \sigma_n^2\mathbf{I}_M$. Consequently, $(\mathbf{y}_t)_{t=1,\ldots,T}$ are i.i.d. M-dimensional random variable whose probability density function (pdf) is mixed circular Gaussian

$$p(\mathbf{y}_t;\boldsymbol{\alpha}) = \frac{1}{L\pi^M\sigma_n^{2M}}\sum_{l=1}^{L} e^{-\|\mathbf{y}_t-\sigma_1 e^{i\phi_1}\epsilon_l\mathbf{a}_1\|^2/\sigma_n^2} \qquad (2.1)$$

with $L = 2$ and $\epsilon_l = \pm 1$ (respectively, $L = 4$ and $\epsilon_l = \pm\sqrt{2}/2 \pm i\sqrt{2}/2$) for BPSK (respectively, QPSK) modulated signals and where $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (\sigma_n, \sigma_1, \phi_1, \theta_1)^T$.

III. Stochastic CRB for BPSK and QPSK Signals

The main result of this paper, proved in Appendix A, is contained in the following theorem.

Theorem 1: The FIM associated with the parameter $(\sigma_n, \sigma_1, \phi_1, \theta_1)$ of stochastic BPSK and QPSK modulated signals is given by the explicit closed-form expressions

$$\mathbf{I}_{\mathrm{F}}^{\mathrm{BPSK}} = T\begin{bmatrix}\mathbf{I}_{\mathrm{F}_1}^{\mathrm{BPSK}} & \mathbf{O}\\ \mathbf{O} & \mathbf{I}_{\mathrm{F}_2}^{\mathrm{BPSK}}\end{bmatrix}$$
$$\mathbf{I}_{\mathrm{F}}^{\mathrm{QPSK}} = T\begin{bmatrix}\mathbf{I}_{\mathrm{F}_1}^{\mathrm{QPSK}} & \mathbf{O}\\ \mathbf{O} & \mathbf{I}_{\mathrm{F}_2}^{\mathrm{QPSK}}\end{bmatrix}$$

with the equation shown at the bottom of the page with $\rho \stackrel{\text{def}}{=} M\sigma_1^2/\sigma_n^2$ and $\mathbf{a}_1' \stackrel{\text{def}}{=} d\mathbf{a}_1/d\theta_1$ and where f_1 and f_2 are the following decreasing function of ρ: $f_1(\rho) \stackrel{\text{def}}{=} (e^{-\rho}/\sqrt{2\pi})\int_{-\infty}^{+\infty}(u^2 e^{-u^2/2}/\cosh(u\sqrt{2\rho}))du$, $f_2(\rho) \stackrel{\text{def}}{=} (e^{-\rho}/\sqrt{2\pi})\int_{-\infty}^{+\infty}(e^{-u^2/2}/\cosh(u\sqrt{2\rho}))du$.

We note the similarity of the 2×2 top left corner of these FIMs with those derived in [10] used for the estimation of the SNR of BPSK and QPSK modulated signals. Because these Fisher information matrices are block diagonal, the following explicit expressions for the CRB for the parameter DOA alone are easily derived:

$$\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1) = \frac{1}{T}\left(\frac{1}{\gamma_1}\frac{\sigma_n^2}{\sigma_1^2}\right)\left(\frac{1}{1-f_2(\rho)}\right) \qquad (3.1)$$

$$\mathrm{CRB}_{\mathrm{QPSK}}(\theta_1) = \frac{1}{T}\left(\frac{1}{\gamma_1}\frac{\sigma_n^2}{\sigma_1^2}\right)\left(\frac{1}{1-f_2\left(\frac{\rho}{2}\right)}\right) \qquad (3.2)$$

where γ_1 is the purely geometrical factor $2\mathbf{a}_1'^H\mathbf{\Pi}_{\mathbf{a}_1}^{\perp}\mathbf{a}_1'$ with $\mathbf{\Pi}_{\mathbf{a}_1}^{\perp} \stackrel{\text{def}}{=} \mathbf{I}_M - \mathbf{a}_1\mathbf{a}_1^H/M$. We note that thanks to the decreasing function f_2, $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1) < \mathrm{CRB}_{\mathrm{QPSK}}(\theta_1)$.

$$\mathbf{I}_{\mathrm{F}_1}^{\mathrm{BPSK}} = \begin{bmatrix}\frac{4M}{\sigma_n^2}\left(1-\frac{2\sigma_1^2}{\sigma_n^2}f_1(\rho)\right) & \frac{4M\sigma_1}{\sigma_n^3}f_1(\rho)\\ \frac{4M\sigma_1}{\sigma_n^3}f_1(\rho) & \frac{2M}{\sigma_n^2}(1-f_1(\rho))\end{bmatrix}$$

$$\mathbf{I}_{\mathrm{F}_2}^{\mathrm{BPSK}} = \begin{bmatrix}\frac{2M\sigma_1^2}{\sigma_n^2}(1-f_2(\rho)) & \frac{2\sigma_1^2(i\mathbf{a}_1'^H\mathbf{a}_1)}{\sigma_n^2}(1-f_2(\rho))\\ \frac{2\sigma_1^2(i\mathbf{a}_1'^H\mathbf{a}_1)}{\sigma_n^2}(1-f_2(\rho)) & \frac{2\sigma_1^2\|\mathbf{a}_1'\|^2}{\sigma_n^2}(1-f_2(\rho))\end{bmatrix}$$

$$\mathbf{I}_{\mathrm{F}_1}^{\mathrm{QPSK}} = \begin{bmatrix}\frac{4M}{\sigma_n^2}\left(1-\frac{2\sigma_1^2}{\sigma_n^2}f_1\left(\frac{\rho}{2}\right)\right) & \frac{4M\sigma_1}{\sigma_n^3}f_1\left(\frac{\rho}{2}\right)\\ \frac{4M\sigma_1}{\sigma_n^3}f_1\left(\frac{\rho}{2}\right) & \frac{2M}{\sigma_n^2}\left(1-f_1\left(\frac{\rho}{2}\right)\right)\end{bmatrix}$$

$$\mathbf{I}_{\mathrm{F}_2}^{\mathrm{QPSK}} = \begin{bmatrix}\frac{2M\sigma_1^2}{\sigma_n^2}\left(1-(1+\rho)f_2\left(\frac{\rho}{2}\right)\right) & \frac{2\sigma_1^2(i\mathbf{a}_1'^H\mathbf{a}_1)}{\sigma_n^2}\left(1-(1+\rho)f_2\left(\frac{\rho}{2}\right)\right)\\ \frac{2\sigma_1^2(i\mathbf{a}_1'^H\mathbf{a}_1)}{\sigma_n^2}\left(1-(1+\rho)f_2\left(\frac{\rho}{2}\right)\right) & \frac{2\sigma_1^2\|\mathbf{a}_1'\|^2}{\sigma_n^2}\left(1-(1+\frac{\rho}{M}\frac{|\mathbf{a}_1^H\mathbf{a}_1'|^2}{\|\mathbf{a}_1'\|^2})f_2\left(\frac{\rho}{2}\right)\right)\end{bmatrix}$$

In the absence of phase offset or after correcting it (i.e., parameter ϕ_1 is known), these CRBs for θ_1 become[1]

$$\mathrm{CRB}^{\mathrm{CO}}_{\mathrm{BPSK}}(\theta_1)=\frac{1}{T}\left(\frac{1}{2\|\mathbf{a}'_1\|^2}\frac{\sigma_n^2}{\sigma_1^2}\right)\left(\frac{1}{1-f_2(\rho)}\right) \quad (3.3)$$

$$\mathrm{CRB}^{\mathrm{CO}}_{\mathrm{QPSK}}(\theta_1)=\frac{1}{T}\left(\frac{1}{2\|\mathbf{a}'_1\|^2}\frac{\sigma_n^2}{\sigma_1^2}\right)\left(\frac{1}{1-\left(1+\frac{\rho}{M}\frac{|\mathbf{a}_1^H\mathbf{a}'_1|^2}{\|\mathbf{a}'_1\|^2}\right)f_2\left(\frac{\rho}{2}\right)}\right). \quad (3.4)$$

Comparing (3.3) to (3.1), we note that the phase information for a BPSK source is quite informative for all SNRs because $2\|\mathbf{a}'_1\|^2 > \gamma_1$ for the DOA perspective contrary to a QPSK source, as shown in Section V-A (Fig. 3).

IV. Comparison With Related CRBs

Depending on the presence of different *a priori* information on the parameters and on the distribution of the sources, several CRBs can be considered.

