Asymptotically Minimum Variance Second-Order Estimation for Noncircular Signals with Application to DOA Estimation

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***Abstract*—This paper addresses asymptotically minimum variance (AMV) algorithms within the class of algorithms based on second-order statistics for estimating direction-of-arrival (DOA) parameters of possibly spatially correlated (even coherent) narrowband noncircular sources impinging on arbitrary array structures. To reduce the computational complexity due to the nonlinear minimization required by the matching approach, the covariance matching estimation technique (COMET) is included in the algorithm. Numerical examples illustrate the performance of the AMV algorithm.**

***Index Terms*—Asymptotically minimum variance, complex noncircular, DOA estimation, second-order statistics-based algorithms.**

I. Introduction

THERE is considerable literature about second-order statistics-based algorithms for estimating directions of arrival (DOA) of narrowband sources impinging on an array of sensors. The interest in these algorithms stems from a large number of applications including mobile communications systems [1]. In this application, after frequency down-shifting the sensor signals to baseband, the in-phase and quadrature components are paired to obtain complex signals. In addition, complex noncircular signals [2], for example, binary phase shift keying (BPSK) modulated signals, are often used. However, only a few contributions, such as [3]–[6], have been devoted to noncircular signals.

The DOA second-order algorithms devoted to complex circular signals rely on the positive definite Hermitian covariance matrix $\mathrm{E}(\mathbf{y}_t\mathbf{y}_t^H)$, and naturally, they can be used in the context of noncircular signals. Because the second-order statistical characteristics are also contained in the complex symmetric covariance matrix $\mathrm{E}(\mathbf{y}_t\mathbf{y}_t^T)$ for noncircular signals, a potential performance improvement ought to be obtained if these two covariance matrices are used. In the context of spatially uncorrelated amplitude modulated or BPSK-modulated sources impinging on a linear uniform array, a significant performance improvement has been already observed by simulations in [5] and [6] thanks to a MUSIC-like algorithm and a root-MUSIC like algorithm, respectively.

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To improve the performance of these algorithms and to extend DOA estimation to spatially correlated or even coherent arbitrary noncircular sources and to arbitrary array structures, we propose to consider asymptotically (in the number of measurements) minimum variance algorithms in the class of algorithms based on the two covariance matrices. We extend to complex noncircular processes the result of Porat and Friedlander [7], which is devoted to the estimating of moving average (MA) and autoregressive (AR) MA parameters of real non-Gaussian processes from sample high-order statistics. After a general lower bound is derived for the covariance of the estimated DOAs, it is shown that a generalized covariance matching algorithm attains this bound. Furthermore, the ideas of the covariance matching estimation technique (COMET) [8] are exploited to reduce the dimension of the optimization problem.

The paper is organized as follows. Section II presents the asymptotically minimum variance second-order estimator for stationary complex noncircular processes with special attention to the statistics involved. As an application, the estimation of the DOA parameters is considered in Section III. The asymptotic performance is analyzed in Section IV. Finally, illustrative examples with comparisons with the asymptotically minimum variance (AMV) estimators based on the first covariance matrix only are given in Section V.

The following notations are used throughout the paper. Matrices and vectors are represented by bold uppercase and bold lowercase characters, respectively. Vectors are by default in column orientation, whereas T, H, and $*$ stand for transpose, conjugate transpose, and conjugate, respectively. $\mathbf{e}_{K,k}$ is the kth unit vector in $\mathcal{R}^K$. $\mathrm{vec}(\cdot)$ is the "vectorization" operator that turns a matrix into a vector by stacking the columns of the matrix one below another, and v(.) denotes the operator obtained from vec(.) by eliminating all supradiagonal elements of the matrix. They are used in conjunction with the Kronecker product $\mathbf{A}\otimes\mathbf{B}$ as the block matrix whose (i,j) block element is $a_{i,j}\mathbf{B}$ and with the vec-permutation matrix $\mathbf{K}$ that transforms vec(.) to vec($.^T$) for any square matrix. The notation $f(x)=o(x)$ means that $\lim_{x\to 0}(f(x)/x)=0$.

II. Asymptotic Minimum Variance Second-Order Estimator

We consider a zero-mean strict-sense stationary M-variate complex, possibly noncircular process $\mathbf{y}_t$ whose structured covariance matrices $\mathbf{R}(\Theta)\stackrel{\text{def}}{=}\mathrm{E}(\mathbf{y}_t\mathbf{y}_t^H)$ and $\mathbf{R}'(\Theta)\stackrel{\text{def}}{=}\mathrm{E}(\mathbf{y}_t\mathbf{y}_t^T)$ are parameterized by the real parameter $\Theta\in\mathcal{R}^L$.

These covariance matrices are classically estimated by $\mathbf{R}_T = (1/T)\sum_{t=1}^T \mathbf{y}_t\mathbf{y}_t^H$ and $\mathbf{R}'_T = (1/T)\sum_{t=1}^T \mathbf{y}_t\mathbf{y}_t^T$, respectively. This parameter is supposed identifiable from $(\mathbf{R}(\Theta), \mathbf{R}'(\Theta))$ in the following sense:

$$\mathbf{R}(\Theta) = \mathbf{R}(\Theta') \quad \text{and} \quad \mathbf{R}'(\Theta) = \mathbf{R}'(\Theta') \Rightarrow \Theta = \Theta'.$$

To consider the asymptotic performance of a second-order algorithm, we adopt a functional analysis that consists of recognizing that the whole process of constructing an estimate Θ_T of Θ is equivalent to defining a functional relation linking this estimate Θ_T to the statistics $(\mathbf{R}_T, \mathbf{R}'_T)$ from which it is inferred. This functional dependence is denoted

$$(\mathbf{R}_T, \mathbf{R}'_T) \longmapsto \Theta_T = \text{alg}\,(\mathbf{R}_T, \mathbf{R}'_T)\,.$$

By assumption, $\Theta = \text{alg}(\mathbf{R}(\Theta), \mathbf{R}'(\Theta))$, and therefore, the different algorithms alg(.) constitute distinct extensions of the mapping $(\mathbf{R}(\Theta), \mathbf{R}'(\Theta)) \longmapsto \Theta$ generated by any unstructured Hermitian matrix $\mathbf{R}_T$ and complex symmetric matrix $\mathbf{R}'_T$.

