

Stochastic Cramér–Rao Bound for Noncircular Signals with Application to DOA Estimation

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Abstract—After providing an extension of the Slepian–Bangs formula for general noncircular complex Gaussian distributions, this paper focuses on the stochastic Cramér–Rao bound (CRB) on direction-of-arrival (DOA) estimation accuracy for noncircular sources. We derive an explicit expression of the CRB for DOA parameters alone in the case of noncircular complex Gaussian sources by two different methods. One of them consists of computing the asymptotic covariance matrix of the maximum likelihood (ML) estimator, and the other is obtained directly from our extended Slepian–Bangs formula. Some properties of this CRB are proved, and finally, it is numerically compared with the CRBs under circular complex Gaussian and complex discrete distributions of sources.

Index Terms—Maximum likelihood estimation, noncircular signals, stochastic CRB.

I. INTRODUCTION

DETERMINISTIC and stochastic Cramér–Rao bound (CRBs) play an important role in parametric estimation because the statistical performances of numerous estimation methods are known to be comparable with these bounds under certain mild conditions. Moreover, the stochastic CRB can be achieved asymptotically (in the number of measurements) by the stochastic ML method. Most of the contributions on the stochastic CRB are dedicated to Gaussian distributions for which a particularly convenient CRB formula was obtained for real Gaussian distributions by Slepian [1] and Bangs [2], which are referred to as the Slepian–Bangs formula, then extended to circular complex Gaussian distributions (see, e.g., [3, rel. (B.3.25)]). As is well known, the importance of the Gaussian CRB formulas lies in the fact that Gaussian data are rather frequently encountered in applications. Another important point is that under rather general conditions, the real [resp. circular complex] Gaussian CRB matrix is the largest of all CRB matrices among the class of arbitrary real [resp. circular complex] distributions with given mean and covariance matrices (see, e.g., [3, p. 293]). However, noncircular complex signals are frequently encountered in digital communications. For example, binary phase shift keying (BPSK) is often used, and no closed-form expression of the CRB is available for

these signals. Consequently, for noncircular complex signals, we need an upper bound of this CRB, but to the best of our knowledge, the Slepian–Bangs formula has yet to be extended to noncircular complex Gaussian distributions.

The first contribution of this paper is to give an extended Slepian–Bangs formula based on the work of [4]. Then, we concentrate on direction-of-arrival (DOA) estimation. For noncircular Gaussian sources, an explicit expression of the CRB for DOA parameters alone is derived from two different methods. One of them is obtained in an indirect manner by an asymptotic analysis of the ML estimator by slight modifications of the proof given by Stoica *et al.* [7], and the other is obtained directly from our extended Slepian–Bangs formula by following along the lines of the paper by Stoica *et al.* [5]. We prove that this CRB generally outperforms the circular complex Gaussian CRB associated with the same Hermitian covariance matrix. Next, we prove that this CRB decreases monotonically as the noncircularity rate increases in the particular case of one source. Finally, numerical comparison of the CRB under BPSK and noncircular Gaussian distributions are given. In particular, we show that for one source, the CRB under the BPSK distribution and under the noncircular complex Gaussian distribution approximately coincide, but for two equipowered uncorrelated BPSK sources, the CRB under the BPSK distribution outperforms the CRB under the noncircular complex Gaussian distribution, and the difference between them is more prominent for small DOA and phase separations.

The following notations are used throughout the paper. Matrices and vectors are represented by bold uppercase and bold lowercase characters, respectively. Vectors are by default in column orientation, whereas T , H , and $*$ stand for transpose, conjugate transpose, and conjugate, respectively. \odot is the Hadamard matrix product (i.e., $(\mathbf{A} \odot \mathbf{B})_{i,j} = (\mathbf{A})_{i,j}(\mathbf{B})_{i,j}$), and \perp is the ortho-complement of a projector matrix. $\text{Tr}(\cdot)$, $\det(\cdot)$, $\ln(\cdot)$, $\Re(\cdot)$, and $\Im(\cdot)$ denote the trace, the determinant, the logarithm, and the real and the imaginary part operator, respectively.

II. STOCHASTIC CRB FOR NONCIRCULAR GAUSSIAN SIGNALS

We consider a n -variate complex Gaussian random variable (RV) $\mathbf{z} \stackrel{\text{def}}{=} \mathbf{x} + i\mathbf{y}$, whose structured mean $\mathbf{m}_z \stackrel{\text{def}}{=} \mathbf{m}_x + i\mathbf{m}_y \stackrel{\text{def}}{=} \mathbf{E}(\mathbf{x}) + i\mathbf{E}(\mathbf{y})$ and covariance matrices $\mathbf{R}_z \stackrel{\text{def}}{=} \mathbf{E}[(\mathbf{z} - \mathbf{m}_z)(\mathbf{z} - \mathbf{m}_z)^H]$ and $\mathbf{R}'_z \stackrel{\text{def}}{=} \mathbf{E}[(\mathbf{z} - \mathbf{m}_z)(\mathbf{z} - \mathbf{m}_z)^T]$ are parameterized by the real parameter $\Theta \in \mathcal{R}^L$. Considering the Fisher information matrix, we prove the following result.

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Result 1: The Fisher information matrix corresponding to the nonnegative definite and noncircular complex Gaussian distribution is given (elementwise) by

$$(\mathbf{I}_F)_{k,l} = \left(\frac{\partial \mathbf{m}_{\tilde{z}}}{\partial \theta_k} \right)^H \mathbf{R}_{\tilde{z}}^{-1} \frac{\partial \mathbf{m}_{\tilde{z}}}{\partial \theta_l} + \frac{1}{2} \text{Tr} \left[\frac{\partial \mathbf{R}_{\tilde{z}}}{\partial \theta_k} \mathbf{R}_{\tilde{z}}^{-1} \frac{\partial \mathbf{R}_{\tilde{z}}}{\partial \theta_l} \mathbf{R}_{\tilde{z}}^{-1} \right] \quad (2.1)$$

$$\text{with } \mathbf{m}_{\tilde{z}} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{m}_z \\ \mathbf{m}_z^* \end{pmatrix} \text{ and } \mathbf{R}_{\tilde{z}} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{R}_z & \mathbf{R}'_z \\ \mathbf{R}'_z^* & \mathbf{R}_z^* \end{bmatrix}.$$

Proof: Because the nonsingular n -variate complex Gaussian RV \mathbf{z} is simply a $2n$ -variate real Gaussian RV $(\mathbf{x}^T, \mathbf{y}^T)^T$, with mean $(\mathbf{m}_x^T, \mathbf{m}_y^T)^T$ and arbitrary non-negative definite symmetric covariance matrix $\mathbf{\Gamma}_{2r}$, the real Slepian–Bangs formula (see e.g., [3, rel. (B.3.3)]) can be applied:

$$(\mathbf{I}_F)_{k,l} = \frac{\partial}{\partial \theta_k} (\mathbf{m}_x^T, \mathbf{m}_y^T) \mathbf{\Gamma}_{2r}^{-1} \frac{\partial}{\partial \theta_l} \begin{pmatrix} \mathbf{m}_x \\ \mathbf{m}_y \end{pmatrix} + \frac{1}{2} \text{Tr} \left[\frac{\partial \mathbf{\Gamma}_{2r}}{\partial \theta_k} \mathbf{\Gamma}_{2r}^{-1} \frac{\partial \mathbf{\Gamma}_{2r}}{\partial \theta_l} \mathbf{\Gamma}_{2r}^{-1} \right]. \quad (2.2)$$

Then, thanks to the relation $\mathbf{R}_{\tilde{z}} = \mathbf{M} \mathbf{\Gamma}_{2r} \mathbf{M}^H$ with $\mathbf{M} \stackrel{\text{def}}{=} 1/2 \begin{bmatrix} \mathbf{I}_n & \mathbf{I}_n \\ -i\mathbf{I}_n & i\mathbf{I}_n \end{bmatrix}$ proved in [4], using

$$\begin{pmatrix} \mathbf{m}_x \\ \mathbf{m}_y \end{pmatrix} = \mathbf{M} \begin{pmatrix} \mathbf{m}_z \\ \mathbf{m}_z^* \end{pmatrix} \text{ and } \frac{\partial \mathbf{\Gamma}_{2r}}{\partial \theta_k} \mathbf{\Gamma}_{2r}^{-1} \frac{\partial \mathbf{\Gamma}_{2r}}{\partial \theta_l} \mathbf{\Gamma}_{2r}^{-1} = \mathbf{M}^{-1} \frac{\partial \mathbf{R}_{\tilde{z}}}{\partial \theta_k} \mathbf{R}_{\tilde{z}}^{-1} \frac{\partial \mathbf{R}_{\tilde{z}}}{\partial \theta_l} \mathbf{R}_{\tilde{z}}^{-1} \mathbf{M}$$

in (2.2), result 1 is proved. ■

Remark: We note that for circular complex Gaussian RVs, $\mathbf{R}_{\tilde{z}} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{R}_z & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_z^* \end{bmatrix}$; consequently, (2.1) reduces to the circular complex Gaussian Slepian–Bangs formula [3, rel. B.3.25].