A. Data-Aided CRB

If we assume that transmitted symbols are known at the array receiver, the modulation can be removed perfectly and the resulting signal $\mathbf{y}_t$ is circular Gaussian distributed with mean $\sigma_1 e^{i\phi_1}\epsilon_t\mathbf{a}_1$ and covariance $\sigma_n^2\mathbf{I}_M$. Consequently, $(\mathbf{y}_1^T,\ldots,\mathbf{y}_T^T)^T$ is circular Gaussian distributed as well, with mean $\boldsymbol{\epsilon}\otimes\sigma_1 e^{i\phi_1}\mathbf{a}_1$ and covariance $\sigma_n^2\mathbf{I}_{MT}$, where $\boldsymbol{\epsilon}\stackrel{\mathrm{def}}{=}(\epsilon_1,\ldots,\epsilon_T)^T$. Applying the Slepian–Bangs formula (see, e.g., [5, (B.3.25)]), we obtain

$$\left(\mathbf{I}_{\mathrm{F}}^{\mathrm{DA}}\right)_{k,l}=\frac{1}{\sigma_n^4}\mathrm{Tr}\left(\frac{\partial\sigma_n^2\mathbf{I}_{MT}}{\partial\alpha_k}\frac{\partial\sigma_n^2\mathbf{I}_{MT}}{\partial\alpha_l}\right)+\frac{2}{\sigma_n^2}\sum_{t=1}^{T}|\epsilon_t|^2\Re\left(\frac{\partial\sigma_1 e^{-i\phi_1}\mathbf{a}_1^H}{\partial\alpha_k}\frac{\partial\sigma_1 e^{i\phi_1}\mathbf{a}_1}{\partial\alpha_l}\right)$$

then the following FIM is straightforwardly derived by noting that $(1/T)\sum_{t=1}^T|\epsilon_t|^2=1$ and $\|\mathbf{a}_1\|^2=M$, which implies $\mathbf{a}_1^H\mathbf{a}'_1+(\mathbf{a}_1^H\mathbf{a}'_1)^H=0$ and consequently $\Re(\mathbf{a}_1^H\mathbf{a}'_1)=0$ and $\Im(\mathbf{a}_1^H\mathbf{a}'_1)=i\mathbf{a}_1'^H\mathbf{a}_1$

$$\mathbf{I}_{\mathrm{F}}^{\mathrm{DA}}=T\begin{bmatrix}\frac{4M}{\sigma_n^2} & 0 & 0 & 0\\ 0 & \frac{2M}{\sigma_n^2} & 0 & 0\\ 0 & 0 & \frac{2M\sigma_1^2}{\sigma_n^2} & \frac{2\sigma_1^2(i\mathbf{a}_1'^H\mathbf{a}_1)}{\sigma_n^2}\\ 0 & 0 & \frac{2\sigma_1^2(i\mathbf{a}_1'^H\mathbf{a}_1)}{\sigma_n^2} & \frac{2\sigma_1^2\|\mathbf{a}'_1\|^2}{\sigma_n^2}\end{bmatrix}$$

and consequently

$$\mathrm{CRB}_{\mathrm{DA}}(\theta_1)=\frac{1}{T}\left(\frac{1}{\gamma_1}\frac{\sigma_n^2}{\sigma_1^2}\right).$$

Because $\lim_{\rho\to\infty}f_1(\rho)=\lim_{\rho\to\infty}f_2(\rho)=0$, we note that the FIMs $\mathbf{I}_{\mathrm{F}}^{\mathrm{BPSK}}$ and $\mathbf{I}_{\mathrm{F}}^{\mathrm{QPSK}}$ of Section III approach $\mathbf{I}_{\mathrm{F}}^{\mathrm{DA}}$ and consequently $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ and $\mathrm{CRB}_{\mathrm{QPSK}}(\theta_1)$ approach $\mathrm{CRB}^{\mathrm{DA}}(\theta_1)$ for large SNR values. On the contrary, because $2\|\mathbf{a}'_1\|^2 > \gamma_1$, $\mathrm{CRB}^{\mathrm{CO}}_{\mathrm{BPSK}}(\theta_1)$ and $\mathrm{CRB}^{\mathrm{CO}}_{\mathrm{QPSK}}(\theta_1)$ are lower than $\mathrm{CRB}^{\mathrm{DA}}(\theta_1)$ for large SNR values. The phase information is quite informative compared with the training symbols from the DOA perspective.

B. Deterministic CRB

For the deterministic or conditional model of the signal waveform, the CRB for θ_1 does not depend on the realization of $(\epsilon_t)_{t=1,\ldots,T}$ because $(1/T)\sum_{t=1}^T|\epsilon_t|^2=1$ and is given by (see, e.g., [4, (2.11)])

$$\mathrm{CRB}_{\mathrm{DET}}(\theta_1)=\frac{1}{T}\left(\frac{1}{\gamma_1}\frac{\sigma_n^2}{\sigma_1^2}\right).$$

We note that if side information is available such as the constant modulus of BPSK and QPSK modulation [11] and some training symbols among the T symbols ϵ_t [9, (50)], the previous CRB for θ_1 is preserved. Furthermore, the data-aided and deterministic CRB are the same, implying that knowing the signal or not is not important.

C. Stochastic Complex Gaussian CRB

Because the BPSK (respectively, QPSK) modulation is noncircular (respectively, circular) complex to the second-order, it makes sense to compare the stochastic CRBs (3.1) and (3.2) to the CRBs associated with, respectively, noncircular[2] (NCG) [6, (3.14)] or circular (CG) complex Gaussian distribution that can be considered as upper bounds on the true stochastic CRBs (see, e.g., [5, p. 293]). More precisely, after recalling these CRBs under Gaussian distributions for the convenience of the reader

$$\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)=\frac{1}{T}\left(\frac{1}{\gamma_1}\left[\frac{\sigma_n^2}{\sigma_1^2}+\frac{1}{2M}\frac{\sigma_n^4}{\sigma_1^4}\right]\right)$$

$$\mathrm{CRB}_{\mathrm{CG}}(\theta_1)=\frac{1}{T}\left(\frac{1}{\gamma_1}\left[\frac{\sigma_n^2}{\sigma_1^2}+\frac{1}{M}\frac{\sigma_n^4}{\sigma_1^4}\right]\right)$$

we have

$$\frac{\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)}{\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)}=\frac{1}{(1-f_2(\rho))\left(1+\frac{1}{2\rho}\right)}$$

$$\frac{\mathrm{CRB}_{\mathrm{QPSK}}(\theta_1)}{\mathrm{CRB}_{\mathrm{CG}}(\theta_1)}=\frac{1}{\left(1-f_2\left(\frac{\rho}{2}\right)\right)\left(1+\frac{1}{\rho}\right)}.$$

We note that these ratios depend on $\rho\stackrel{\mathrm{def}}{=}M\sigma_1^2/\sigma_n^2$ only and tend to one when ρ tends to ∞. However, this dependence in ρ is not monotone, as is numerically shown in the next section.

V. Numerical Examples

The purpose of this section is to illustrate the results of Section IV and to extend them to the case of two independent BPSK distributed sources. We consider throughout this section one or two independent sources impinging on a uniform linear array

[1]Where the exponent CO of $\mathrm{CRB}^{\mathrm{CO}}_{\mathrm{BPSK}}(\theta_1)$ and $\mathrm{CRB}^{\mathrm{CO}}_{\mathrm{QPSK}}(\theta_1)$ means coherent.

[2]Because $\mathrm{E}(\epsilon_t^2)=\mathrm{E}|\epsilon_t^2|$ for the BPSK modulation, we consider the noncircular complex Gaussian distribution associated with $\mathrm{E}(\epsilon_t^2)=\mathrm{E}|\epsilon_t^2|=1$, i.e., with $\rho_1=1$ in [6, (3.14)] where the noncircularity rate ρ_1 and the circularity phase ϕ_1 of ϵ_t are defined here by $\mathrm{E}(\epsilon_t^2)=\rho_1 e^{2i\phi_1}\mathrm{E}|\epsilon_t^2|$.

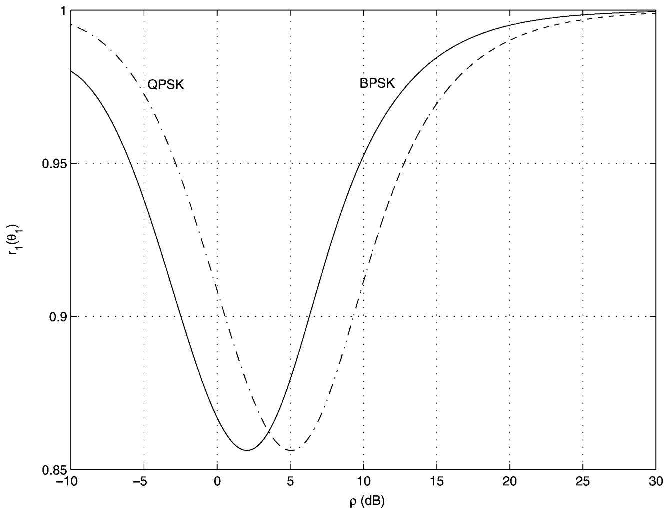

Fig. 1. Ratios $r_1(\theta_1) \stackrel{\text{def}}{=} \text{CRB}_{\text{BPSK}}(\theta_1)/\text{CRB}_{\text{NCG}}(\theta_1)$ and $r_1(\theta_1) \stackrel{\text{def}}{=} \text{CRB}_{\text{QPSK}}(\theta_1)/\text{CRB}_{\text{CG}}(\theta_1)$ as a function of $\rho \stackrel{\text{def}}{=} M\sigma_1^2/\sigma_n^2$.

of M sensors spaced a half-wavelength apart for which $\mathbf{a}_k = (1, e^{i\theta_k}, \ldots, e^{i(M-1)\theta_k})^T$.

A. Single Source Case

The first experiment illustrates the results of Section IV. Fig. 1 shows the ratios $\text{CRB}_{\text{BPSK}}(\theta_1)/\text{CRB}_{\text{NCG}}(\theta_1)$ and $\text{CRB}_{\text{QPSK}}(\theta_1)/\text{CRB}_{\text{CG}}(\theta_1)$ as a function of $\rho \stackrel{\text{def}}{=} M\sigma_1^2/\sigma_n^2$. We see from that figure that the CRBs under the noncircular (respectively, circular) complex Gaussian distribution are tight upper bounds on the CRBs under the BPSK (respectively, QPSK) distribution at very low and very high SNRs only.