To extend the ideas of Porat and Friedlander [7] concerning asymptotically minimum variance second-order estimators to complex noncircular processes, two conditions must be satisfied. First, the covariance $\mathbf{C}_{\substack{r\\r'}}(\Theta)$ of the asymptotic distribution of $(\mathbf{R}_T, \mathbf{R}'_T)$ must be regular. Second, the involved second-order algorithm considered as a mapping must be complex differentiable w.r.t. $(\mathbf{R}_T, \mathbf{R}'_T)$ at the point $(\mathbf{R}(\Theta), \mathbf{R}'(\Theta))$. While these two conditions are satisfied for a second-order algorithms based on $\mathbf{R}_T$ only, neither of these two conditions are satisfied in our situation for the following reasons. First, because $\mathbf{R}'_T$ is symmetric, the rank of $\mathbf{C}_{\substack{r\\r'}}(\Theta)$, which is the rank of the set of the entries of $(\mathbf{R}_T, \mathbf{R}'_T)$, is not full. Consequently, $\mathbf{C}_{\substack{r\\r'}}(\Theta)$ is singular. Second, because $\mathbf{R}'_T$ is complex non-Hermitian, an algorithm considered to be a mapping is not complex differentiable w.r.t. $\mathbf{R}'_T$ at point $\mathbf{R}'(\Theta)$.

To satisfy these two conditions, we must eliminate the common terms in $\mathbf{R}'_T$ and add complex conjuguate associated terms. Below, we consider the equivalent to $(\mathbf{R}_T, \mathbf{R}'_T)$ statistics $\mathbf{s}_T$ constituted by $\mathbf{r}_T \stackrel{\text{def}}{=} \text{vec}(\mathbf{R}_T)$, $\tilde{\mathbf{r}}'_T \stackrel{\text{def}}{=} \text{v}(\mathbf{R}'_T)$, and $\tilde{\mathbf{r}}'^*_T \stackrel{\text{def}}{=} \text{v}(\mathbf{R}'^*_T)$

$$\mathbf{s}_T \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{r}_T \\ \tilde{\mathbf{r}}'_T \\ \tilde{\mathbf{r}}'^*_T \end{pmatrix}$$

and the associated mapping

$$\mathbf{s}_T \longmapsto \Theta_T = \text{alg}(\mathbf{s}_T).$$

$\mathbf{r}(\Theta)$, $\tilde{\mathbf{r}}'(\Theta)$ and $\mathbf{s}(\Theta)$ are defined in the same way from $\mathbf{R}(\Theta)$ and $\mathbf{R}'(\Theta)$. Because $\text{vec}(\mathbf{R}_T^*) = \text{vec}(\mathbf{R}_T^T) = \mathbf{K}\text{vec}(\mathbf{R}_T)$, $\mathbf{s}^* = \mathbf{P}\mathbf{s}$, where $\mathbf{P}$ is the permutation matrix $\begin{pmatrix} \mathbf{K} & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I} \\ \mathbf{O} & \mathbf{I} & \mathbf{O} \end{pmatrix}$. Consequently, any mapping alg(.) differentiable w.r.t. $(\Re(\mathbf{s}), \Im(\mathbf{s}))$ becomes differentiable w.r.t. $\mathbf{s}$ alone if $\delta\mathbf{s}$ is structured as $\delta\mathbf{s} = \begin{pmatrix} \delta\mathbf{r} \\ \delta\tilde{\mathbf{r}} \\ \delta\tilde{\mathbf{r}}'^* \end{pmatrix}$, in which case

$$\begin{aligned} \text{alg}\,[\mathbf{s}(\Theta) + \delta\mathbf{s}] &= \text{alg}\,[\mathbf{s}(\Theta)] + [\mathbf{D}_s, \mathbf{D}_s^*] \begin{bmatrix} \delta\mathbf{s} \\ \delta\mathbf{s}^* \end{bmatrix} + o(\delta\mathbf{s}) \\ &= \Theta + \mathbf{D}_s^{\text{alg}}\delta\mathbf{s} + o(\delta\mathbf{s}) \end{aligned}$$

where $\mathbf{D}_s$ and $\mathbf{D}_s^*$ denote the Jacobian matrices of this differential at point $\mathbf{s}(\Theta)$, with $\mathbf{D}_s^{\text{alg}} \stackrel{\text{def}}{=} \mathbf{D}_s + \mathbf{D}_s^*\mathbf{K}$. In addition, because $\text{alg}[\mathbf{s}(\Theta)] = \Theta$ for all Θ, we have with $\mathbf{S} \stackrel{\text{def}}{=} (d\mathbf{s}(\Theta)/d\Theta)$

$$\begin{aligned} \text{alg}\,[\mathbf{s}(\Theta + \delta\Theta)] &= \text{alg}\,[\mathbf{s}(\Theta) + \mathbf{S}\delta\Theta + o(\delta\Theta)] \\ &= \Theta + \mathbf{D}_s^{\text{alg}}\mathbf{S}\delta\Theta + o(\delta\Theta) \\ &= \Theta + \delta\Theta. \end{aligned}$$

Therefore, $\mathbf{D}_s^{\text{alg}}$ is a left inverse of $\mathbf{S}$

$$\mathbf{D}_s^{\text{alg}}\mathbf{S} = \mathbf{I}_L \tag{2.1}$$

and this time, the rank of the set of the entries of $\mathbf{s}_T$ is generally $M^2 + M(M+1)$; therefore, the covariance $\mathbf{C}_s(\Theta)$ of the asymptotic distribution of $\mathbf{s}_T$ is a Hermitian positive definite matrix. Therefore, we obtain, by application of [7, Th. 2], extended to the complex case.