III. APPLICATION TO DOA ESTIMATION FOR NONCIRCULAR SOURCES

In the following, we will be concerned with the signal model

$$\mathbf{z}_t = \mathbf{A} \mathbf{s}_t + \mathbf{n}_t, \quad t = 1, \dots, T$$

where $(\mathbf{z}_t)_{t=1, \dots, T}$ represents the independent identically distributed M -vectors of observed complex envelope at the sensor output. $\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_K]$ is the steering matrix where each vector \mathbf{a}_k is parameterized by the real scalar parameter θ_k . $\mathbf{s}_t = (s_{t,1}, \dots, s_{t,K})^T$ and \mathbf{n}_t model signals transmitted by K sources and additive measurement noise, respectively. \mathbf{s}_t and \mathbf{n}_t are multivariate independent, complex zero-mean. \mathbf{n}_t is assumed circular complex Gaussian, spatially uncorrelated with $\text{E}(\mathbf{n}_t \mathbf{n}_t^H) = \sigma_n^2 \mathbf{I}_M$, whereas \mathbf{s}_t is either noncircular complex Gaussian or complex discrete distributed and possibly spatially correlated or even coherent with $\mathbf{R}_s \stackrel{\text{def}}{=} \text{E}(\mathbf{s}_t \mathbf{s}_t^H)$ and

$\mathbf{R}'_s \stackrel{\text{def}}{=} \text{E}(\mathbf{s}_t \mathbf{s}_t^T)$. Consequently, this leads to the covariance matrices of \mathbf{z}_t :

$$\mathbf{R}_z(\Theta) = \mathbf{A} \mathbf{R}_s \mathbf{A}^H + \sigma_n^2 \mathbf{I}_M \text{ and } \mathbf{R}'_z(\Theta) = \mathbf{A} \mathbf{R}'_s \mathbf{A}^T.$$

If no *a priori* information is available, $(\mathbf{R}_z(\Theta), \mathbf{R}'_z(\Theta))$ is generically parametrized by the $L = K + K^2 + K(K+1) + 1$ real parameters $\Theta = (\Theta_1^T, \Theta_2^T)^T$ with $\Theta_1 \stackrel{\text{def}}{=} (\theta_1, \dots, \theta_K)^T$, and

$$\begin{aligned} \Theta_2 \stackrel{\text{def}}{=} & ((\Re([\mathbf{R}_s]_{i,j}), \Im([\mathbf{R}_s]_{i,j}), \Re([\mathbf{R}'_s]_{i,j}), \\ & \Im([\mathbf{R}'_s]_{i,j}))_{1 \leq j < i \leq K}, ([\mathbf{R}_s]_{i,i}, \Re([\mathbf{R}'_s]_{i,i}), \\ & \Im([\mathbf{R}'_s]_{i,i}))_{i=1, \dots, K}, \sigma_n^2)^T. \end{aligned}$$

The parameter Θ is supposed to be identifiable from $(\mathbf{R}_z(\Theta), \mathbf{R}'_z(\Theta))$ in the following sense:

$$\mathbf{R}_z(\Theta) = \mathbf{R}_z(\Theta') \text{ and } \mathbf{R}'_z(\Theta) = \mathbf{R}'_z(\Theta') \Rightarrow \Theta = \Theta'. \quad (3.1)$$

A. Indirect Derivation of the Stochastic CRB for Noncircular Sources

To derive the stochastic CRB of the parameter Θ_1 alone, we consider the asymptotic covariance of the ML estimator. We first note that the probability density function (PDF) of \mathbf{z} that is considered to be a $2M$ -variate real Gaussian RV is given by an expression that is similar to that of the PDF in the circular case, provided it is expressed as a function of $\tilde{\mathbf{z}} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{z} \\ \mathbf{z}^* \end{pmatrix}$. From [4, rel. (15)], we have

$$p(\mathbf{x}, \mathbf{y}) = p'(\tilde{\mathbf{z}}) = (\pi)^{-M} [\det(\mathbf{R}_{\tilde{z}})]^{-1/2} \exp \left[-\frac{1}{2} \tilde{\mathbf{z}}^H \mathbf{R}_{\tilde{z}}^{-1} \tilde{\mathbf{z}} \right] \quad (3.2)$$

where

$$\mathbf{R}_{\tilde{z}} \stackrel{\text{def}}{=} \text{E}(\tilde{\mathbf{z}}_t \tilde{\mathbf{z}}_t^H) = \tilde{\mathbf{A}} \mathbf{R}_s \tilde{\mathbf{A}}^H + \sigma_n^2 \mathbf{I}_{2M} \text{ with } \mathbf{R}_s = \begin{bmatrix} \mathbf{R}_s & \mathbf{R}'_s \\ \mathbf{R}'_s^* & \mathbf{R}_s^* \end{bmatrix} \quad (3.3)$$

and $\tilde{\mathbf{A}} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^* \end{bmatrix}$. Then, classically (see, e.g., [6] and [7]), after dropping the constants, the log-likelihood function can be written as

$$L(\Theta_1, \Theta_2) = -\frac{T}{2} (\ln[\det(\mathbf{R}_{\tilde{z}})] + \text{Tr}(\mathbf{R}_{\tilde{z}}^{-1} \mathbf{R}_{\tilde{z},T})) \quad (3.4)$$

where $\mathbf{R}_{\tilde{z},T} \stackrel{\text{def}}{=} (1/T) \sum_{t=1}^T \tilde{\mathbf{z}}_t \tilde{\mathbf{z}}_t^H$, where the parameters Θ_1 and Θ_2 are imbedded in the covariance matrix $\mathbf{R}_{\tilde{z}}$. In (3.4), $\mathbf{R}_{\tilde{z}}$ depends on \mathbf{R}_s , which is structured via (3.3). Due to these constraints, the ML estimation of (Θ_1, Θ_2) becomes a constrained optimization problem, which is not standard. Despite this difficulty, we prove in the following that the ML estimate of the DOA parameters Θ_1 and source and noise covariance parameters Θ_2 may be obtained in a separable form. We are restricted here to the case where $K < M$ and \mathbf{A} is full column rank.

Result 2: If the sample covariance matrix $\mathbf{R}_{\tilde{z},T}$ is positive definite, the joint ML estimates that maximize the log-likelihood function (3.4) subject to the constraints (3.3) are given by the following:

¹We note that contrary to $\mathbf{\Gamma}_{2r}$, $\mathbf{R}_{\tilde{z}}$ is block structured, where \mathbf{R}_z and \mathbf{R}'_z are, respectively, Hermitian complex and symmetric complex. Consequently, the sample matrix $\mathbf{R}_{\tilde{z},T}$ is not described by a traditional $2n$ -variate complex Wishart distribution.