Figs. 2 and 3 show the different CRBs[3] as a function of the SNR for the BPSK and QPSK modulations, respectively. From these figures, we see that the CRBs achieved in the absence of phase offset outperform all the other CRBs except for very low SNRs. It is shown that there is no significant difference between the bounds for non-data-aided and data-aided estimations except at low SNR.

B. Two Sources Case

We consider now two independent BPSK or QPSK distributed sources. Because the pdf of $\mathbf{y}_t$ is a mixture of 4 or 16 Gaussian pdfs, the associated stochastic CRB appears to be prohibitive to compute. Consequently, we use a numerical approximation derived from the strong law of large numbers, i.e.,

$$\text{CRB}_{\text{PSK}}(\theta_1, \theta_2) = \left(\mathbf{I}_{\text{F}}^{-1}\right)_{([4\ 7],[4\ 7])}$$

$$\frac{1}{T}\left(\mathbf{I}_{\text{F}}\right)_{k,l} = \lim_{T'\to\infty} \frac{1}{T'} \sum_{t=1}^{T'} \left(\frac{\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial \alpha_k}\right)\left(\frac{\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial \alpha_l}\right) \quad (5.1)$$

[3]All the CRBs are computed for $T = 1$. That means that the actual CRBs associated with the signal model defined in Section II are obtained from the results given in this section by dividing by T.

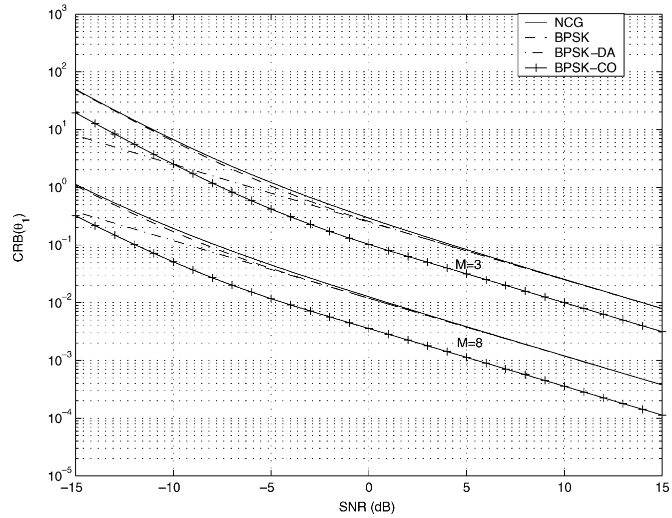

Fig. 2. Normalized ($T = 1$) CRBs for BPSK modulation: $\text{CRB}_{\text{NCG}}(\theta_1)$, $\text{CRB}_{\text{BPSK}}(\theta_1)$, $\text{CRB}_{\text{BPSK}}^{\text{DA}}(\theta_1)$, and $\text{CRB}_{\text{BPSK}}^{\text{CO}}(\theta_1)$ as a function of the SNR.

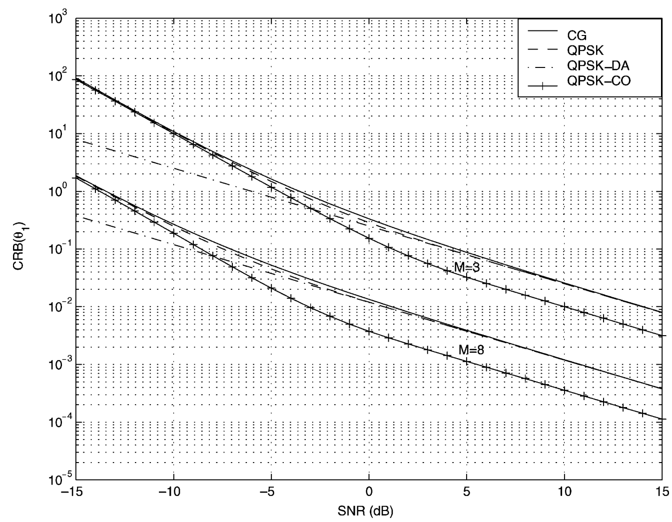

Fig. 3. Normalized ($T = 1$) CRBs for QPSK modulation: $\text{CRB}_{\text{CG}}(\theta_1)$, $\text{CRB}_{\text{QPSK}}(\theta_1)$, $\text{CRB}_{\text{QPSK}}^{\text{DA}}(\theta_1)$, and $\text{CRB}_{\text{QPSK}}^{\text{CO}}(\theta_1)$ as a function of the SNR.

where $\left(\mathbf{I}_{\text{F}}^{-1}\right)_{([4\ 7],[4\ 7])}$ denotes the 2×2 submatrix of $\mathbf{I}_{\text{F}}^{-1}$ derived from the rows and columns 4 and 7

$$p(\mathbf{y}_t;\boldsymbol{\alpha}) = \frac{1}{L^2\pi^M\sigma_n^{2M}} \sum_{j=1}^{L^2} e^{-\|\mathbf{y}_t - \mathbf{A}\mathbf{s}_j\|^2/\sigma_n^2}$$

$$\text{with } \mathbf{s}_j \stackrel{\text{def}}{=} (\sigma_1\eta_{j,1}e^{i\phi_1}, \sigma_2\eta_{j,2}e^{i\phi_2})^T$$

with $L = 2$ (respectively, $L = 4$) for BPSK (respectively, QPSK) modulated signals where $(\eta_{j,1}, \eta_{j,2})_{j=1\ldots,L^2} = (\pm 1, \pm 1)$ (respectively, $(\pm\sqrt{2}/2 \pm i\sqrt{2}/2, \pm\sqrt{2}/2 \pm i\sqrt{2}/2)$) and where $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (\sigma_n, \sigma_1, \phi_1, \theta_1, \sigma_2, \phi_2, \theta_2)^T$ and $\mathbf{A} \stackrel{\text{def}}{=} (\mathbf{a}_1, \mathbf{a}_2)$. At high SNRs (more precisely for $M(\sigma_1^2/\sigma_n^2) \gg 1$ and $M(\sigma_2^2/\sigma_n^2) \gg 1$) it is proved in Appendix B that the FIMs associated with BPSK and QPSK signals are approximated by the same following explicit expression as shown in (5.2) at the bottom of the next page.

We clearly see that the entries corresponding to sources 1 and 2 are decoupled. Consequently, for large SNRs and independent sources, the CRB for the DOA of one source is independent of the parameters of the other source and

$$\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1,\theta_2)\approx \mathrm{CRB}_{\mathrm{QPSK}}(\theta_1,\theta_2)\approx \frac{1}{T}\begin{bmatrix}\frac{1}{\gamma_1}\frac{\sigma_n^2}{\sigma_1^2} & 0\\ 0 & \frac{1}{\gamma_2}\frac{\sigma_n^2}{\sigma_2^2}\end{bmatrix}$$

where $\gamma_i \stackrel{\mathrm{def}}{=} 2\mathbf{a}_i'^H \mathbf{\Pi}_{\mathbf{a}_i}^{\perp}\mathbf{a}_i'$, $i=1,2$. This quite curious result reminds one of a similar result in cissoid parameter estimation (see, e.g., [12]) where the asymptotic CRB of the frequencies is independent of the frequency separation. Furthermore, the CRBs $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_k)\approx \mathrm{CRB}_{\mathrm{QPSK}}(\theta_k)$ and $\mathrm{CRB}_{\mathrm{BPSK}}^{\mathrm{CO}}(\theta_k)\approx \mathrm{CRB}_{\mathrm{QPSK}}^{\mathrm{CO}}(\theta_k)$ for each DOA are those of the single source case (because the 4×4 top left corner of the FIM (5.2) is the FIM given in Theorem 1 for high SNR). We note that this property is quite different from the behavior of the CRB under the Gaussian distribution and the deterministic CRB, for which the CRB for the DOA of one source depends on the DOA separation. More precisely, it is proved [4, R9] that these latter two CRBs tend to the same limit as all SNRs increase. For independent sources, they are given by (from, e.g., [4, (2.13)])

$$\mathrm{CRB}_{\mathrm{DET}}(\theta_1,\theta_2) = \mathrm{CRB}_{\mathrm{CG}}(\theta_1,\theta_2)=\frac{1}{T}\begin{bmatrix}\frac{1}{\beta_1}\frac{\sigma_n^2}{\sigma_1^2} & 0\\ 0 & \frac{1}{\beta_2}\frac{\sigma_n^2}{\sigma_2^2}\end{bmatrix}$$

with

$$\beta_k \stackrel{\mathrm{def}}{=} 2\left(\|\mathbf{a}_k'\|^2-\gamma_k(\theta_1,\theta_2)\right),\; k=1,2$$

where

$$\gamma_k(\theta_1,\theta_2) \stackrel{\mathrm{def}}{=} \frac{M(|\mathbf{a}_k'^H\mathbf{a}_k|^2+|\mathbf{a}_k'^H\mathbf{a}_{3-k}|^2)-2\Re(\mathbf{a}_k^H\mathbf{a}_k'\mathbf{a}_{3-k}^H\mathbf{a}_k\mathbf{a}_k'^H\mathbf{a}_{3-k})}{M^2-|\mathbf{a}_1^H\mathbf{a}_2|^2},\quad k=1,2.$$

If the transmitted symbols are known, the derivation of the FIM $\mathbf{I}_{\mathrm{F}}^{\mathrm{DA}}$ follows the same lines that for the single source case. And more specifically, because $\lim_{T\to\infty}(1/T)\sum_{t=1}^{T}\epsilon_{t,1}\epsilon_{t,2}=\mathrm{E}(\epsilon_{t,1}\epsilon_{t,2})=0$, the parameter associated with the two sources is decoupled, and it is straightforward to prove that the asymptotic (for $T\gg 1$) FIM $\mathbf{I}_{\mathrm{F}}^{\mathrm{DA}}$ is given by (5.2) as well.