Theorem 1: The asymptotic covariance matrix $\mathbf{C}_\Theta$ of an estimator of Θ given by an arbitrary second-order algorithm is bounded below by the real symmetric matrix $(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S})^{-1}$:

$$\mathbf{C}_\Theta = \mathbf{D}_s^{\text{alg}}\mathbf{C}_s(\Theta)\left(\mathbf{D}_s^{\text{alg}}\right)^H \geq \left(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S}\right)^{-1}. \tag{2.2}$$

Proof: From (2.1), we get

$$\begin{aligned} 0 &\leq \left[\mathbf{D}_s^{\text{alg}} - \left(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S}\right)^{-1}\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\right]\mathbf{C}_s(\Theta) \\ &\quad \times \left[\mathbf{D}_s^{\text{alg}} - \left(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S}\right)^{-1}\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\right]^H \\ &= \mathbf{D}_s^{\text{alg}}\mathbf{C}_s(\Theta)\left(\mathbf{D}_s^{\text{alg}}\right)^H - \mathbf{D}_s^{\text{alg}}\mathbf{S}\left(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S}\right)^{-1} \\ &\quad - \left(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S}\right)^{-1}\mathbf{S}^H\left(\mathbf{D}_s^{\text{alg}}\right)^H + \left(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S}\right)^{-1} \\ &= \mathbf{D}_s^{\text{alg}}\mathbf{C}_s(\Theta)\left(\mathbf{D}_s^{\text{alg}}\right)^H - \left(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S}\right)^{-1} \end{aligned}$$

and furthermore, because $\mathbf{s}^* = \mathbf{P}\mathbf{s}$ implies $\mathbf{S}^* = \mathbf{P}\mathbf{S}$ and $\mathbf{C}_s^T(\Theta) = \mathbf{P}\mathbf{C}_s(\Theta)\mathbf{P}(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S})^T = \mathbf{S}^T(\mathbf{C}_s^{-1}(\Theta))^T\mathbf{S}^* = \mathbf{S}^H\mathbf{P}(\mathbf{C}_s^T(\Theta))^{-1}\mathbf{P}\mathbf{S} = \mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S}$, the Hermitian matrix $(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S})^{-1}$ is real symmetric. ■

Furthermore, we prove that this lowest bound is asymptotically tight, i.e., there exists an algorithm alg(.) whose covariance of the asymptotic distribution of Θ_T satisfies (2.2) with equality. Therefore, [7, Th. 3] extends to the complex noncircular case.

Theorem 2: The following nonlinear least square algorithm is an AMV second-order algorithm.

$$\Theta_T = \arg\min_{\alpha\in\mathcal{R}^L}\left[\mathbf{s}_T - \mathbf{s}(\alpha)\right]^H\mathbf{C}_s^{-1}(\alpha)\left[\mathbf{s}_T - \mathbf{s}(\alpha)\right]. \tag{2.3}$$

Proof: By a perturbation analysis, $\Theta_T = \Theta + \delta\Theta$ is associated with $\mathbf{s}_T = \mathbf{s}(\Theta) + \delta\mathbf{s}$ (with $\delta\mathbf{s}$ strutured). If $V(\alpha) \stackrel{\text{def}}{=} [\mathbf{s}(\Theta) - \mathbf{s}(\alpha)]^H\mathbf{C}_s^{-1}(\alpha)[\mathbf{s}(\Theta) - \mathbf{s}(\alpha)]$ and

$V_T(\alpha) \stackrel{\text{def}}{=} [\mathbf{s}_T - \mathbf{s}(\alpha)]^H \mathbf{C}_s^{-1}(\alpha)[\mathbf{s}_T - \mathbf{s}(\alpha)]$, we have $(V(\alpha)/d\alpha)_{|\alpha=\Theta} = \mathbf{0}$ and $(V_T(\alpha)/d\alpha)_{|\alpha=\Theta+\delta\Theta} = \mathbf{0}$. Expanding these two derivatives, we straightforwardly obtain $(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S} + \mathbf{S}^T\mathbf{C}_s^{-1}(\Theta)^*\mathbf{S}^*)\delta\Theta + o(\delta\Theta) = \mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\delta\mathbf{s} + \mathbf{S}^T\mathbf{C}_s^{-1}(\Theta)^*\delta\mathbf{s}^* + o(\delta\mathbf{s})$. Consequently, the algorithm (2.3) satisfies

$$\begin{aligned}\text{alg}\,[\mathbf{s}(\Theta)+\delta\mathbf{s}] &= \Theta + \left(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S} + \mathbf{S}^T\mathbf{C}_s^{-1}(\Theta)^*\mathbf{S}^*\right)^{-1} \\ &\quad\times \left(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta), \mathbf{S}^T\mathbf{C}_s^{-1}(\Theta)^*\right) \\ &\quad\times \begin{pmatrix}\delta\mathbf{s}\\ \delta\mathbf{s}^*\end{pmatrix} + o(\delta\mathbf{s}) \\ &= \Theta + \left(\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S}\right)^{-1} \\ &\quad\times \mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\delta\mathbf{s} + o(\delta\mathbf{s})\end{aligned}$$

by using $\mathbf{S}^* = \mathbf{P}\mathbf{S}$ and $\mathbf{C}_s^T(\Theta) = \mathbf{C}_s^*(\Theta) = \mathbf{P}\mathbf{C}_s(\Theta)\mathbf{P}$ in the second equality. Consequently, the derivative of the mapping alg(.) involved by (2.3) is $\mathbf{D}_s^{\text{alg}} = (\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S})^{-1}\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)$ and $\mathbf{C}_\Theta = \mathbf{D}_s^{\text{alg}}\mathbf{C}_s(\Theta)(\mathbf{D}_s^{\text{alg}})^H = (\mathbf{S}^H\mathbf{C}_s^{-1}(\Theta)\mathbf{S})^{-1}$. ■

In practice, it is difficult to optimize the nonlinear function (2.3), where it involves the computation of $\mathbf{C}_s^{-1}(\alpha)$. Porat and Friedlander proved for the real case in [9] that the lowest bound (2.2) is also obtained if an arbitrary consistent estimate $\mathbf{C}_{s,T}$ of $\mathbf{C}_s(\alpha)$ is used in (2.3). This property extends to the complex noncircular case and to any Hermitian positive definite weighting matrix, and therefore, we prove the following theorem.