$\hat{\Theta}_{1,ML}$ is obtained by the minimizing with respect to Θ_1

$$F_T(\Theta_1) = \ln[\det(\tilde{\mathbf{A}}\hat{\mathbf{R}}_{\tilde{s},ML}\tilde{\mathbf{A}}^H + \hat{\sigma}_{n,ML}^2\mathbf{I}_{2M})] \quad (3.5)$$

where $\hat{\mathbf{R}}_{\tilde{s},ML}$, and $\hat{\sigma}_{n,ML}^2$ are given by

$$\hat{\mathbf{R}}_{\tilde{s},ML} = [\tilde{\mathbf{A}}^H(\Theta_1)\tilde{\mathbf{A}}(\Theta_1)]^{-1}\tilde{\mathbf{A}}^H(\Theta_1) \times [\mathbf{R}_{z,T} - \hat{\sigma}_{n,ML}^2\mathbf{I}_{2M}]\tilde{\mathbf{A}}(\Theta_1)[\tilde{\mathbf{A}}^H(\Theta_1)\tilde{\mathbf{A}}(\Theta_1)]^{-1} \quad (3.6)$$

and

$$\hat{\sigma}_{n,ML}^2 = \frac{1}{M-K} \text{Tr} \left(\mathbf{\Pi}_{\tilde{\mathbf{A}}(\Theta_1)}^\perp \mathbf{R}_{z,T} \right)$$

where $\mathbf{\Pi}_{\tilde{\mathbf{A}}(\Theta_1)}$ is the projection matrix $\mathbf{A}(\Theta_1)[\mathbf{A}^H(\Theta_1)\mathbf{A}(\Theta_1)]^{-1}\mathbf{A}^H(\Theta_1)$. Furthermore

$$\hat{\mathbf{R}}_{s,ML} = [\mathbf{A}^H(\hat{\Theta}_{1,ML})\mathbf{A}(\hat{\Theta}_{1,ML})]^{-1}\mathbf{A}^H(\hat{\Theta}_{1,ML}) \times [\mathbf{R}_{z,T} - \hat{\sigma}_{n,ML}^2\mathbf{I}_M]\mathbf{A}(\hat{\Theta}_{1,ML})[\mathbf{A}^H(\hat{\Theta}_{1,ML})\mathbf{A}(\hat{\Theta}_{1,ML})]^{-1} \quad (3.7)$$

and

$$\hat{\mathbf{R}}'_{s,ML} = [\mathbf{A}^H(\hat{\Theta}_{1,ML})\mathbf{A}(\hat{\Theta}_{1,ML})]^{-1}\mathbf{A}^H(\hat{\Theta}_{1,ML})\mathbf{R}'_{z,T} \mathbf{A}^*(\hat{\Theta}_{1,ML})[\mathbf{A}^T(\hat{\Theta}_{1,ML})\mathbf{A}^*(\hat{\Theta}_{1,ML})]^{-1}. \quad (3.8)$$

where $\mathbf{R}_{z,T} \stackrel{\text{def}}{=} (1/T)\sum_{t=1}^T \mathbf{z}_t\mathbf{z}_t^H$, and $\mathbf{R}'_{z,T} \stackrel{\text{def}}{=} (1/T)\sum_{t=1}^T \mathbf{z}_t\mathbf{z}_t^T$.

Proof: Maximizing the log-likelihood (3.4) without any constraint on the Hermitian matrix $\mathbf{R}_{\tilde{s}}$ reduces to a standard maximization problem. Its solution is given (e.g., in [6] and [7]) by the minimization of (3.5), where $\hat{\mathbf{R}}_{\tilde{s},ML}$ is given by (3.6) and $\hat{\sigma}_{n,ML}^2$ by

$$\hat{\sigma}_{n,ML}^2 = \frac{1}{2M-2K} \text{Tr} \left(\mathbf{\Pi}_{\tilde{\mathbf{A}}(\Theta_1)}^\perp \mathbf{R}_{z,T} \right).$$

Because $\mathbf{R}_{z,T}$, $\tilde{\mathbf{A}}^H(\Theta_1)\tilde{\mathbf{A}}(\Theta_1)$, and then $[\tilde{\mathbf{A}}^H(\Theta_1)\tilde{\mathbf{A}}(\Theta_1)]^{-1}$ and $[\tilde{\mathbf{A}}^H(\Theta_1)\tilde{\mathbf{A}}(\Theta_1)]^{-1}\tilde{\mathbf{A}}^H(\Theta_1)$ are all partitioned of the form $\begin{bmatrix} (\diamond) & (\times) \\ (\times)^* & (\diamond)^* \end{bmatrix}$, (3.6) is also partitioned of this form, where the matrices (\diamond) and (\times) are given by (3.7) and (3.8), respectively.

Finally, because $\mathbf{\Pi}_{\tilde{\mathbf{A}}(\Theta_1)} = \begin{bmatrix} \mathbf{\Pi}_{\mathbf{A}(\Theta_1)} & \mathbf{O} \\ \mathbf{O} & \mathbf{\Pi}_{\mathbf{A}(\Theta_1)}^* \end{bmatrix}$ and $\mathbf{R}_{z,T} = \begin{bmatrix} \mathbf{R}_{z,T} & \mathbf{R}'_{z,T} \\ \mathbf{R}'_{z,T} & \mathbf{R}_{z,T} \end{bmatrix}$, result 2 is proved. ■

Because the dimension of Θ that parametrizes our model is fixed, it follows from the standard statistical theory of ML estimator (see, e.g., [9]) that the ML estimator of Θ_1 asymptotically (in the number of measurements) achieves the CRB for Θ_1 estimation. Consequently, an explicit expression of the CRB of Θ_1 alone can be derived thanks to an asymptotic analysis of the ML estimate of Θ_1 given by result 2. Thus, by adapting the proof given in [7], the following result is proved.

Result 3: The normalized (i.e., for $T = 1$) DOA-related block of CRB for noncircular complex Gaussian (NCG) sources

is given by the explicit expression in (3.9), shown at the bottom of the page, where $\mathbf{D} \stackrel{\text{def}}{=} d\mathbf{A}(\Theta_1)/d\Theta_1$.

Proof: Deriving the covariance \mathbf{C}_{Θ_1} of the asymptotic distribution of the minimizing $\Theta_{1,T}$ of the function $F_T(\Theta_1)$ [see (3.5)], depending on the unknown Θ_1 and the statistics $\mathbf{R}_{z,T}$ is a standard problem. Its solution is given by (see, e.g., [5])

$$\mathbf{C}_{\Theta_1} = [\mathbf{F}''(\Theta_1)]^{-1} \times \left(\lim_{T \rightarrow \infty} \text{E} \left([\mathbf{F}'_T(\Theta_1)] [\mathbf{F}'_T(\Theta_1)]^T \right) \right) [\mathbf{F}''(\Theta_1)]^{-1} \quad (3.10)$$

where $\mathbf{F}'_T(\Theta_1)$ is the gradient of $\mathbf{F}_T(\Theta_1)$, and $\mathbf{F}''(\Theta_1)$ is the limit of the Hessian of $\mathbf{F}_T(\Theta_1)$ when $T \rightarrow \infty$. Because it is proved in result 2 that the constrained ML estimate of Θ_1 [i.e., with $\mathbf{R}_{\tilde{s}}$ satisfying the constraints (3.3)] coincides with the unconstrained ML estimate of Θ_1 (i.e., where $\mathbf{R}_{\tilde{s}}$ is an arbitrary Hermitian matrix), we can follow along the lines of the derivation given in [7], where $\mathbf{R}_z = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \sigma_n^2\mathbf{I}_M$ is replaced here by $\mathbf{R}_{\tilde{z}} = \mathbf{A}\mathbf{R}_{\tilde{s}}\tilde{\mathbf{A}}^H + \sigma_n^2\mathbf{I}_{2M}$. $\mathbf{F}'_T(\Theta_1)$ and $\mathbf{F}''(\Theta_1)$; then, $\mathbf{C}_{\Theta_1}^{\text{NCG}}$ are derived in Appendix A thanks to slight modifications w.r.t. [7]. More precisely, [7, rel. (3.7)] is generalized to