The second experiment considers two independent and equipowered BPSK or QPSK distributed sources. Fig. 4 compares $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ given by (5.1) with the CRB under the noncircular complex Gaussian distribution. And to be fair, this comparison must be done under the same *a priori* that the two sources are independent with $(\rho_k=1)_{k=1,2}$, i.e., with the same parameter $\boldsymbol{\alpha} \stackrel{\mathrm{def}}{=} (\sigma_n,\sigma_1,\phi_1,\theta_1,\sigma_2,\phi_2,\theta_2)^T$. For that reason, we use the nonexplicit expression of $\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)$ obtained in [13] which can take this *a priori* information into account.

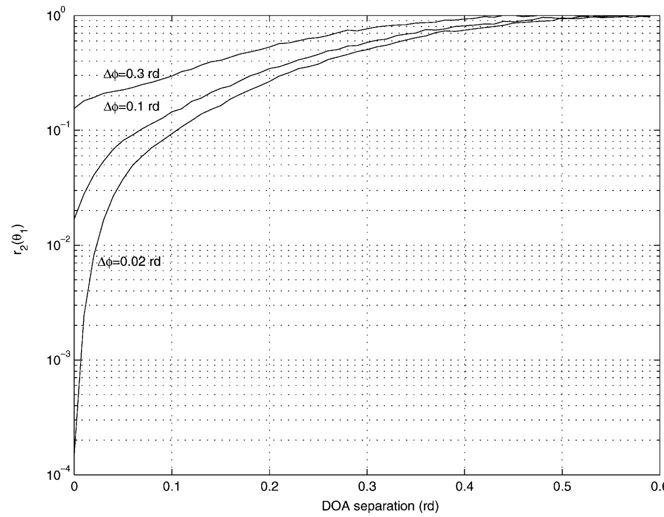

Fig. 4. Ratio $r_2(\theta_1) \stackrel{\mathrm{def}}{=} \mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)/\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)$ as a function of the DOA separation for different values of the phase separation $\Delta\phi$ for $M=6$ and SNR$=20$ dB.

Fig. 4 exhibits the ratio $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)/\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)$ as a function of the DOA separation $\theta_2-\theta_1$ for three values of the phase separation $\Delta\phi \stackrel{\mathrm{def}}{=} \phi_2-\phi_1$. We see that the CRB is very sensitive to the phase separation except for large DOA separation. This figure shows that, contrary to the single source case, the CRB under the noncircular complex Gaussian distribution is a very loose upper bound on the CRB under the BPSK distribution except for large values of the DOA and phase separation.

Fig. 5 exhibits the domain of validity of the high SNR approximation. We see from this figure that this domain depends not only on M, SNR, and DOA separation, but also on the distributed sources. It is shown that this domain reduces for QPSK sources compared with BPSK sources. For example, for $M=6$, the threshold is about 5 dB [respectively, 8 dB] for $\Delta\theta=0.1rd$ and 8 dB [respectively, 12 dB] for $\Delta\theta=0.5rd$ for BPSK [respectively, QPSK] sources. The larger the DOA separation is or the larger M is, the larger is the domain of validity of the approximation.

$$T\begin{bmatrix}
\frac{4M}{\sigma_n^2} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \frac{2M}{\sigma_n^2} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{2M\sigma_1^2}{\sigma_n^2} & \frac{2\sigma_1^2(i\mathbf{a}_1'^H\mathbf{a}_1)}{\sigma_n^2} & 0 & 0 & 0\\
0 & 0 & \frac{2\sigma_1^2(i\mathbf{a}_1'^H\mathbf{a}_1)}{\sigma_n^2} & \frac{2\sigma_1^2\|\mathbf{a}_1'\|^2}{\sigma_n^2} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \frac{2M}{\sigma_n^2} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{2M\sigma_2^2}{\sigma_n^2} & \frac{2\sigma_2^2(i\mathbf{a}_2'^H\mathbf{a}_2)}{\sigma_n^2}\\
0 & 0 & 0 & 0 & 0 & \frac{2\sigma_2^2(i\mathbf{a}_2'^H\mathbf{a}_2)}{\sigma_n^2} & \frac{2\sigma_2^2\|\mathbf{a}_2'\|^2}{\sigma_n^2}
\end{bmatrix}. \tag{5.2}$$

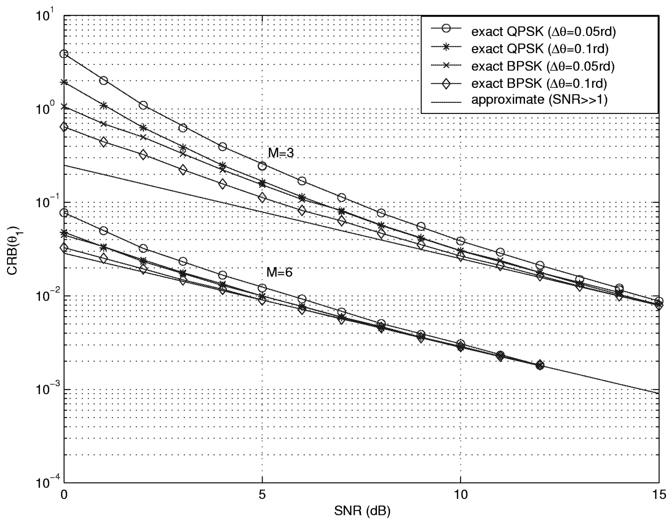

Fig. 5. Approximate and exact value (obtained due to (5.1) with $T' = 10\,000$) of $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ and $\mathrm{CRB}_{\mathrm{QPSK}}(\theta_1)$ as a function of the SNR for different values of the DOA separation.

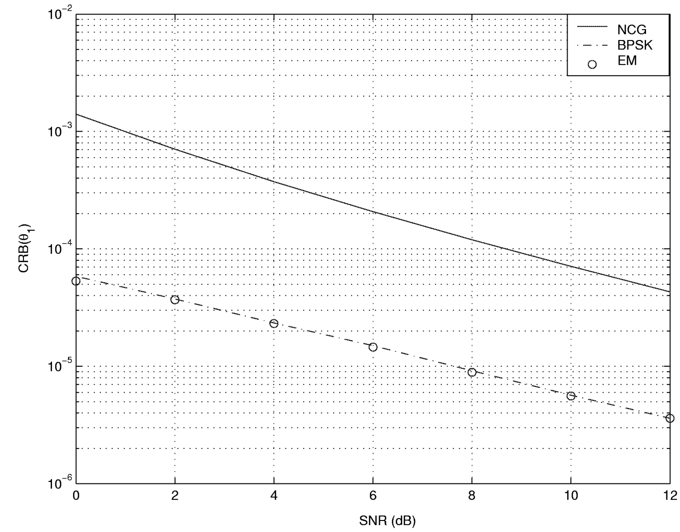

Fig. 7. $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$, $\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)$, and estimated MSE $\mathrm{E}(\theta_{1,T} - \theta_1)^2$ given by the deterministic EM algorithm (five iterations) for $\Delta\theta = 0.3rd$ and $\Delta\phi = 0.1rd$, versus SNR.

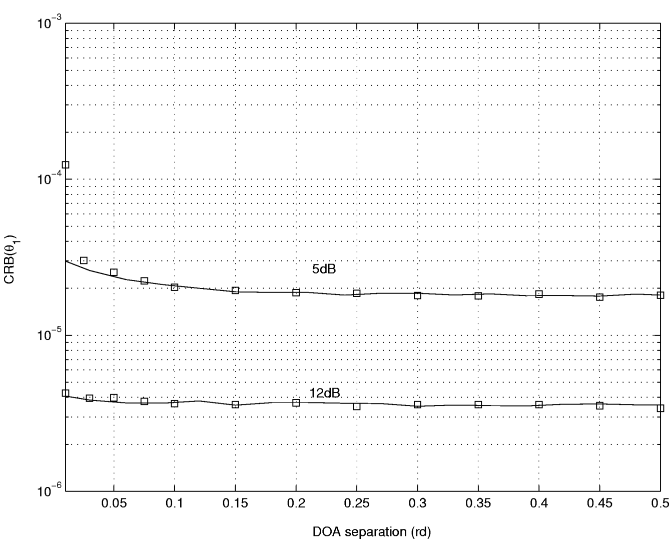

Fig. 6. $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ and estimated MSE $\mathrm{E}(\theta_{1,T} - \theta_1)^2$ given by the deterministic EM algorithm (ten iterations) as a function of the DOA separation for $\Delta\phi = 0.1rd$.