Theorem 3: The covariance of the asymptotic distribution of Θ_T given by an arbitrary nonlinear least square algorithm

$$\Theta_T = \arg\min_{\alpha\in\mathcal{R}^L} [\mathbf{s}_T - \mathbf{s}(\alpha)]^H \mathbf{W}(\alpha)[\mathbf{s}_T - \mathbf{s}(\alpha)] \tag{2.4}$$

is preserved if the Hermitian positive definite weighting matrix $\mathbf{W}(\alpha)$ is replaced by an arbitrary consistent estimate $\mathbf{W}_T$ that satisfies $\mathbf{W}_T = \mathbf{W}(\Theta) + O(\mathbf{s}_T - \mathbf{s}(\Theta))$.

Proof: Following a perturbation analysis similar to those of the proof of Theorem 2, it is straightforward to show that the differential $\mathbf{D}_s^{\text{alg}} = (\mathbf{S}^H\mathbf{W}(\Theta)\mathbf{S})^{-1}\mathbf{S}^H\mathbf{W}(\Theta)$ of the mapping alg(.) involved by (2.4) is preserved. ■

Therefore, the minimization (2.3) can be preferably replaced by the following:

$$\Theta_T = \arg\min_{\alpha\in\mathcal{R}^L} [\mathbf{s}_T - \mathbf{s}(\alpha)]^H \mathbf{C}_{s,T}^{-1}[\mathbf{s}_T - \mathbf{s}(\alpha)]. \tag{2.5}$$

III. Application to Estimation of DOA

In the following, we will be concerned with the signal model

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{n}_t, \quad t = 1,\dots,T$$

where $(\mathbf{y}_t)_{t=1,\dots,T}$ represents the independent identically distributed M-vectors of observed complex envelope at the sensor output. $\mathbf{A} = [\mathbf{a}_1,\dots,\mathbf{a}_K]$ is the steering matrix where each vector $\mathbf{a}_k$ is parameterized by the real scalar parameter θ_k to avoid unnecessary notational complexity, but the results presented here apply to a general parameterization. $\mathbf{x}_t = (x_{t,1},\dots,x_{t,K})^T$ and $\mathbf{n}_t$ model signals transmitted by K sources and additive measurement noise, respectively. $\mathbf{x}_t$ and $\mathbf{n}_t$ are multivariate independent, zero-mean, complex wide-sense stationary. $\mathbf{n}_t$ is assumed to be Gaussian complex circular and spatially uncorrelated with $\text{E}(\mathbf{n}_t\mathbf{n}_t^H) = \sigma_n^2\mathbf{I}_M$, whereas $\mathbf{x}_t$ is complex circular or not, Gaussian or not, and possibly spatially correlated or even coherent with $\mathbf{R}_x \stackrel{\text{def}}{=} \text{E}(\mathbf{x}_t\mathbf{x}_t^H)$ and $\mathbf{R}'_x \stackrel{\text{def}}{=} \text{E}(\mathbf{x}_t\mathbf{x}_t^T)$. Consequently, this leads to the covariance matrices of $\mathbf{y}_t$

$$\mathbf{R}(\Theta) = \mathbf{A}\mathbf{R}_x\mathbf{A}^H + \sigma_n^2\mathbf{I}_M \quad \text{and} \quad \mathbf{R}'(\Theta) = \mathbf{A}\mathbf{R}'_x\mathbf{A}^T.$$

$(\mathbf{R}(\Theta), \mathbf{R}'(\Theta))$ is generically parametrized by the $L = K + K^2 + K(K+1) + 1$ real parameters $\Theta = (\Theta_1, \Theta_2)$ with $\Theta_1 \stackrel{\text{def}}{=} (\theta_1,\dots,\theta_K)^T$ and $\Theta_2 \stackrel{\text{def}}{=} ((\Re([\mathbf{R}_x]_{i,j}), \Im([\mathbf{R}_x]_{i,j}), \Re([\mathbf{R}'_x]_{i,j}), \Im([\mathbf{R}'_x]_{i,j}))_{1\le j<i\le K}, ([\mathbf{R}_x]_{i,i}, \Re([\mathbf{R}'_x]_{i,i}), \Im([\mathbf{R}'_x]_{i,i}))_{i=1,\dots,K}, \sigma_n^2)^T$.

For performance analysis, some extra hypotheses are needed. The rank of $\mathbf{R}_x$ is denoted $\tilde{K}$. Clearly, $\tilde{K} \le K$, and strict inequality implies linear dependence among the signal waveforms emanating from, e.g., specular multipath or smart jamming in communication applications. We suppose that the signal waveforms are linearly issued from $\tilde{K}$ independent signals $(\tilde{x}_{t,k})_{k=1,\dots,\tilde{K}}$, i.e., there exists a full colomn rank matrix $\mathbf{B}$ such that $\mathbf{x}_t = \mathbf{B}\tilde{\mathbf{x}}_t$. The fourth-order cumulants of these $\tilde{K}$ sources are denoted by $\kappa_{\tilde{x}_k} \stackrel{\text{def}}{=} \text{Cum}(\tilde{x}_{t,k}, \tilde{x}_{t,k}^*, \tilde{x}_{t,k}, \tilde{x}_{t,k}^*)$, $\kappa'_{\tilde{x}_k} \stackrel{\text{def}}{=} \text{Cum}(\tilde{x}_{t,k}, \tilde{x}_{t,k}, \tilde{x}_{t,k}, \tilde{x}_{t,k})$ and $\kappa''_{\tilde{x}_k} \stackrel{\text{def}}{=} \text{Cum}(\tilde{x}_{t,k}, \tilde{x}_{t,k}^*, \tilde{x}_{t,k}^*, \tilde{x}_{t,k}^*)$.