$$(\mathbf{F}'_T(\Theta_1))_k = 2\Re \left[\mathbf{h}_{k,T}^H \mathbf{R}_{z,T} \mathbf{\Pi}_{\tilde{\mathbf{A}}}^\perp \mathbf{d}_k + \mathbf{g}_{k,T}^H \mathbf{R}'_{z,T} \mathbf{\Pi}_{\tilde{\mathbf{A}}}^\perp \mathbf{d}_k \right], k = 1, \dots, K \quad (3.11)$$

and [7, rel. (3.8)] is generalized to

$$\mathbf{F}''(\Theta_1) = \frac{2}{\sigma_n^2} \times \Re \left[\mathbf{D}^H \mathbf{\Pi}_{\tilde{\mathbf{A}}}^\perp \mathbf{D} \odot \left([\mathbf{R}_s \mathbf{A}^H, \mathbf{R}'_s \mathbf{A}^T] \mathbf{R}_{\tilde{z}}^{-1} \begin{bmatrix} \mathbf{A} \mathbf{R}_s \\ \mathbf{A}^* \mathbf{R}'_s \end{bmatrix} \right)^T \right] \quad (3.12)$$

where $\mathbf{h}_{k,T}^H$ and $\mathbf{g}_{k,T}^H$ are, respectively, the k th row of the matrices

$$\left[\left(\tilde{\mathbf{A}}^H \mathbf{R}_{z,T} \tilde{\mathbf{A}} \right)_{(1:K,1:K)}^{-1} - \frac{1}{\sigma_n^2} (\mathbf{A}^H \mathbf{A})^{-1} \right] \mathbf{A}^H \text{ and } \left(\tilde{\mathbf{A}}^H \mathbf{R}_{z,T} \tilde{\mathbf{A}} \right)_{(1:K,K+1:2K)}^{-1} \mathbf{A}^T.$$

Remark 1: We note that for circular complex Gaussian (CG) sources, $\mathbf{R}'_s = \mathbf{O}$, and $\mathbf{R}_{\tilde{z}} = \begin{bmatrix} \mathbf{R}_z & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_z^* \end{bmatrix}$. Consequently, (3.9) reduces to

$$\mathbf{C}_{\Theta_1}^{\text{CG}} = \frac{\sigma_n^2}{2} \left\{ \Re \left[\mathbf{D}^H \mathbf{\Pi}_{\tilde{\mathbf{A}}}^\perp \mathbf{D} \odot (\mathbf{R}_s \mathbf{A}^H \mathbf{R}_z^{-1} \mathbf{A} \mathbf{R}_s)^T \right] \right\}^{-1}$$

indirectly derived in [7] and then directly derived from the circular complex Stepan-Bangs formula in [5].

The next result compares the CRBs $\mathbf{C}_{\Theta_1}^{\text{NCG}}$ and $\mathbf{C}_{\Theta_1}^{\text{CG}}$ associated with sources with the same first covariance matrix \mathbf{R}_s .

$$\mathbf{C}_{\Theta_1}^{\text{NCG}} = \frac{\sigma_n^2}{2} \left\{ \Re \left[\mathbf{D}^H \mathbf{\Pi}_{\tilde{\mathbf{A}}}^\perp \mathbf{D} \odot \left([\mathbf{R}_s \mathbf{A}^H, \mathbf{R}'_s \mathbf{A}^T] \mathbf{R}_{\tilde{z}}^{-1} \begin{bmatrix} \mathbf{A} \mathbf{R}_s \\ \mathbf{A}^* \mathbf{R}'_s \end{bmatrix} \right)^T \right] \right\}^{-1} \quad (3.9)$$

Result 4: The DOA-related block of CRB for noncircular complex Gaussian sources is upper bounded by the associated CRB for circular complex Gaussian sources corresponding to the same first covariance matrix \mathbf{R}_s .

$$\mathbf{C}_{\Theta_1}^{\text{NCG}} \leq \mathbf{C}_{\Theta_1}^{\text{CG}}. \quad (3.13)$$

Proof: First, from [7, lemma A.4], we have $\mathbf{B}_1 - \mathbf{B}_2 \geq \mathbf{O}$ with $\mathbf{B}_1 \stackrel{\text{def}}{=} [\mathbf{R}_s \mathbf{A}^H, \mathbf{R}'_s \mathbf{A}^T] \mathbf{R}_z^{-1} \begin{bmatrix} \mathbf{A} \mathbf{R}_s \\ \mathbf{A}^* \mathbf{R}'_s \end{bmatrix}$ and $\mathbf{B}_2 \stackrel{\text{def}}{=} \mathbf{R}_s \mathbf{A}^H \mathbf{R}_z^{-1} \mathbf{A} \mathbf{R}_s$, and this inequality applies to the transpose of these matrices: $\mathbf{B}_1^T - \mathbf{B}_2^T \geq \mathbf{O}$. Then, because $\mathbf{B}_3 \stackrel{\text{def}}{=} \mathbf{D}^H \mathbf{\Pi}_A \mathbf{D} \geq \mathbf{O}$, we have, thanks to a standard result of linear algebra (see, e.g., [3, App. A, result R.19], $\mathbf{B}_3 \odot (\mathbf{B}_1^T - \mathbf{B}_2^T) \geq \mathbf{O}$. This inequality is extended to the associated real symmetric matrices $\Re[\mathbf{B}_3 \odot \mathbf{B}_1^T] - \Re[\mathbf{B}_3 \odot \mathbf{B}_2^T] \geq \mathbf{O}$; then, by inversion, $\{\Re[\mathbf{B}_3 \odot \mathbf{B}_1^T]\}^{-1} - \{\Re[\mathbf{B}_3 \odot \mathbf{B}_2^T]\}^{-1} \leq \mathbf{O}$. ■

In the particular case of one source, we prove the following:

Result 5: The CRB of θ_1 for a noncircular complex Gaussian source decreases monotonically as the noncircularity rate increases and is given by the expression

$$C_{\theta_1}^{\text{NCG}} = \frac{1}{\alpha_1} \left[\frac{2r_1^{-1} + \|\mathbf{a}_1\|^{-2} r_1^{-2} + \|\mathbf{a}_1\|^2 - \|\mathbf{a}_1\|^2 \rho_1^2}{\|\mathbf{a}_1\|^2 r_1 + 1 + (1 - \|\mathbf{a}_1\|^2 r_1) \rho_1^2} \right] \quad (3.14)$$

where the noncircularity rate ρ_1 is defined by $E(s_{t,1}^2) = \rho_1 e^{i\phi_1} E|s_{t,1}^2|$ and satisfies $0 \leq \rho_1 \leq 1$. ϕ_1 is the circularity phase of $s_{t,1}$ (it will be used in Section III-C). The SNR is defined by $r_1 \stackrel{\text{def}}{=} \sigma_1^2 / \sigma_n^2$, and α_1 is the purely geometrical factor $2\mathbf{a}_1^H \mathbf{\Pi}_{\mathbf{a}_1} \mathbf{a}_1'$ with $\mathbf{a}_1' \stackrel{\text{def}}{=} d\mathbf{a}_1/d\theta_1$.

Proof: First, note that the structure of the inverse of \mathbf{R}_z in (3.9) is preserved, i.e., $\mathbf{R}_z^{-1} = \begin{bmatrix} \mathbf{G} & \mathbf{G}' \\ \mathbf{G}'^* & \mathbf{G}^* \end{bmatrix}$ with $\mathbf{G} = (\mathbf{R}_z - \mathbf{R}_z' \mathbf{R}_z'^{-1} \mathbf{R}_z'^*)^{-1}$ and $\mathbf{G}' = -\mathbf{G} \mathbf{R}_z' \mathbf{R}_z'^{-1}$. With $\mathbf{R}_z = \sigma_1^2 \mathbf{a}_1 \mathbf{a}_1^H + \sigma_n^2 \mathbf{I}_M$ and $\mathbf{R}_z' = \sigma_1^2 \rho_1 e^{i\phi_1} \mathbf{a}_1 \mathbf{a}_1^T$, (3.14) follows thanks to straightforward but tedious calculations. The monotony of $C_{\theta_1}^{\text{NCG}}$ with ρ_1 is proved in Appendix B. ■

Consequently, for one source, the CRB decreases from $C_{\theta_1} = (1/\alpha_1 r_1) (1 + 1/\|\mathbf{a}_1\|^2 r_1)$ ($\rho_1 = 0$, circular case) to $C_{\theta_1} = (1/\alpha_1 r_1) (1 + 1/2\|\mathbf{a}_1\|^2 r_1)$ ($\rho_1 = 1$, unfiltered BPSK case).