Since the CRB under the noncircular [respectively, circular] Gaussian distribution is a very loose upper bound on the CRB under the discrete BPSK [respectively, QPSK] distribution, specifically for small DOA or phase separation, the ML estimators that take these discrete distributions into account outperform the stochastic ML estimator under the circular Gaussian distribution (see, e.g., [4]) and the weighed subspace fitting estimator (see, e.g., [14]), which both reach $\mathrm{CRB}_{\mathrm{CG}}(\theta_1)$. Consequently, the EM approaches [15] that are iterative procedures capable of implementing the stochastic ML estimator under these discrete distributions outperform the ML estimator under noncircular or circular Gaussian distribution. Fig. 6 exhibits $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ and the estimated mean square error (MSE) $\mathrm{E}(\theta_{1,T} - \theta_1)^2$ given by the deterministic EM algorithm initialized by the estimate given by the MUSIC-like algorithm described in [16], as a function of the DOA separation for two SNRs. We see that contrary to $\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)$, $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ does not increase significantly when decreasing the DOA separation. Fig. 7 compares the MSE $\mathrm{E}(\theta_{1,T} - \theta_1)^2$ given by the deterministic EM algorithm (initialized as in Fig. 6) to $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ and $\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)$, as a function of the SNR. We see from this figure that the EM estimate reaches $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$, which largely outperforms $\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)$. To show the asymptotic domain (domain of sample size and SNR for which $\mathrm{E}(\theta_{1,T} - \theta_1)^2 \approx \mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$), Figs. 8 and 9 compare these CRBs to the estimated mean square error given by the deterministic EM algorithm initialized by values $\theta_{1,0}$ and $\theta_{2,0}$ in the vicinity of θ_1 and θ_2. We see that the estimates given by the ML estimator under the discrete distribution reach $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ in a very large domain (from SNR $= -5$ dB for $T = 500$ and from $T = 5$ for SNR $= 10$ dB) in our scenario.

VI. Conclusion

This paper developed explicit expressions for the stochastic CRB of the DOA parameter estimates for BPSK and QPSK modulated signals corrupted by additive circular complex Gaussian noise. These stochastic CRBs have been compared, on the one hand, with those obtained with different *a priori* knowledge and, on the other hand, with CRBs under the noncircular and circular complex Gaussian distribution and with different deterministic CRBs.

For a single source, we have proved that the CRBs under the noncircular (respectively, circular) complex Gaussian distribution are tight upper bounds on the CRBs under the BPSK (respectively, QPSK) distribution at very low and very high SNR only. And for the case of two independent BPSK (respectively, QPSK) distributed sources, we have exhibited an important difference of behavior of the stochastic CRB compared to those obtained under the noncircular (respectively, circular) Gaussian distribution. Because we have proved that the stochastic CRB for the DOA of one source is independent of the parameters

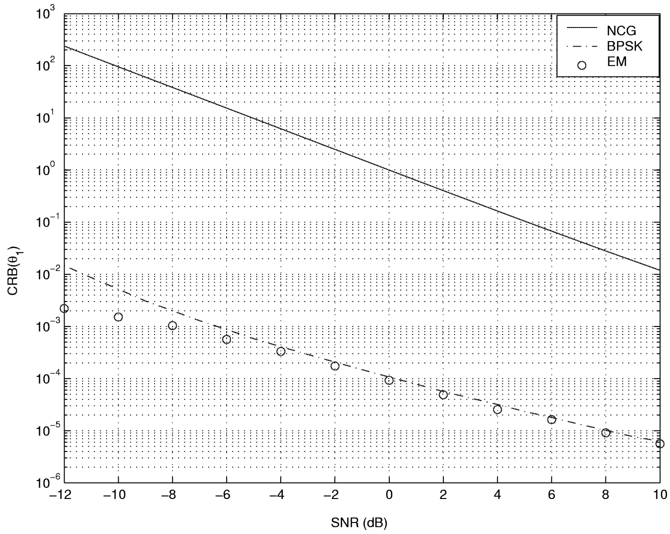

Fig. 8. $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$, $\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)$, and estimated (1000 independent runs) mean square error $\mathrm{E}(\theta_{1,T}-\theta_1)^2$ given by the deterministic EM algorithm (ten iterations) for $T=500$, $M=6$, $\Delta\theta=0.05rd$, and $\Delta\phi=0.1rd$, versus SNR.

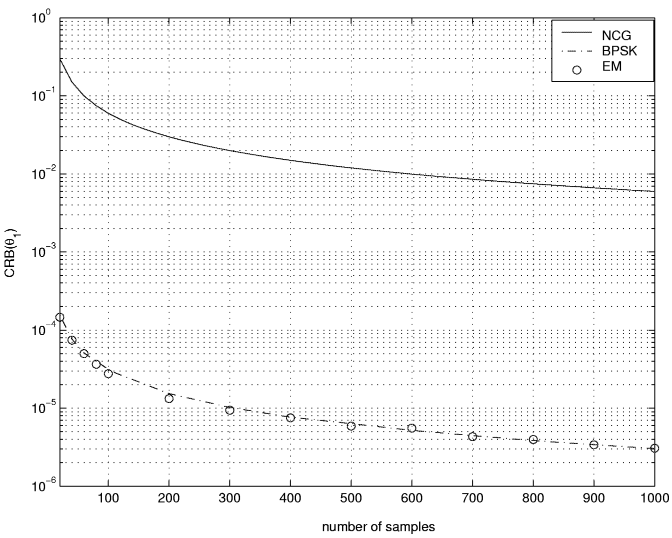

Fig. 9. $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$, $\mathrm{CRB}_{\mathrm{NCG}}(\theta_1)$, and estimated (1000 independent runs) mean square error $\mathrm{E}(\theta_{1,T}-\theta_1)^2$ given by the deterministic EM algorithm (ten iterations), for SNR $=10$ dB, $M=6$, $\Delta\theta=0.05rd$, and $\Delta\phi=0.1rd$, versus T.

of the other source over wide SNR ranges, the CRB under the noncircular (respectively, circular) complex Gaussian distribution is a very loose upper bound on the CRB under the BPSK (respectively, QPSK) distribution except for large values of the DOA and phase separation. Consequently, ML implementations such as the EM approaches outperform the ML estimator under the circular Gaussian distribution, specifically for small DOA or phase separation.

Appendix A
Proof of Theorem 1

The Fisher information matrix is given (elementwise) by

$$\frac{1}{T}\left(\mathbf{I}_{\mathrm{F}}\right)_{k,l}=-\mathrm{E}\left(\frac{\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial\alpha_k\partial\alpha_l}\right)\qquad k,l=1,\ldots,4 \quad \text{(A.1)}$$

where the pdf (2.1) is written after straightforward manipulations as

$$p_{\mathrm{BPSK}}(\mathbf{y}_t;\boldsymbol{\alpha})=\frac{1}{\pi^M\sigma_n^{2M}}e^{-(\|\mathbf{y}_t\|^2+M\sigma_1^2)/\sigma_n^2}\cosh\left(\frac{\sigma_1}{\sigma_n^2}g_1(\mathbf{y}_t)\right)$$
$$p_{\mathrm{QPSK}}(\mathbf{y}_t;\boldsymbol{\alpha})=\frac{1}{\pi^M\sigma_n^{2M}}e^{-(\|\mathbf{y}_t\|^2+M\sigma_1^2)/\sigma_n^2}\cdot\cosh\left(\frac{\sigma_1}{\sigma_n^2\sqrt{2}}g_1(\mathbf{y}_t)\right)\cosh\left(\frac{\sigma_1}{\sigma_n^2\sqrt{2}}k_1(\mathbf{y}_t)\right)$$

with $g_1(\mathbf{y}_t)\stackrel{\text{def}}{=}2\Re(e^{i\phi_1}\mathbf{y}_t^H\mathbf{a}_1)$ and $k_1(\mathbf{y}_t)\stackrel{\text{def}}{=}2\Im(e^{i\phi_1}\mathbf{y}_t^H\mathbf{a}_1)$.

We evaluate (A.1) for the BPSK modulation by taking partial derivatives as follows:

$$\frac{\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial\sigma_1^2}=-\frac{2M}{\sigma_n^2}+\frac{g_1^2(\mathbf{y}_t)}{\sigma_n^4}\frac{1}{\cosh^2\left(\frac{\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^2}\right)}$$

$$\frac{\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial\sigma_1\partial\sigma_n}=\frac{4\sigma_1M}{\sigma_n^3}-\frac{2\sigma_1g_1^2(\mathbf{y}_t)}{\sigma_n^5}\frac{1}{\cosh^2\left(\frac{\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^2}\right)}-\frac{2g_1(\mathbf{y}_t)}{\sigma_n^3}\tanh\left(\frac{\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^2}\right)$$

$$\frac{\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial\sigma_n^2}=\frac{2\sigma_n^2M-6(\sigma_1^2M+|\mathbf{y}_t|^2)}{\sigma_n^4}+\frac{4\sigma_1^2g_1^2(\mathbf{y}_t)}{\sigma_n^6}\frac{1}{\cosh^2\left(\frac{\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^2}\right)}+\frac{6\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^4}\tanh\left(\frac{\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^2}\right)$$

$$\frac{\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial\phi_1^2}=\frac{\sigma_1^2k_1^2(\mathbf{y}_t)}{\sigma_n^4}\frac{1}{\cosh^2\left(\frac{\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^2}\right)}-\frac{\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^2}\tanh\left(\frac{\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^2}\right)$$