We note that $\mathbf{s}(\Theta)$ is linear with respect to Θ_2. Consequently (see e.g., [8]), there exists [1] a known matrix $\mathbf{\Psi}(\Theta_1)$ of the unknown DOA parameters Θ_1:

$$\mathbf{s}(\Theta) = \mathbf{\Psi}(\Theta_1)\Theta_2.$$

Because we suppose[2] in this paper that Θ is identifiable from $(\mathbf{R}(\Theta), \mathbf{R}'(\Theta))$, Θ must be identifiable from $\mathbf{s}(\Theta)$, and necessarily, $\mathbf{\Psi}(\Theta_1)$ has column full rank [8]. Under these conditions, the minimization (2.5) with respect to Θ_2 is immediate if Θ_2 is not restricted to be real. With a geometric procedure, we obtain

$$\widehat{\Theta}_2 = \left[\mathbf{\Psi}^H(\Theta_1)\mathbf{W}\mathbf{\Psi}(\Theta_1)\right]^{-1}\mathbf{\Psi}^H(\Theta_1)\mathbf{W}\mathbf{s}_T \tag{3.1}$$

with $\mathbf{W} \stackrel{\text{def}}{=} \mathbf{C}_{s,T}^{-1}$. Because $\text{vec}(\mathbf{y}_t\mathbf{y}_t^H) = \mathbf{y}_t^* \otimes \mathbf{y}_t$ and $\text{v}(\mathbf{y}_t\mathbf{y}_t^T) = \mathbf{U}(\mathbf{y}_t\otimes\mathbf{y}_t)$, where $\mathbf{U}$ is the $(M(M+1))/(2)\times M^2$ selection matrix that satisfies $\text{v}(.) = \mathbf{U}\text{vec}(.)$ for all $M\times M$ matrices, $\mathbf{s}_T$ can be written as

$$\mathbf{s}_T = \frac{1}{T}\sum_{t=1}^T \mathbf{s}(t) \quad \text{with} \quad \mathbf{s}(t) \stackrel{\text{def}}{=} \begin{pmatrix}\mathbf{y}_t^*\otimes\mathbf{y}_t\\ \mathbf{U}(\mathbf{y}_t\otimes\mathbf{y}_t)\\ \mathbf{U}(\mathbf{y}_t^*\otimes\mathbf{y}_t^*)\end{pmatrix}.$$

Consequently, $\mathbf{s}_T$ is the mean of the T independent equidistributed random variables $\mathbf{s}(t)$. Therefore, $\text{Cov}(\mathbf{s}_T) =$

[1]An explicit expression for $\mathbf{\Psi}(\Theta_1)$ will depend on the parameterization of $\mathbf{R}(\Theta)$ and $\mathbf{R}'(\Theta)$.

[2]We note that sufficient conditions for the identifiability will be application specific since they will depend on the structure of the array and the spatial correlation and the type of noncircularity of the sources.

$(1/T)\mathrm{Cov}(\mathbf{s}(t)) = (1/T)\mathrm{E}[(\mathbf{s}(t)-\mathrm{E}(\mathbf{s}(t)))(\mathbf{s}(t)-\mathrm{E}(\mathbf{s}(t)))^H]$, and

$$\mathbf{C}_{s,T} = \frac{1}{T}\sum_{t=1}^{T}\left[\left(\mathbf{s}(t)-\frac{1}{T}\sum_{t=1}^{T}\mathbf{s}(t)\right)\left(\mathbf{s}(t)-\frac{1}{T}\sum_{t=1}^{T}\mathbf{s}(t)\right)^H\right]$$

is a consistent estimate of $\mathbf{C}_s(\Theta)$ structured as $\mathbf{s}_T\mathbf{s}_T^H$ for the real/imaginary part point of view. With arguments similar to that of COMET [8], we prove that $\widehat{\Theta}_2$ is real-valued.

Proof: If $\mathbf{J}$ denotes the linear invertible transformation that is associated with $\mathbf{s}_T$, the real-valued vector γ_T comprised of the real and imaginary parts of $\mathbf{s}_T$, $\gamma_T = \mathbf{J}\mathbf{s}_T$, and $\widehat{\Theta}_2$ given by (3.1) assumes the form $[(\mathbf{J}\mathbf{\Psi})^H(\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^H)^{-1}(\mathbf{J}\mathbf{\Psi})]^{-1}(\mathbf{J}\mathbf{\Psi})^H(\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^H)^{-1}\mathbf{J}\mathbf{s}_T$, where $\mathbf{J}\mathbf{s}_T$ is real, and so is $\mathbf{J}\mathbf{\Psi}$. We must still examine $\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^H$. Because $\mathbf{J}\mathbf{s}_T\mathbf{s}_T^H\mathbf{J}^H = \gamma_T\gamma_T^H$ is real-valued and because $\mathbf{C}_{s,T}$ is structured as $\mathbf{s}_T\mathbf{s}_T^H$, the matrix $\mathbf{J}\mathbf{W}^{-1}\mathbf{J}^H = \mathbf{J}\mathbf{C}_{s,T}\mathbf{J}^H$ is real-valued. ■

Thus, $\widehat{\Theta}_2$ given by (3.1) is the real value that minimizes (2.5). $\Theta_{1,T}$ is obtained by subtituting $\widehat{\Theta}_2$ in (2.5):

$$\Theta_{1,T} = \arg\max_{\alpha_1\in\mathcal{R}^K} V'(\alpha_1) \tag{3.2}$$

with

$$V'(\alpha_1) \stackrel{\text{def}}{=} \mathbf{s}_T^H\mathbf{W}\mathbf{\Psi}(\alpha_1)\left[\mathbf{\Psi}^H(\alpha_1)\mathbf{W}\mathbf{\Psi}(\alpha_1)\right]^{-1}\mathbf{\Psi}^H(\alpha_1)\mathbf{W}\mathbf{s}_T.$$

This COMET estimate is in general obtained by maximizing a multidimensional nonlinear cost function. See [8] for some implementational aspects (scoring technique, initialization of the multidimensional search, regularization of the sample covariance matrices...).

To evaluate the improvement provided by the use of the covariance matrix $\mathbf{R}'_T$ compared with the case in which only $\mathbf{R}_T$ is considered, we first consider AMV second-order algorithms based on $\mathbf{R}_T$ only.