B. Direct Derivation of the Stochastic CRB for Noncircular Sources

To directly prove result 3 from the Fisher information matrix (2.1), we first note that thanks to the proof of result 2, the constrained ML estimate of Θ_1 coincides with the unconstrained ML estimate of Θ_1 . Consequently the associated CRB's of Θ_1 coincide for these two models. Using the unconstrained model, let $\Theta = (\Theta_1^T, \Theta_2^T)^T$ with here $\Theta_2 \stackrel{\text{def}}{=} (\boldsymbol{\rho}^T, \sigma_n^2)^T$ where

$$\boldsymbol{\rho} \stackrel{\text{def}}{=} \left(\Re([\mathbf{R}_s]_{i,j}), \Im([\mathbf{R}_s]_{i,j}), \Re([\mathbf{R}'_s]_{i,j}), \Im([\mathbf{R}'_s]_{i,j}), 1 \leq j < i \leq K, \Re([\mathbf{R}_s]_{i,i}), \Re([\mathbf{R}'_s]_{i,i}), \Im([\mathbf{R}'_s]_{i,i}), i=1, \dots, 2K \right)^T.$$

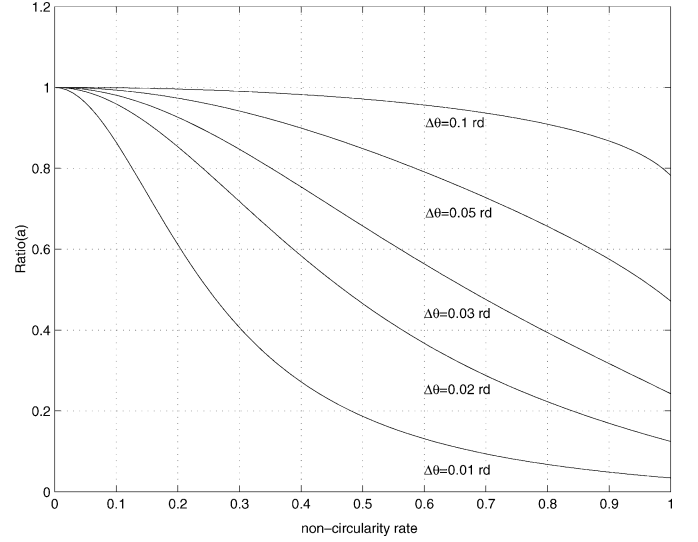


Fig. 1. Ratio $r_a \stackrel{\text{def}}{=} \left(\mathbf{C}_{\Theta_1}^{\text{NCG}} \right)_{(1,1)} / \left(\mathbf{C}_{\Theta_1}^{\text{CG}} \right)_{(1,1)}$ as a function of the noncircularity rate for different values of DOA separation ($\Delta\theta$) for $\phi_1 = \pi/2$ and $\phi_2 = \pi/3$.

With this unconstrained model, we can follow along the lines of the derivation given in [5] where $\mathbf{R}_z = \mathbf{A} \mathbf{R}_s \mathbf{A}^H + \sigma_n^2 \mathbf{I}_M$ is replaced here by $\mathbf{R}_z = \tilde{\mathbf{A}} \mathbf{R}_s \tilde{\mathbf{A}}^H + \sigma_n^2 \mathbf{I}_{2M}$ because the key point of the derivation, i.e., the relation $\text{vec}(\mathbf{R}_s) = \mathbf{J} \boldsymbol{\rho}$ where \mathbf{J} is a constant nonsingular complex matrix, is preserved. In addition, (3.9) is proved in Appendix C thanks to slight modifications of the direct derivation given in [5].

We note that the validity conditions of result 2 are “ $K < M$ and \mathbf{A} is full column rank,” whereas the identifiability condition (3.1) does not impose such conditions if *a priori* knowledge is available. For example, in the case of a uniform linear array and K independent sources of maximum noncircularity rates ($\sigma_k, k = 1, \dots, K$), it is shown in the simulations of [1] that up to $k = 2M - 2$ sources can be identified. In these cases, we have to resort to the CRB derived from the closed-form expression of the asymptotic minimum variance for complex noncircular Gaussian signals [3]. This remark extends to noncircular Gaussian signals; the discussion considered in [2] extends to circular Gaussian signals.

C. Illustrative Examples

The purpose of this section is to illustrate results 3–5 and to compare these CRBs with the CRB associated with BPSK distributed sources. We consider throughout this section one or two independent and equipowered sources with identical noncircularity rate. These sources impinge on a uniform linear array of M sensors separated by a half-wavelength for which $\mathbf{a}_k = (1, e^{i\theta_k}, \dots, e^{i(M-1)\theta_k})^T$, where $\theta_k = \pi \sin(\alpha_k)$ with α_k , which are the DOAs relative to the normal of array broadside.

The first experiment illustrates results 3 and 4. We consider two noncircular complex Gaussian sources with $M = 6$ and SNR = 20 dB. Figs. 1–3 exhibit the dependence of

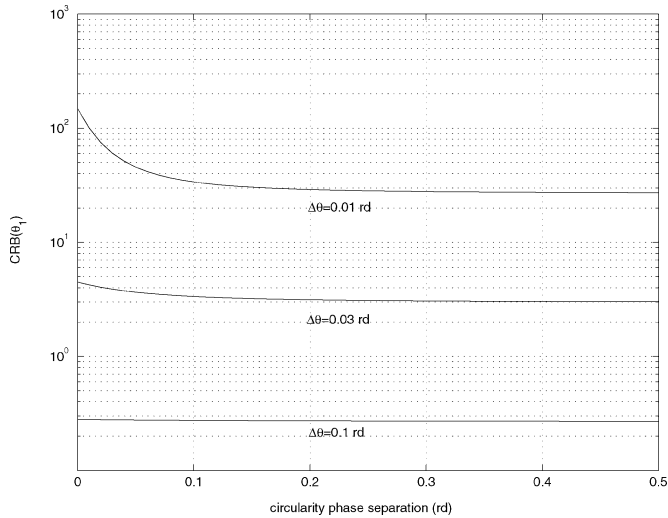


Fig. 2. $(C_{\Theta_1}^{NCG})_{(1,1)}$ as a function of the circularity phase separation for different values of DOA separation ($\Delta\theta$) for $\rho_1 = \rho_2 = 1$.

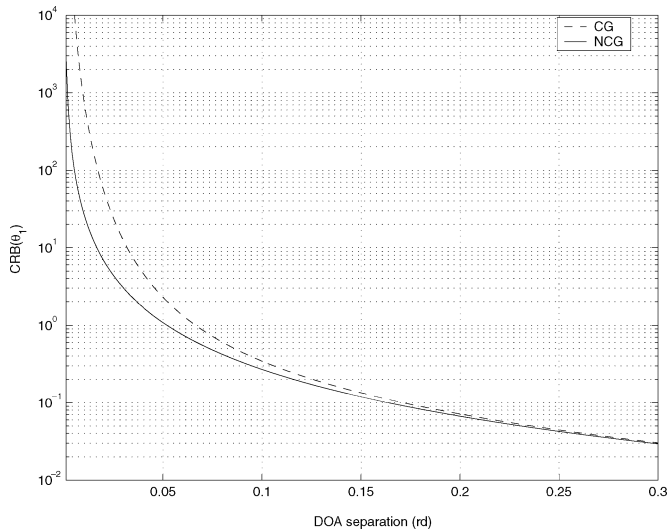


Fig. 3. $(C_{\Theta_1}^{NCG})_{(1,1)}$ and $(C_{\Theta_1}^{CG})_{(1,1)}$ as a function of the DOA separation for $\rho_1 = \rho_2 = 1$ and $\phi_1 = \pi/2$ and $\phi_2 = \pi/3$.