$$\frac{\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial\sigma_1\partial\phi_1}=-\frac{\sigma_1g_1(\mathbf{y}_t)k_1(\mathbf{y}_t)}{\sigma_n^4}\frac{1}{\cosh^2\left(\frac{\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^2}\right)}-\frac{k_1(\mathbf{y}_t)}{\sigma_n^2}\tanh\left(\frac{\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^2}\right)$$

$$\frac{\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial\sigma_n\partial\phi_1}=\frac{2\sigma_1^2g_1(\mathbf{y}_t)k_1(\mathbf{y}_t)}{\sigma_n^5}\frac{1}{\cosh^2\left(\frac{\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^2}\right)}+\frac{2\sigma_1k_1(\mathbf{y}_t)}{\sigma_n^3}\tanh\left(\frac{\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^2}\right)$$

$$\frac{\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial\sigma_1\partial\theta_1}=\frac{\sigma_1g_1(\mathbf{y}_t)g_1'(\mathbf{y}_t)}{\sigma_n^4}\frac{1}{\cosh^2\left(\frac{\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^2}\right)}+\frac{g_1'(\mathbf{y}_t)}{\sigma_n^2}\tanh\left(\frac{\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^2}\right)$$

$$\frac{\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial\sigma_n\partial\theta_1}=-\frac{2\sigma_1^2g_1(\mathbf{y}_t)g_1'(\mathbf{y}_t)}{\sigma_n^5}\frac{1}{\cosh^2\left(\frac{\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^2}\right)}-\frac{2\sigma_1g_1'(\mathbf{y}_t)}{\sigma_n^3}\tanh\left(\frac{\sigma_1g_1(\mathbf{y}_t)}{\sigma_n^3}\right)$$

$$\frac{\partial^2 \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial\theta_1^2} = \frac{\sigma_1^2 {g_1'}^2(\mathbf{y}_t)}{\sigma_n^4}\frac{1}{\cosh^2\left(\frac{\sigma_1 g_1(\mathbf{y}_t)}{\sigma_n^2}\right)} + \frac{\sigma_1 g_1''(\mathbf{y}_t)}{\sigma_n^2}\tanh\left(\frac{\sigma_1 g_1(\mathbf{y}_t)}{\sigma_n^2}\right)$$

$$\frac{\partial^2 \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial\theta_1\partial\phi_1} = -\frac{\sigma_1^2 g_1'(\mathbf{y}_t)k_1(\mathbf{y}_t)}{\sigma_n^4}\frac{1}{\cosh^2\left(\frac{\sigma_1 g_1(\mathbf{y}_t)}{\sigma_n^2}\right)} - \frac{\sigma_1 k_1'(\mathbf{y}_t)}{\sigma_n^2}\tanh\left(\frac{\sigma_1 g_1(\mathbf{y}_t)}{\sigma_n^2}\right)$$

with $g_1'(\mathbf{y}_t) \stackrel{\text{def}}{=} 2\Re(e^{i\phi_1}\mathbf{y}_t^H\mathbf{a}_1')$, $g_1''(\mathbf{y}_t) \stackrel{\text{def}}{=} 2\Re(e^{i\phi_1}\mathbf{y}_t^H\mathbf{a}_1'')$ and $k_1'(\mathbf{y}_t) \stackrel{\text{def}}{=} 2\Im(e^{i\phi_1}\mathbf{y}_t^H\mathbf{a}_1')$.

Using the regularity condition $(\partial/\partial\alpha_k)\int p(\mathbf{y};\boldsymbol{\alpha})d\mathbf{y} = \int(\partial p(\mathbf{y};\boldsymbol{\alpha})/\partial\alpha_k)d\mathbf{y}$ (see, e.g., [17, rel. (a.9)]) which is fulfilled for finite mixtures of Gaussian distributions, the following property holds: $\mathrm{E}\left(\partial\ln p(\mathbf{y}_t;\boldsymbol{\alpha})/\partial\sigma_1\right) = 0$. With $\partial\ln p(\mathbf{y}_t;\boldsymbol{\alpha})/\partial\sigma_1 = -2M\sigma_1/\sigma_n^2 + (g_1(\mathbf{y}_t)/\sigma_n^2)\tanh(\sigma_1 g_1(\mathbf{y}_t)/\sigma_n^2)$, we obtain

$$\mathrm{E}\left(g_1(\mathbf{y}_t)\tanh\left(\frac{\sigma_1 g_1(\mathbf{y}_t)}{\sigma_n^2}\right)\right) = 2M\sigma_1. \quad \text{(A.2)}$$

This identity enables us to straightforwardly derive the terms $(\mathbf{I}_\mathrm{F})_{(\sigma_n,\sigma_n)}$, $(\mathbf{I}_\mathrm{F})_{(\sigma_n,\sigma_1)}$ and $(\mathbf{I}_\mathrm{F})_{(\sigma_1,\sigma_1)}$ of $\mathbf{I}_\mathrm{F}^{\mathrm{BPSK}}$ thanks to the definition of the function $f_1(\rho)$.

To evaluate $(\mathbf{I}_\mathrm{F})_{(\phi_1,\phi_1)}$, we note that $g_1(\mathbf{y}_t) = 2M\sigma_1\epsilon_t + \left(e^{i\phi_1}\mathbf{n}_t^H\mathbf{a}_1 + e^{-i\phi_1}\mathbf{a}_1^H\mathbf{n}_t\right)$ and $k_1(\mathbf{y}_t) = (1/i)\left(e^{i\phi_1}\mathbf{n}_t^H\mathbf{a}_1 - e^{-i\phi_1}\mathbf{a}_1^H\mathbf{n}_t\right)$. Because ϵ_t and the couple $\left(e^{i\phi_1}\mathbf{n}_t^H\mathbf{a}_1 + e^{-i\phi_1}\mathbf{a}_1^H\mathbf{n}_t, (1/i)(e^{i\phi_1}\mathbf{n}_t^H\mathbf{a}_1 - e^{-i\phi_1}\mathbf{a}_1^H\mathbf{n}_t)\right)$ are independent, and that these last two Gaussian random variables are uncorrelated and consequently independent, the three random variables ϵ_t, $e^{i\phi_1}\mathbf{n}_t^H\mathbf{a}_1 + e^{-i\phi_1}\mathbf{a}_1^H\mathbf{n}_t$ and $(1/i)\left(e^{i\phi_1}\mathbf{n}_t^H\mathbf{a}_1 - e^{-i\phi_1}\mathbf{a}_1^H\mathbf{n}_t\right)$ are collectively independent and thus $g_1(\mathbf{y}_t)$ and $k_1(\mathbf{y}_t)$ are independent. Therefore

$$\mathrm{E}\left(\frac{k_1^2(\mathbf{y}_t)}{\cosh^2\left(\frac{\sigma_1 g_1(\mathbf{y}_t)}{\sigma_n^2}\right)}\right) = \mathrm{E}\left(k_1^2(\mathbf{y}_t)\right)\mathrm{E}\left(\frac{1}{\cosh^2\left(\frac{\sigma_1 g_1(\mathbf{y}_t)}{\sigma_n^2}\right)}\right)$$

with $\mathrm{E}\left(k_1^2(\mathbf{y}_t)\right) = 2M\sigma_n^2$. With $\mathrm{E}\left(1/\cosh^2(\sigma_1 g_1(\mathbf{y}_t)/\sigma_n^2)\right) = f_2(\rho)$ and thanks to the identity (A.2), $(\mathbf{I}_\mathrm{F})_{(\phi_1,\phi_1)}$ is straightforwardly derived.

Noting that $g_1'(\mathbf{y}_t) = e^{i\phi_1}\mathbf{n}_t^H\mathbf{a}_1' + e^{-i\phi_1}{\mathbf{a}_1'}^H\mathbf{n}_t$, thanks to ${\mathbf{a}_1'}^H\mathbf{a}_1 + \mathbf{a}_1^H\mathbf{a}_1' = 0$ derived from $\|\mathbf{a}_1\|^2 = M$. Consequently for the same reason that $g_1(\mathbf{y}_t)$ and $k_1(\mathbf{y}_t)$, $g_1(\mathbf{y}_t)$ and $g_1'(\mathbf{y}_t)$ are independent as well. And because $k_1(\mathbf{y}_t)$ and $g_1'(\mathbf{y}_t)$ are zero-mean, the expectations of the two terms of $\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})/\partial\sigma_1\partial\phi_1$, $\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})/\partial\sigma_n\partial\phi_1$, $\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})/\partial\sigma_1\partial\theta_1$ and $\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})/\partial\sigma_n\partial\theta_1$ vanish and therefore $(\mathbf{I}_\mathrm{F})_{(\sigma_1,\phi_1)} = (\mathbf{I}_\mathrm{F})_{(\sigma_n,\phi_1)} = (\mathbf{I}_\mathrm{F})_{(\sigma_1,\theta_1)} = (\mathbf{I}_\mathrm{F})_{(\sigma_n,\theta_1)} = 0$.

Considering the first term of $\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})/\partial\theta_1^2$, in the same way

$$\mathrm{E}\left(\frac{{g_1'}^2(\mathbf{y}_t)}{\cosh^2\left(\frac{\sigma_1 g_1(\mathbf{y}_t)}{\sigma_n^2}\right)}\right) = \mathrm{E}\left({g_1'}^2(\mathbf{y}_t)\right)\mathrm{E}\left(\frac{1}{\cosh^2\left(\frac{\sigma_1 g_1(\mathbf{y}_t)}{\sigma_n^2}\right)}\right)$$

with $\mathrm{E}({g_1'}^2(\mathbf{y}_t)) = 2\|\mathbf{a}_1'\|^2\sigma_n^2$.