IV. Performance Analysis

A. AMV Estimator Based on $\mathbf{R}_T$ Only

We suppose here that Θ is identifiable from $\mathbf{R}(\Theta)$ only. In this case, the asymptotic minimum variance of the estimated parameters relies on the following standard central limit theorem applied to the independent equidistributed complex noncircular random variables $\mathbf{y}_t^*\otimes\mathbf{y}_t$. Thanks to simple algebraic manipulations of $\mathbf{C}_r = \mathrm{E}((\mathbf{y}_t^*\otimes\mathbf{y}_t - \mathrm{vec}(\mathbf{R}(\Theta)))(\mathbf{y}_t^*\otimes\mathbf{y}_t - \mathrm{vec}(\mathbf{R}(\Theta)))^H)$ and $\mathbf{C}'_r = \mathrm{E}((\mathbf{y}_t^*\otimes\mathbf{y}_t - \mathrm{vec}(\mathbf{R}(\Theta)))(\mathbf{y}_t^*\otimes\mathbf{y}_t - \mathrm{vec}(\mathbf{R}(\Theta)))^T)$, we straightforwardly prove the following lemma.

Lemma 1: $\sqrt{T}\,(\mathrm{vec}(\mathbf{R}_T) - \mathrm{vec}(\mathbf{R}(\Theta)))$ converges in distribution to the zero-mean complex noncircular Gaussian distribution of covariances $\mathbf{C}_r$ and $\mathbf{C}'_r = \mathbf{C}_r\mathbf{K}$, where[3]

$$\begin{aligned}\mathbf{C}_r = {}&(\mathbf{A}^*\otimes\mathbf{A})\mathbf{C}_{r_x}(\mathbf{A}^T\otimes\mathbf{A}^H) + \sigma_n^4\mathbf{I}_{M^2}\\ &+\sigma_n^2\mathbf{I}_M\otimes\mathbf{A}\mathbf{R}_x\mathbf{A}^H + \mathbf{A}^*\mathbf{R}_x^*\mathbf{A}^T\otimes\sigma_n^2\mathbf{I}_M\end{aligned} \tag{4.3}$$

with[4]

$$\mathbf{C}_{r_x} = \mathbf{R}_x^*\otimes\mathbf{R}_x + \mathbf{K}(\mathbf{R}'_x\otimes\mathbf{R}'^*_x) + \mathbf{Q}_x \quad \text{and}$$

$$\mathbf{Q}_x = (\mathbf{B}^*\otimes\mathbf{B})\left(\sum_{k=1}^{\tilde{K}}\kappa_{\tilde{x}_k}(\mathbf{e}_{\tilde{K},k}\otimes\mathbf{e}_{\tilde{K},k})\left(\mathbf{e}_{\tilde{K},k}^T\otimes\mathbf{e}_{\tilde{K},k}^T\right)\right)\times(\mathbf{B}^T\otimes\mathbf{B}^H).$$

First, we note that Theorems 1–3 apply to the statistics $\mathbf{r}_T$ because a second-order algorithm based on $\mathbf{R}_T$ only is a mapping $\mathbf{R}_T\to\Theta_T = \mathrm{alg}(\mathbf{R}_T)$, which is complex differentiable w.r.t. $\mathbf{R}_T$ at the point $\mathbf{R}(\Theta)$, and the covariance $\mathbf{C}_r(\Theta)$ of the asymptotic distribution of $\mathbf{R}_T$ is regular. By application of Theorem 1 applied to the statistics $\mathbf{r}_T$, the covariance of the asymptotic distribution of the minimum variance second-order DOA estimator (3.2) based on $\mathbf{R}_T$ only is given by the top left $K\times K$ "DOA corner" of $(\mathbf{S}^H\mathbf{C}_r^{-1}(\Theta)\mathbf{S})^{-1}$, where $\mathbf{C}_r(\Theta)$ is given by (4.3). If we note here that $\mathbf{S}\stackrel{\text{def}}{=}(d\mathbf{r}/d\Theta) = [\mathbf{S}_1,\mathbf{\Psi}]$ with $\mathbf{S}_1\stackrel{\text{def}}{=}(\partial\mathbf{r}/\partial\Theta_1)$ and $\mathbf{\Psi}$ given by $\mathbf{r} = \mathbf{\Psi}(\Theta_1)\Theta_2$, the partitioned matrix inversion lemma gives

$$\begin{aligned}&\left(\mathbf{S}^H\mathbf{C}_r^{-1}\mathbf{S}\right)^{-1}_{(1:K,1:K)}\\ &=\left(\mathbf{S}_1^H\mathbf{C}_r^{-1}\mathbf{S}_1 - \mathbf{S}_1^H\mathbf{C}_r^{-1}\mathbf{\Psi}\left[\mathbf{\Psi}^H\mathbf{C}_r^{-1}\mathbf{\Psi}\right]^{-1}\mathbf{\Psi}^H\mathbf{C}_r^{-1}\mathbf{S}_1\right)^{-1}\\ &=\left(\mathbf{S}_1^H\mathbf{C}_r^{-\frac{1}{2}}\mathbf{P}^{\perp}_{\mathbf{C}_r^{-\frac{1}{2}}\mathbf{\Psi}}\mathbf{C}_r^{-\frac{1}{2}}\mathbf{S}_1\right)^{-1}\end{aligned}$$

where $\mathbf{P}^{\perp}_{\mathbf{C}_r^{-1/2}\mathbf{\Psi}}$ denotes the projector onto the ortho-complement of the colomns of $\mathbf{C}_r^{-1/2}\mathbf{\Psi}$. Consequently, we prove the following theorem.

Theorem 4: For Gaussian or non-Gaussian and complex circular or noncircular sources, the covariance of the asymptotic distribution of the minimum variance second-order DOA estimator based on $\mathbf{R}_T$ only has the common closed-form expression

$$\mathbf{C}_{\Theta_1} = \left(\mathbf{S}_1^H\mathbf{C}_r^{-\frac{1}{2}}\mathbf{P}^{\perp}_{\mathbf{C}_r^{-\frac{1}{2}}\mathbf{\Psi}}\mathbf{C}_r^{-\frac{1}{2}}\mathbf{S}_1\right)^{-1}. \tag{4.4}$$

This expression (4.4) extends to non-Gaussian and/or complex noncircular sources, which is the expression of the asymptotic covariance given in [8] for Gaussian complex circular sources. On the other hand, we note that this expression is no longer equal to the Cramér–Rao bound because this AMV second-order estimator based on $\mathbf{R}_T$ only is no longer efficient for non-Gaussian and/or complex noncircular sources.