$(C_{\Theta_1}^{NCG})_{(1,1)}^2$ with the noncircularity rate $\rho_1 = \rho_2$, the circularity phase separation $\phi_2 - \phi_1$, and the DOA separation $\theta_2 - \theta_1$, respectively. Fig. 1 shows that $(C_{\Theta_1}^{NCG})_{(1,1)}$ decreases as the noncircularity rate increases (this extends to two equipowered sources result 5 proved in the one source case). Furthermore, this decrease is more prominent for low DOA separations. Fig. 2 shows that $(C_{\Theta_1}^{NCG})_{(1,1)}$ is sensitive to the circularity phase separation for low DOA separations, and Fig. 3 illustrates the inequality (3.13) of result 4. It shows that the difference between these two values is very sensitive for very low DOA separations only. Fig. 4 compares the noncircular complex Gaussian CRB $C_{\Theta_1}^{NCG}$ with the noncircular complex Gaussian CRB $C_{\Theta_1}^{NCG'}$

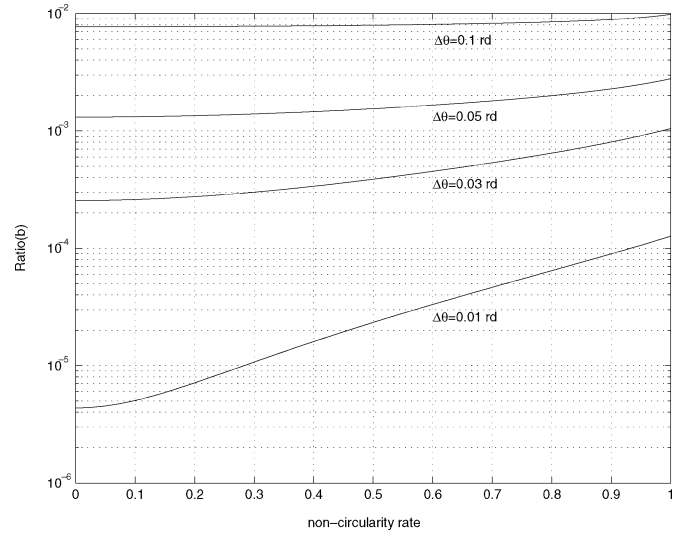


Fig. 4. Ratio $r_b \stackrel{\text{def}}{=} (C_{\Theta_1}^{NCG'})_{(1,1)} / (C_{\Theta_1}^{NCG})_{(1,1)}$ as a function of the noncircularity rate for different values of DOA separation ($\Delta\theta$) for $\phi_1 = \pi/2$ and $\phi_2 = \pi/3$.

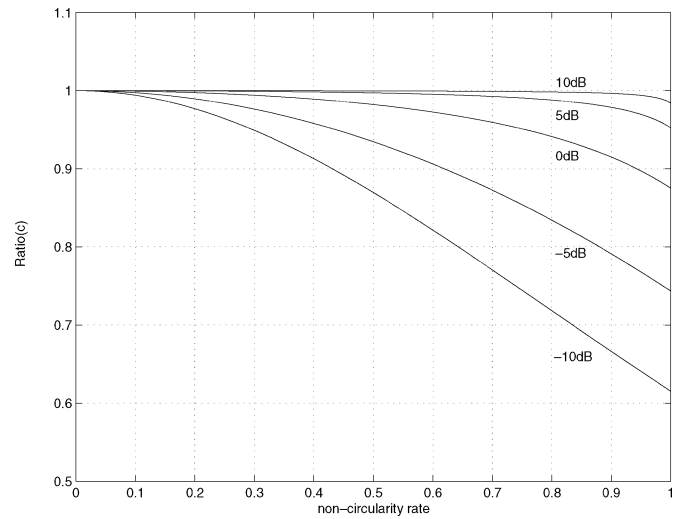


Fig. 5. Ratio $r_c \stackrel{\text{def}}{=} (C_{\Theta_1}^{NCG}) / (C_{\Theta_1}^{CG})$ as a function of the noncircularity rate ρ_1 for different values of the SNR r_1 .

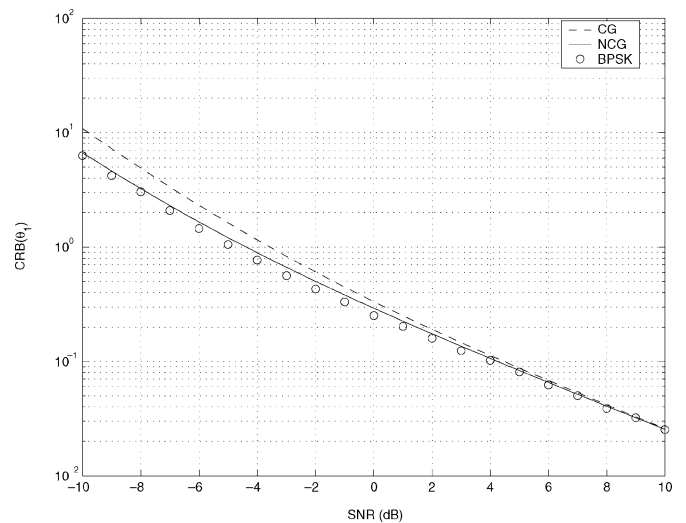


Fig. 6. $C_{\Theta_1}^{NCG}$, $C_{\Theta_1}^{CG}$, and $C_{\Theta_1}^{BPSK}$ as a function of the SNR.

²All the CRBs are computed for $T = 1$. That means that the actual CRBs associated with the signal model defined in Section III are obtained from the results given in this section by dividing by T .

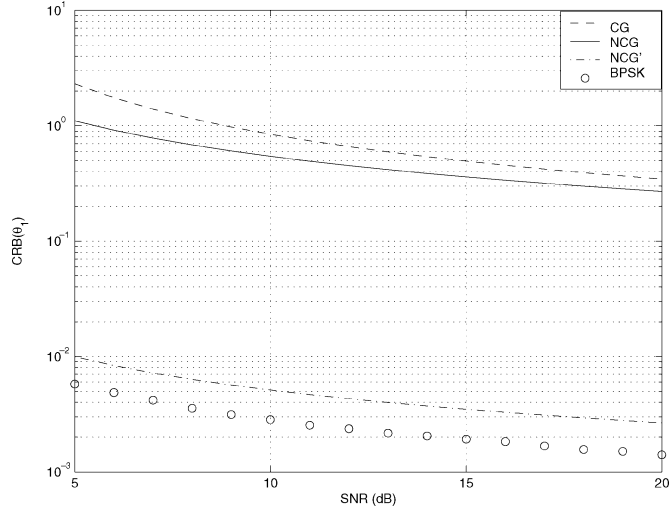


Fig. 7. $(\mathbf{C}_{\Theta_1}^{\text{BPSK}})_{(1,1)}$ (o o o), and $(\mathbf{C}_{\Theta_1}^{\text{NCG}'})_{(1,1)}$, $(\mathbf{C}_{\Theta_1}^{\text{NCG}})_{(1,1)}$, $(\mathbf{C}_{\Theta_1}^{\text{CG}})_{(1,1)}$ as a function of the SNR for a DOA separation of $0.1rd$, $\rho_1 = \rho_2 = 1$ and $\phi_1 = \pi/2$, $\phi_2 = \pi/3$.

under the *a priori* information that the two sources are independent,³ given in [11], by a nonexplicit expression. Fig. 4 shows that this *a priori* information is quite informative, but this information gain decreases as the noncircularity rate increases. This is particularly prominent for low DOA separations.