For the first term of $\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})/\partial\theta_1\partial\phi_1$, we note that the random variables ϵ_t and $\left(e^{i\phi_1}\mathbf{n}_t^H\mathbf{a}_1 + e^{-i\phi_1}\mathbf{a}_1^H\mathbf{n}_t, (g_1'(\mathbf{y}_t), k_1(\mathbf{y}_t))\right)$ are independent. And since the zero-mean Gaussian random variables $e^{i\phi_1}\mathbf{n}_t^H\mathbf{a}_1 + e^{-i\phi_1}\mathbf{a}_1^H\mathbf{n}_t$ and $(g_1'(\mathbf{y}_t), k_1(\mathbf{y}_t))$ are independent too, the three random variables ϵ_t, $e^{i\phi_1}\mathbf{n}_t^H\mathbf{a}_1 + e^{-i\phi_1}\mathbf{a}_1^H\mathbf{n}_t$ and $(g_1'(\mathbf{y}_t), k_1(\mathbf{y}_t))$ are collectively independent. Consequently the sum $g_1(\mathbf{y}_t)$ of these first two random variables is independent of the last random variable $(g_1'(\mathbf{y}_t), k_1(\mathbf{y}_t))$ and thus

$$\mathrm{E}\left(\frac{g_1'(\mathbf{y}_t)k_1(\mathbf{y}_t)}{\cosh^2\left(\frac{\sigma_1 g_1(\mathbf{y}_t)}{\sigma_n^2}\right)}\right) = \mathrm{E}(g_1'(\mathbf{y}_t)k_1(\mathbf{y}_t))\,\mathrm{E}\left(\frac{1}{\cosh^2\left(\frac{\sigma_1 g_1(\mathbf{y}_t)}{\sigma_n^2}\right)}\right)$$

with $\mathrm{E}\left(g_1'(\mathbf{y}_t)k_1(\mathbf{y}_t)\right) = -2\sigma_n^2(i{\mathbf{a}_1'}^H\mathbf{a}_1)$ (see Section IV).

Finally, regarding the second term of $\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})/\partial\theta_1^2$ and $\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})/\partial\theta_1\partial\phi_1$, we have to elaborate a little bit. Because $g_1''(\mathbf{y}_t) = -2\|\mathbf{a}_1'\|^2\sigma_1\epsilon_t + x_t''$ with $x_t'' \stackrel{\text{def}}{=} e^{i\phi_1}\mathbf{n}_t^H\mathbf{a}_1'' + e^{-i\phi_1}{\mathbf{a}_1''}^H\mathbf{n}_t$ and $g_1(\mathbf{y}_t) = 2M\sigma_1\epsilon_t + x_t$ with $x_t \stackrel{\text{def}}{=} e^{i\phi_1}\mathbf{n}_t^H\mathbf{a}_1 + e^{-i\phi_1}\mathbf{a}_1^H\mathbf{n}_t$ where (x_t'', x_t) is zero-mean Gaussian distributed, we have[4]

$$g_1''(\mathbf{y}_t) = -\frac{\|\mathbf{a}_1'\|^2}{M}g_1(\mathbf{y}_t) + x_t'$$

with

$$x_t' \stackrel{\text{def}}{=} x_t'' + \frac{\|\mathbf{a}_1'\|^2}{M}x_t$$

where (x_t, x_t') is zero-mean Gaussian distributed with

$$\begin{aligned}\mathrm{E}(x_t x_t') &= \mathrm{E}(x_t x_t'') + \frac{\|\mathbf{a}_1'\|^2}{M}\mathrm{E}(x_t^2)\\ &= -2\|\mathbf{a}_1'\|^2\sigma_n^2 + \frac{\|\mathbf{a}_1'\|^2}{M}(2M\sigma_n^2) = 0\end{aligned}$$

and thus x_t and x_t' are independent. Because $g_1(\mathbf{y}_t) = 2M\sigma_1\epsilon_t + x_t$ where the discrete random variable ϵ_t is independent of the noise random variables x_t', the random variables $g_1(\mathbf{y}_t)$ and x_t' are independent and then

$$\begin{aligned}&\mathrm{E}\left(g_1''(\mathbf{y}_t)\tanh\left(\frac{\sigma_1 g_1(\mathbf{y}_t)}{\sigma_n^2}\right)\right)\\ &= \mathrm{E}\left(-\frac{\|\mathbf{a}_1'\|^2}{M}g_1(\mathbf{y}_t)\tanh\left(\frac{\sigma_1 g_1(\mathbf{y}_t)}{\sigma_n^2}\right)\right) + \mathrm{E}(x_t')\mathrm{E}\left(\tanh\left(\frac{\sigma_1 g_1(\mathbf{y}_t)}{\sigma_n^2}\right)\right)\\ &= -\frac{\|\mathbf{a}_1'\|^2}{M}\mathrm{E}\left(g_1(\mathbf{y}_t)\tanh\left(\frac{\sigma_1 g_1(\mathbf{y}_t)}{\sigma_n^2}\right)\right)\end{aligned}$$

and $(\mathbf{I}_\mathrm{F})_{(\theta_1,\theta_1)}$ follows from identity (A.2). The same approach applies to evaluate $\mathrm{E}\left(k_1'(\mathbf{y}_t)\tanh(\sigma_1 g_1(\mathbf{y}_t)/\sigma_n^2)\right)$ and gives the term $(\mathbf{I}_\mathrm{F})_{(\theta_1,\phi_1)}$.

For the QPSK modulation, evaluating the partial derivatives $\partial^2\ln p(\mathbf{y}_t;\boldsymbol{\alpha})/\partial\alpha_k\partial\alpha_l$ and taking their expectation are derived in the same way, provided the log-likelihoods associated with $g_1(\mathbf{y}_t)$ and $k_1(\mathbf{y}_t)$ are gathered as well as the hypothesis of independence of $\Re(\epsilon_t)$ and $\Im(\epsilon_t)$ is taken into account.

[4]Because $\|\mathbf{a}_1\|^2 = M$ implies $d^2\|\mathbf{a}_1\|^2/d\theta_1^2 = \mathbf{a}_1^H\mathbf{a}_1'' + {\mathbf{a}_1''}^H\mathbf{a}_1 + 2\|\mathbf{a}_1'\|^2 = 0$.

APPENDIX B
PROOF OF (5.2)

The derivation of (5.2) results from the following alternative form of the FIM:

$$\frac{1}{T}(\mathbf{I}_{\mathrm{F}})_{k,l} = \mathrm{E}\left(\frac{\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial \alpha_k}\frac{\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial \alpha_l}\right) \quad k,l=1,\ldots,7. \tag{B.1}$$

We consider only the terms of (B.1) associated with ϕ_1 and ϕ_2 because the other terms follow along the same lines. For the BPSK derivation, the pdf of $\mathbf{y}_t$ is a mixture of four Gaussian pdfs and

$$\frac{\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial \phi_k} = -\frac{1}{\sigma_n^2}\frac{\sum_{j=1}^{4}\frac{\partial h_j}{\partial \phi_k}e^{-h_j/\sigma_n^2}}{\sum_{j=1}^{4}e^{-h_j/\sigma_n^2}} \qquad k=1,2 \tag{B.2}$$

with

$$h_1 = g_1+g_2+g_{1,2},\; h_2 = g_1-g_2-g_{1,2},$$
$$h_3 = -g_1+g_2-g_{1,2} \text{ and } h_4 = -g_1-g_2+g_{1,2}$$

where the random variable $(g_k)_{k=1,2}$ are defined by $g_k \stackrel{\text{def}}{=} 2\Re(e^{i\phi_k}\mathbf{y}_t^H\mathbf{a}_k) = \sigma_k(e^{i\phi_k}\mathbf{y}_t^H\mathbf{a}_k + e^{-i\phi_k}\mathbf{a}_k^H\mathbf{y}_t)$ with $\mathbf{y}_t = \sigma_1 e^{i\phi_1}\epsilon_{t,1}\mathbf{a}_1 + \sigma_2 e^{i\phi_2}\epsilon_{t,2}\mathbf{a}_2 + \mathbf{n}_t$, $k=1,2$ and $g_{1,2} \stackrel{\text{def}}{=} \sigma_1\sigma_2(e^{i(\phi_1-\phi_2)}\mathbf{a}_2^H\mathbf{a}_1 + e^{-i(\phi_1-\phi_2)}\mathbf{a}_1^H\mathbf{a}_2)$.