Remark 1: The expression of $\mathbf{C}_{\Theta_1}$ is generally sensitive to the noncircularity and the distribution of the sources. Furthermore, we note that a parameterization of $\mathbf{R}_x$ and $\mathbf{R}'_x$ may be introduced to incorporate *a priori* knowledge on the spatial correlation of the sources. For example, if the sources are supposed to be spatially uncorrelated, $\mathbf{R}_x$ will be parameterized by $([\mathbf{R}_x]_{i,i})_{i=1,\ldots,K}$, and if, moreover, they are independent, $\mathbf{R}_x$ and $\mathbf{R}'_x$ will be parameterized by $([\mathbf{R}_x]_{i,i},\Re([\mathbf{R}'_x]_{i,i}),\Im([\mathbf{R}'_x]_{i,i}))_{i=1,\ldots,K}$ only. Consequently, the expression of $\mathbf{C}_{\Theta_1}$ is generally sensitive to this *a priori* information as well.

Remark 2: Note that the derivative $\mathbf{D}_r^{\mathrm{AMV}_1}$ of the mapping that associates the estimate $\Theta_{1,T}$ to $\mathbf{R}_T$ depends on the

[3]Because $\mathrm{vec}^T(\mathbf{y}_t\mathbf{y}_t^H - \mathbf{R}(\Theta)) = \mathrm{vec}^H(\mathbf{y}_t\mathbf{y}_t^H - \mathbf{R}(\Theta))\mathbf{K}$, $\mathbf{C}'_r = \mathbf{C}_r\mathbf{K}$, and the noncircular complex Gaussian asymptotic distribution of $\mathbf{R}_T$ is characterized by $\mathbf{C}_r$ only.

[4]If the K sources are independent, $\mathbf{Q}_x$ is reduced to $\mathbf{Q}_x = \sum_{k=1}^{K}\kappa_{x_k}(\mathbf{e}_{K,k}\otimes\mathbf{e}_{K,k})(\mathbf{e}_{K,k}^T\otimes\mathbf{e}_{K,k}^T)$.

noncircularity and the distribution of the sources through the expression of the weighting matrix $\mathbf{C}_r^{-1}$ [see (4.3)]. Consequently, the lemma proved in [10], which states that the constraints $\mathbf{D}_r^{\mathrm{AMV}_1}(\mathbf{A}^* \otimes \mathbf{A}) = \mathbf{O}$ or $\mathbf{D}_r^{\mathrm{AMV}_1}(\mathbf{a}_k^* \otimes \mathbf{a}_k) = \mathbf{0}$, $k = 1, \ldots, K$ that satisfy the derivative $\mathbf{D}_r^{\mathrm{AMV}_1}$ if the sources are not supposed to be spatially uncorrelated or respectively supposed spatially uncorrelated, does not allow us to conclude that the expression of $\mathbf{C}_{\Theta_1}$ is generally in sensitive to the noncircularity and the distribution of the sources.

Remark 3: Note that in the particular case of one source, the numerical value of $\mathbf{C}_\Theta$ is block diagonal $\begin{bmatrix} C_{\Theta_1} & \mathbf{0}^T \\ \mathbf{0} & \mathbf{C}_{\Theta_2} \end{bmatrix}$, where C_{Θ_1} does not depend on the noncircularity and the distribution of the source, but we have not succeeded in proving these properties analytically.

B. AMV EStimator Based on ($\mathbf{R}_T$, $\mathbf{R}'_T$)

To extend Lemma 1 to the statistic $\mathbf{s}_T$, we need to consider the asymptotic joint distribution of $\mathrm{vec}(\mathbf{R}_T)$ and $\mathrm{vec}(\mathbf{R}'_T)$. The standard central limit theorem of the previous section extends similarly to the independent equidistributed complex noncircular random variables $\begin{bmatrix} \mathbf{y}_t^* \otimes \mathbf{y}_t \\ \mathbf{y}_t \otimes \mathbf{y}_t \end{bmatrix}$. From simple algebraic manipulations of $\mathbf{C}_{\substack{r\\r'}} = \mathrm{E}\left(\begin{pmatrix} \mathbf{y}_t^*\otimes\mathbf{y}_t - \mathrm{vec}(\mathbf{R}(\Theta)) \\ \mathbf{y}_t\otimes\mathbf{y}_t - \mathrm{vec}(\mathbf{R}'(\Theta)) \end{pmatrix}\begin{pmatrix} \mathbf{y}_t^*\otimes\mathbf{y}_t - \mathrm{vec}(\mathbf{R}(\Theta)) \\ \mathbf{y}_t\otimes\mathbf{y}_t - \mathrm{vec}(\mathbf{R}'(\Theta)) \end{pmatrix}^H\right)$ and $\mathbf{C}'_{\substack{r\\r'}} = \mathrm{E}\left(\begin{pmatrix} \mathbf{y}_t^*\otimes\mathbf{y}_t - \mathrm{vec}(\mathbf{R}(\Theta)) \\ \mathbf{y}_t\otimes\mathbf{y}_t - \mathrm{vec}(\mathbf{R}'(\Theta)) \end{pmatrix}\begin{pmatrix} \mathbf{y}_t^*\otimes\mathbf{y}_t - \mathrm{vec}(\mathbf{R}(\Theta)) \\ \mathbf{y}_t\otimes\mathbf{y}_t - \mathrm{vec}(\mathbf{R}'(\Theta)) \end{pmatrix}^T\right)$, we straightforwardly prove the following lemma.