The second experiment illustrates result 5, where a noncircular complex Gaussian source and $M = 3$ are considered. Fig. 5 shows that the CRB decreases monotonically as the noncircularity rate increases but it is relatively insensitive to the increase of ρ_1 , except for very low SNR (i.e., for $\|\mathbf{a}_1\|^2 r_1 \approx 1$).

The two last experiments compare the numerical values of the noncircular complex Gaussian CRB $\mathbf{C}_{\Theta_1}^{\text{NCG}'}$ ⁴ with those of the CRB $\mathbf{C}_{\Theta_1}^{\text{BPSK}}$ associated with several BPSK distributed sources. Because the associated PDF of \mathbf{z}_t is a mixture of 2^K Gaussian PDFs, this latter CRB appears to be prohibitive to compute, and we use a numerical approximation derived from the strong law of large numbers, i.e.,

$$\mathbf{C}_{\Theta_1}^{\text{BPSK}} = (\mathbf{I}_F^{-1})_{(1:K,1:K)} \quad \text{with}$$

$$(\mathbf{I}_F)_{k,l} = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \left(\frac{\partial \ln p(\mathbf{z}_t; \Theta)}{\partial \theta_k} \right) \left(\frac{\partial \ln p(\mathbf{z}_t; \Theta)}{\partial \theta_l} \right)$$

where

$$p(\mathbf{z}_t; \Theta) = \frac{1}{2^K \pi^M \sigma_n^{2M}} \sum_{l=0}^{2^K-1} e^{-\|\mathbf{z}_t - \mathbf{A}(\Theta_1) \mathbf{s}_l\|^2 / \sigma_n^2} \quad \text{with}$$

$$\mathbf{s}_l \stackrel{\text{def}}{=} (a_1 \epsilon_{1,l} e^{i\phi_1}, \dots, a_K \epsilon_{k,l} e^{i\phi_K})^T$$

where $\epsilon_{k,l} = -1, +1$ are given by the dyadic expansion $l = \sum_{k=1}^K ((\epsilon_{k,l} + 1)/2) 2^{k-1}$, $l = 0, \dots, 2^K - 1$ for K independent unfiltered (i.e., $\rho_k = 1$) BPSK sources

³We note that the explicit expression (3.9) does not take account of this *a priori* information because it has been derived without any constraint on \mathbf{R}_s and \mathbf{R}'_s .

⁴We note that in the one source case, $\mathbf{C}_{\Theta_1}^{\text{NCG}'}$ and $\mathbf{C}_{\Theta_1}^{\text{NCG}}$ coincide.

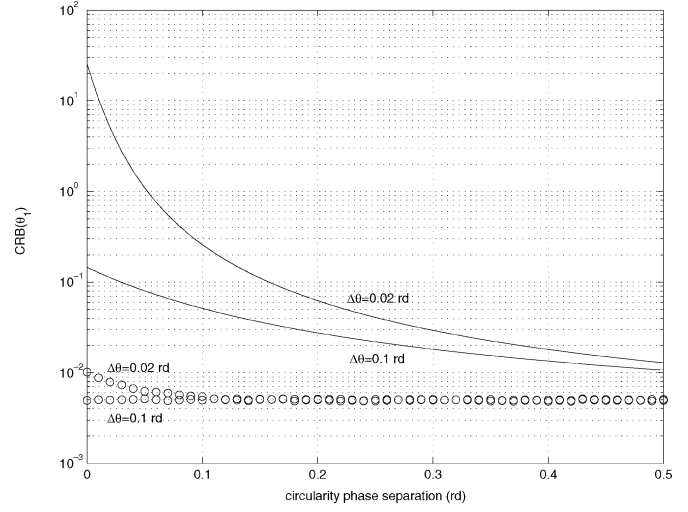


Fig. 8. $(\mathbf{C}_{\Theta_1}^{\text{BPSK}})_{(1,1)}$ (o o o) and $(\mathbf{C}_{\Theta_1}^{\text{NCG}'})_{(1,1)}$ as a function of the circularity phase separation for different values of DOA separation ($\Delta\theta$) for SNR = 20 dB.

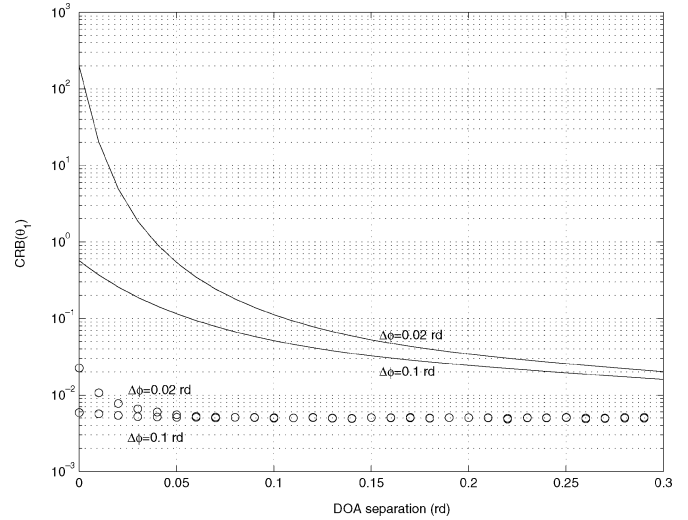


Fig. 9. $(\mathbf{C}_{\Theta_1}^{\text{BPSK}})_{(1,1)}$ (o o o) and $(\mathbf{C}_{\Theta_1}^{\text{NCG}'})_{(1,1)}$ as a function of the DOA separation for different values of circularity phase separation ($\Delta\phi$) for SNR = 20 dB.

for which $\Theta = (\Theta_1^T, \Theta_2^T)^T$, where, here, $\Theta_2 \stackrel{\text{def}}{=} (\sigma_1^2, \dots, \sigma_K^2, \phi_1, \dots, \phi_K, \sigma_n^2)^T$.

In the third experiment, we consider the one source case where $M = 3$. Fig. 6 compares $\mathbf{C}_{\Theta_1}^{\text{NCG}}$, $\mathbf{C}_{\Theta_1}^{\text{CG}}$, and $\mathbf{C}_{\Theta_1}^{\text{BPSK}}$. This figure shows that the CRBs under the BPSK distribution and under the noncircular complex Gaussian distribution (with $\rho_1 = 1$) approximately coincide. They outperform the CRB under the circular complex Gaussian distribution for low SNR only.

The last experiment illustrates the sensitivity of the CRB of the DOA to the distribution of the sources for two sources with $M = 6$ when the *a priori* information that the two sources are independent are taken into account. Figs. 7–9 exhibit the difference between $(\mathbf{C}_{\Theta_1}^{\text{NCG}'})_{(1,1)}$ ⁵ (for $\rho_1 = \rho_2 = 1$) and

⁵We note that comparing directly $\mathbf{C}_{\Theta_1}^{\text{BPSK}}$ to $\mathbf{C}_{\Theta_1}^{\text{NCG}}$ would be unfair because these CRBs are not associated with the same *a priori* information.

$$(\mathbf{I}_F)_{k,l} = \Re \left[\text{Tr} \left(\mathbf{R}_{\tilde{\mathbf{z}}}^{-1/2} \tilde{\mathbf{A}} \mathbf{C}_k \mathbf{D}_k^H \mathbf{R}_{\tilde{\mathbf{z}}}^{-1/2} \mathbf{\Pi}_{\mathbf{R}_{\tilde{\mathbf{z}}}^{-1/2} \tilde{\mathbf{A}} \mathbf{R}_{\tilde{\mathbf{z}}}^{-1/2}}^\perp \mathbf{D}_l \mathbf{C}_l^H \tilde{\mathbf{A}}^H \mathbf{R}_{\tilde{\mathbf{z}}}^{-1/2} \right) \right].$$

$$(\mathbf{I}_F)_{k,l} = \frac{1}{\sigma_n^2} \Re \left[\text{Tr} \left(\begin{pmatrix} \mathbf{d}_k^H & \mathbf{O} \\ \mathbf{O} & \mathbf{d}_k^T \end{pmatrix} \begin{pmatrix} \mathbf{\Pi}_{\mathbf{A}}^\perp & \mathbf{O} \\ \mathbf{O} & \mathbf{\Pi}_{\mathbf{A}^*}^\perp \end{pmatrix} \begin{pmatrix} \mathbf{d}_l & \mathbf{O} \\ \mathbf{O} & \mathbf{d}_l^* \end{pmatrix} \right) \right. \\ \left. \times \begin{pmatrix} \mathbf{c}_l^H & \mathbf{c}_l^T \\ \mathbf{c}_l^H & \mathbf{c}_l^T \end{pmatrix} \begin{pmatrix} \mathbf{A}^H & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^T \end{pmatrix} \mathbf{R}_{\tilde{\mathbf{z}}}^{-1} \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^* \end{pmatrix} \begin{pmatrix} \mathbf{c}_k^* & \mathbf{c}_k' \\ \mathbf{c}_k^* & \mathbf{c}_k' \end{pmatrix} \right].$$