Because the random variables $(\epsilon_{t,1},\epsilon_{t,2})$ and $\mathbf{n}_t$ are independent, we can condition the random variable $(\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})/\partial\alpha_k)(\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})/\partial\alpha_l)$ with respect to the different couples $(\epsilon_{t,1},\epsilon_{t,2}) = (\eta_{l,1},\eta_{l,2})_{l=1,4}$ of symbols to compute the expectation (B.1). In the following, we prove that among the four exponentials in (B.2), three of them are insignificant with respect to one of them that is dominant. For example, for $(\eta_{l,1},\eta_{l,2})_{l=1} = (-1,-1)$, we have

$$g_k = -2M\sigma_k^2 - g_{1,2} + \sigma_k(e^{i\phi_k}\mathbf{n}_t^H\mathbf{a}_k + e^{-i\phi_k}\mathbf{a}_k^H\mathbf{n}_t), \quad k=1,2$$

and for $M(\sigma_1^2/\sigma_n^2) \gg 1$ and $M(\sigma_2^2/\sigma_n^2) \gg 1$

$$-\frac{h_1}{\sigma_n^2} = \frac{2M(\sigma_1^2+\sigma_2^2)}{\sigma_n^2} - n_1' + \frac{g_{1,2}}{\sigma_n^2} \approx \frac{2M(\sigma_1^2+\sigma_2^2)}{\sigma_n^2} + \frac{g_{1,2}}{\sigma_n^2}$$
$$-\frac{h_2}{\sigma_n^2} = \frac{2M(\sigma_1^2-\sigma_2^2)}{\sigma_n^2} - n_2' + \frac{g_{1,2}}{\sigma_n^2} \approx \frac{2M(\sigma_1^2-\sigma_2^2)}{\sigma_n^2} + \frac{g_{1,2}}{\sigma_n^2}$$
$$-\frac{h_3}{\sigma_n^2} = \frac{2M(\sigma_2^2-\sigma_1^2)}{\sigma_n^2} + n_2' + \frac{g_{1,2}}{\sigma_n^2} \approx \frac{2M(\sigma_2^2-\sigma_1^2)}{\sigma_n^2} + \frac{g_{1,2}}{\sigma_n^2}$$
$$-\frac{h_4}{\sigma_n^2} = -\frac{2M(\sigma_1^2+\sigma_2^2)}{\sigma_n^2} + n_1' - \frac{3g_{1,2}}{\sigma_n^2} \approx -\frac{2M(\sigma_1^2+\sigma_2^2)}{\sigma_n^2} - \frac{3g_{1,2}}{\sigma_n^2}$$

where $(n_k')_{k=1,2}$ are zero-mean Gaussian random variables of variance

$$\frac{2}{\sigma_n^2}\left(M(\sigma_1^2+\sigma_2^2) - (-1)^k\Re(\sigma_1\sigma_2 e^{i(\phi_1-\phi_2)}\mathbf{a}_1^H\mathbf{a}_2)\right) > \frac{2M(\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2)}{\sigma_n^2}.$$

Consequently

$$-\frac{h_1}{\sigma_n^2} - \left(-\frac{h_2}{\sigma_n^2}\right) \approx \frac{2M\sigma_2^2}{\sigma_n^2} \gg 1$$
$$-\frac{h_1}{\sigma_n^2} - \left(-\frac{h_3}{\sigma_n^2}\right) \approx \frac{2M\sigma_1^2}{\sigma_n^2} \gg 1$$
$$-\frac{h_1}{\sigma_n^2} - \left(-\frac{h_4}{\sigma_n^2}\right) \approx \frac{4M(\sigma_1^2+\sigma_2^2)}{\sigma_n^2} + \frac{4g_{1,2}}{\sigma_n^2} \geq \frac{4M(\sigma_1^2+\sigma_2^2-2\alpha\sigma_1\sigma_2)}{\sigma_n^2}$$
$$\geq \frac{8M\sigma_1\sigma_2(1-\alpha)}{\sigma_n^2} \geq \frac{8M(1-\alpha)\mathrm{Inf}(\sigma_1^2,\sigma_2^2)}{\sigma_n^2} \gg 1$$

where α defined by $\mathrm{Inf}(g_{1,2}) = -2M\alpha\sigma_1\sigma_2$ is related to the height of first sidelobe of the beam pattern of the array and satisfies for all standard array $\alpha < 0.5$.[5] Therefore it is proved that the term e^{-h_1/σ_n^2} is dominant with respect to the terms e^{-h_2/σ_n^2}, e^{-h_3/σ_n^2} and e^{-h_4/σ_n^2} and

$$\frac{\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial \phi_k} \approx -\frac{1}{\sigma_n^2}\frac{\partial h_1}{\partial \phi_k} = -\frac{1}{\sigma_n^2}\left(\frac{\partial g_k}{\partial \phi_k} + \frac{\partial g_{1,2}}{\partial \phi_k}\right)$$
$$= -\frac{i\sigma_k}{\sigma_n^2}(e^{i\phi_k}\mathbf{n}_t^H\mathbf{a}_k - e^{-i\phi_k}\mathbf{a}_k^H\mathbf{n}_t)$$

and for an arbitrary couple of symbols

$$\frac{\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial \phi_k} \approx -\frac{i\sigma_k}{\sigma_n^2}(e^{i\phi_k}\mathbf{n}_t^H\mathbf{a}_k - e^{-i\phi_k}\mathbf{a}_k^H\mathbf{n}_t)$$
$$(1_{(\epsilon_{t,1},\epsilon_{t,2})=(-1,-1)} - (-1)^k 1_{(\epsilon_{t,1},\epsilon_{t,2})=(-1,+1)}$$
$$+(-1)^k 1_{(\epsilon_{t,1},\epsilon_{t,2})=(+1,-1)} - 1_{(\epsilon_{t,1},\epsilon_{t,2})=(+1,+1)})$$
$$k=1,2.$$

Consequently

$$\frac{\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial \phi_1}\frac{\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial \phi_2}$$
$$\approx -\frac{\sigma_1\sigma_2}{\sigma_n^4}(e^{i\phi_1}\mathbf{n}_t^H\mathbf{a}_1 - e^{-i\phi_1}\mathbf{a}_1^H\mathbf{n}_t)$$
$$\cdot(e^{i\phi_2}\mathbf{n}_t^H\mathbf{a}_2 - e^{-i\phi_2}\mathbf{a}_2^H\mathbf{n}_t)$$
$$(1_{(\epsilon_{t,1},\epsilon_{t,2})=(-1,-1)} - 1_{(\epsilon_{t,1},\epsilon_{t,2})=(-1,+1)}$$
$$- 1_{(\epsilon_{t,1},\epsilon_{t,2})=(+1,-1)} + 1_{(\epsilon_{t,1},\epsilon_{t,2})=(+1,+1)})$$

$$\frac{\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial \phi_k}\frac{\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial \phi_k}$$
$$\approx -\frac{\sigma_k^2}{\sigma_n^4}(e^{i\phi_k}\mathbf{n}_t^H\mathbf{a}_k - e^{-i\phi_k}\mathbf{a}_k^H\mathbf{n}_t)^2$$
$$(1_{(\epsilon_{t,1},\epsilon_{t,2})=(-1,-1)} + 1_{(\epsilon_{t,1},\epsilon_{t,2})=(-1,+1)}$$
$$+ 1_{(\epsilon_{t,1},\epsilon_{t,2})=(+1,-1)} + 1_{(\epsilon_{t,1},\epsilon_{t,2})=(+1,+1)})$$

and because $(\epsilon_{t,1},\epsilon_{t,2})$ and $\mathbf{n}_t$ are independent and the four couples of symbols are equiprobable

$$\mathrm{E}\left(\frac{\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial \phi_1}\frac{\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial \phi_2}\right)$$
$$\approx -\frac{\sigma_1\sigma_2}{\sigma_n^4}\mathrm{E}\big((e^{i\phi_1}\mathbf{n}_t^H\mathbf{a}_1 - e^{-i\phi_1}\mathbf{a}_1^H\mathbf{n}_t)(e^{i\phi_2}\mathbf{n}_t^H\mathbf{a}_2 - e^{-i\phi_2}\mathbf{a}_2^H\mathbf{n}_t)\big)$$
$$(P[(\epsilon_{t,1},\epsilon_{t,2}) = (-1,-1)] - P[(\epsilon_{t,1},\epsilon_{t,2}) = (-1,+1)]$$
$$- P[(\epsilon_{t,1},\epsilon_{t,2}) = (+1,-1)] + P[(\epsilon_{t,1},\epsilon_{t,2}) = (+1,+1)])$$
$$= 0$$

[5]For a uniform linear array of M sensors, $g_{1,2} = 2\sigma_1\sigma_2\cos(\Delta\phi + (M-1)\Delta\theta/2)(\sin(M\Delta\theta/2)/\sin(\Delta\theta/2))$ and $\alpha \approx 0.224$.

$$
\begin{aligned}
&\mathrm{E}\left(\frac{\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial \phi_k}\frac{\partial \ln p(\mathbf{y}_t;\boldsymbol{\alpha})}{\partial \phi_k}\right)\\
&\approx -\frac{\sigma_k^2}{\sigma_n^4}\mathrm{E}\left((e^{i\phi_k}\mathbf{n}_t^H\mathbf{a}_k - e^{-i\phi_k}\mathbf{a}_k^H\mathbf{n}_t)^2\right)\\
&\quad(P[(\epsilon_{t,1},\epsilon_{t,2})=(-1,-1)] + P[(\epsilon_{t,1},\epsilon_{t,2})=(-1,+1)]\\
&\quad + P[(\epsilon_{t,1},\epsilon_{t,2})=(+1,-1)] + P[(\epsilon_{t,1},\epsilon_{t,2})=(+1,+1)])\\
&=\frac{2M\sigma_k^2}{\sigma_n^2}.
\end{aligned}
$$

Using a mixture of 16 Gaussian pdfs, the extension to two QPSK sources follows along the same lines.

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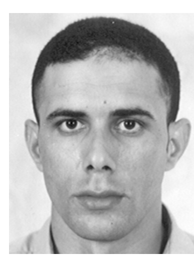

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