$(\mathbf{C}_{\Theta_1}^{\text{BPSK}})_{(1,1)}$ as a function of the SNR, the circularity phase separation $\phi_2 - \phi_1$, and the DOA separation $\theta_2 - \theta_1$, respectively. Fig. 7 shows that for the same *a priori* information, $(\mathbf{C}_{\Theta_1}^{\text{BPSK}})_{(1,1)}$ slightly outperforms $(\mathbf{C}_{\Theta_1}^{\text{NCG}'})_{(1,1)}$ for all SNRs but tremendously outperforms the CRBs $(\mathbf{C}_{\Theta_1}^{\text{CG}})_{(1,1)}$ and $(\mathbf{C}_{\Theta_1}^{\text{NCG}})_{(1,1)}$, which do not take account of this *a priori* information. Figs. 8 and 9 show a weak sensitivity of $(\mathbf{C}_{\Theta_1}^{\text{BPSK}})_{(1,1)}$ to the circularity phase separation $\phi_2 - \phi_1$ and to the DOA separation $\theta_2 - \theta_1$ w.r.t. $(\mathbf{C}_{\Theta_1}^{\text{NCG}'})_{(1,1)}$, which is very sensitive. They also show that the difference between $(\mathbf{C}_{\Theta_1}^{\text{NCG}'})_{(1,1)}$ (for $\rho_1 = \rho_2 = 1$) and $(\mathbf{C}_{\Theta_1}^{\text{BPSK}})_{(1,1)}$ increases as the circularity phase separation or the DOA separation decreases.

IV. CONCLUSION

This paper has provided an extension of the Slepian–Bangs formula for general noncircular complex Gaussian distributions and has then focused on the stochastic CRB on DOA estimation accuracy for noncircular Gaussian sources. An explicit expression of the CRB for DOA parameters alone in the case of noncircular complex Gaussian sources by two different methods has been derived. Some properties of this CRB have been proved, and finally, it has been numerically compared with the CRBs under BPSK distribution.

An issue that was not addressed in this paper is the stochastic CRB of BPSK or QPSK distributed sources and the comparison of these CRBs with those of the noncircular or circular complex Gaussian distribution. A paper has just been submitted to deal with this issue.

APPENDIX A PROOF OF (3.9)

All the steps of the derivation of [7] apply with slight modifications. For the stochastic gradient and the deterministic hessian calculations, [7, rel. B.(16)] and [7, rel. B.(15)] apply where \mathbf{A} and $\mathbf{R}_{z,T}$ are replaced, respectively, by $\tilde{\mathbf{A}}$ and $\mathbf{R}_{\tilde{z},T}$. Using the partitioning of $\tilde{\mathbf{A}}$, $\mathbf{\Pi}_{\tilde{\mathbf{A}}}^\perp$, and $\mathbf{R}_{\tilde{z},T}$, (3.11) and (3.12) follow.

For the derivation of (3.9) from (3.10), all the steps of [7] apply, to prove that $\lim_{T \rightarrow \infty} \mathbb{E} \left([\mathbf{F}'_T(\Theta_1)] [\mathbf{F}'_T(\Theta_1)]^T \right) = \mathbf{F}''(\Theta_1)$, except that here, four terms are concerned from the expression of (3.11).

APPENDIX B

PROOF OF THE MONOTONY OF C_{θ_1} WITH ρ_1

Because (3.14) may be written as the following function of $x \stackrel{\text{def}}{=} \|\mathbf{a}_1\|^2 r_1$

$$C_{\theta_1}^{\text{NCG}} = \frac{\|\mathbf{a}_1\|^2}{\alpha_1 c} \left(-1 + \frac{a+b}{b+\rho_1^2} \right) \quad \text{with} \quad a \stackrel{\text{def}}{=} \frac{(1+x)^2}{x^2} \\ b \stackrel{\text{def}}{=} \frac{1+x}{1-x} \quad \text{and} \quad c \stackrel{\text{def}}{=} 1-x$$

and that $(a+b)/c = (1+x)/x^2(1-x)^2 > 0$, $C_{\theta_1}^{\text{NCG}}$ is a decreasing function of ρ_1 .

APPENDIX C

INDIRECT PROOF OF (3.9)

All the steps of the direct derivation of [5] apply, where [5, rel. (15)] is replaced by

$$\mathbf{R}_s = [\mathbf{c}_1, \dots, \mathbf{c}_K] = \begin{bmatrix} \mathbf{c}_1^H \\ \vdots \\ \mathbf{c}_K^H \end{bmatrix} \quad \text{and} \\ \mathbf{R}'_s = [\mathbf{c}'_1, \dots, \mathbf{c}'_K] = \begin{bmatrix} \mathbf{c}'_1{}^T \\ \vdots \\ \mathbf{c}'_K{}^T \end{bmatrix}$$

and [5, rels. (16)–(18)] become, respectively

$$\frac{d\mathbf{R}_{\tilde{z}}}{d\theta_k} = \mathbf{D}_k \mathbf{C}_k^H \tilde{\mathbf{A}}^H + \tilde{\mathbf{A}} \mathbf{C}_k \mathbf{D}_k^H \quad \text{with} \quad \mathbf{D}_k \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{d}_k & \mathbf{O} \\ \mathbf{O} & \mathbf{d}_k^* \end{pmatrix} \\ \mathbf{d}_k \stackrel{\text{def}}{=} \frac{d\mathbf{a}_k}{d\theta_k} \quad \text{and} \quad \mathbf{C}_k \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{c}_k & \mathbf{c}'_k \\ \mathbf{c}_k^* & \mathbf{c}'_k^* \end{pmatrix} \\ \text{and} \\ \mathbf{Z}_k = \mathbf{R}_{\tilde{z}}^{-1/2} \tilde{\mathbf{A}} \mathbf{C}_k \mathbf{D}_k^H \mathbf{R}_{\tilde{z}}^{-1/2}.$$

Consequently, [5, rel. (30)] becomes the first equation at the top of the page. Then, thanks to $\mathbf{R}_{\tilde{z}}^{-1/2} \mathbf{\Pi}_{\mathbf{R}_{\tilde{z}}^{-1/2} \tilde{\mathbf{A}} \mathbf{R}_{\tilde{z}}^{-1/2}}^\perp = (1/\sigma_n^2) \mathbf{\Pi}_{\tilde{\mathbf{A}}}^\perp = 1/\sigma_n^2 \begin{pmatrix} \mathbf{\Pi}_{\tilde{\mathbf{A}}}^\perp & \mathbf{O} \\ \mathbf{O} & \mathbf{\Pi}_{\tilde{\mathbf{A}^*}^\perp} \end{pmatrix}$, [5, rel. (32)] must be replaced by the second equation at the top of the page. Exploiting the structure $\begin{bmatrix} \mathbf{G} & \mathbf{G}' \\ \mathbf{G}'^* & \mathbf{G}^* \end{bmatrix}$ of $\mathbf{R}_{\tilde{z}}^{-1}$, (3.9) is straightforwardly deduced.

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