Lemma 2: $\sqrt{T}\begin{pmatrix} \mathrm{vec}(\mathbf{R}_T) - \mathrm{vec}(\mathbf{R}(\Theta)) \\ \mathrm{vec}(\mathbf{R}'_T) - \mathrm{vec}(\mathbf{R}'(\Theta)) \end{pmatrix}$ converges in distribution to the zero-mean complex noncircular Gaussian distribution of covariances $\mathbf{C}_{\substack{r\\r'}} = \begin{pmatrix} \mathbf{C}_r & \mathbf{C}_{r,r'} \\ \mathbf{C}_{r,r'}^H & \mathbf{C}_{r'} \end{pmatrix}$ and $\mathbf{C}'_{\substack{r\\r'}} = \begin{pmatrix} \mathbf{C}_r\mathbf{K} & \mathbf{K}\mathbf{C}_{r,r'}^* \\ \mathbf{C}_{r,r'}^H\mathbf{K} & \mathbf{C}'_{r'} \end{pmatrix}$, where $\mathbf{C}_r$ is given by (4.3), and

$$
\begin{aligned}
\mathbf{C}_{r'} =& (\mathbf{A}\otimes\mathbf{A})\mathbf{C}_{r'_x}(\mathbf{A}^H \otimes \mathbf{A}^H) + \sigma_n^4(\mathbf{I}_{M^2} + \mathbf{K}) \\
&+ (\mathbf{I}_{M^2} + \mathbf{K})\left(\sigma_n^2\mathbf{I}_M \otimes \mathbf{A}\mathbf{R}_x\mathbf{A}^H + \mathbf{A}\mathbf{R}_x\mathbf{A}^H \otimes \sigma_n^2\mathbf{I}_M\right) \\
\mathbf{C}'_{r'} =& (\mathbf{A}\otimes\mathbf{A})\mathbf{C}'_{r'_x}(\mathbf{A}^T\otimes\mathbf{A}^T) \\
\mathbf{C}_{r,r'} =& (\mathbf{A}^*\otimes\mathbf{A})\mathbf{C}_{r_x,r'_x}(\mathbf{A}^H\otimes\mathbf{A}^H)
\end{aligned}
$$

with

$$
\begin{aligned}
\mathbf{C}_{r'_x} &= \mathbf{R}_x \otimes \mathbf{R}_x + \mathbf{K}(\mathbf{R}_x \otimes \mathbf{R}_x) + \mathbf{Q}_x \\
\mathbf{C}'_{r'_x} &= \mathbf{R}'_x \otimes \mathbf{R}'_x + \mathbf{K}(\mathbf{R}'_x \otimes \mathbf{R}'_x) + \mathbf{Q}'_x \\
\mathbf{C}_{r_x,r'_x} &= \mathbf{R}'^*_x \otimes \mathbf{R}_x + \mathbf{K}(\mathbf{R}_x \otimes \mathbf{R}'^*_x) + \mathbf{Q}''_x
\end{aligned}
$$

where $\mathbf{Q}_x$ is given in Lemma 1, and[5]

$$
\begin{aligned}
\mathbf{Q}'_x =& (\mathbf{B}\otimes\mathbf{B})\left(\sum_{k=1}^{\tilde{K}} \kappa'_{\tilde{x}_k}(\mathbf{e}_{\tilde{K},k}\otimes\mathbf{e}_{\tilde{K},k})\left(\mathbf{e}_{\tilde{K},k}^T \otimes \mathbf{e}_{\tilde{K},k}^T\right)\right) \\
&\times(\mathbf{B}^T\otimes\mathbf{B}^T) \\
\mathbf{Q}''_x =& (\mathbf{B}^*\otimes\mathbf{B})\left(\sum_{k=1}^{\tilde{K}} \kappa''_{\tilde{x}_k}(\mathbf{e}_{\tilde{K},k}\otimes\mathbf{e}_{\tilde{K},k})\left(\mathbf{e}_{\tilde{K},k}^T \otimes \mathbf{e}_{\tilde{K},k}^T\right)\right) \\
&\times(\mathbf{B}^H\otimes\mathbf{B}^H).
\end{aligned}
$$

Thanks to the standard continuity theorem, the asymptotic behavior of $\mathbf{s}_T$ and ($\mathbf{R}_T$, $\mathbf{R}'_T$) are directly related. Therefore, Lemma 1 extends to the statistic $\mathbf{s}_T$

$$
\sqrt{T}\,(\mathbf{s}_T - \mathbf{s}(\Theta)) \xrightarrow{\mathcal{L}} \mathcal{N}_c\left(\mathbf{0}; \mathbf{C}_s(\Theta), \mathbf{C}'_s(\Theta)\right)
$$

with

$$
\mathbf{C}_s(\Theta) = \begin{pmatrix} \mathbf{C}_r & \mathbf{C}_{r,r'}\mathbf{U}^T & \mathbf{K}\mathbf{C}_{r,r'}^*\mathbf{U}^T \\ \mathbf{U}\mathbf{C}_{r,r'}^H & \mathbf{U}\mathbf{C}_{r'}\mathbf{U}^T & \mathbf{U}\mathbf{C}'_{r'}\mathbf{U}^T \\ \mathbf{U}\mathbf{C}_{r,r'}^T\mathbf{K} & \mathbf{U}\mathbf{C}'^H_{r'}\mathbf{U}^T & \mathbf{U}\mathbf{C}^*_{r'}\mathbf{U}^T \end{pmatrix} \quad \text{and}
$$

$$
\mathbf{C}'_s(\Theta) = \mathbf{C}_s(\Theta)\mathbf{P}. \tag{4.5}
$$

Consequenlty, Theorem 4 extends to the minimum variance second-order DOA estimator (3.2) based on ($\mathbf{R}_T$, $\mathbf{R}'_T$) by direct application of Theorem 1. Following the same procedure used to prove Theorem 4, where, here, $\mathbf{S}_1 \stackrel{\mathrm{def}}{=} (\partial\mathbf{s}/\partial\Theta_1)$, $\mathbf{\Psi}$ given by $\mathbf{s} = \mathbf{\Psi}(\Theta_1)\Theta_2$ and $\mathbf{C}_r(\Theta)$ is replaced by $\mathbf{C}_s(\Theta)$ given in (4.5), we prove the following theorem.

Theorem 5: For Gaussian or non-Gaussian and complex circular or noncircular sources, the covariance of the asymptotic distribution of the minimum variance second-order DOA estimator based on $\mathbf{R}_T$ and $\mathbf{R}'_T$ has the common closed-form expression

$$
\mathbf{C}_{\Theta_1} = \left(\mathbf{S}_1^H\mathbf{C}_s^{-\frac{1}{2}}\mathbf{P}^{\perp}_{\mathbf{C}_s^{-\frac{1}{2}}\mathbf{\Psi}}\mathbf{C}_s^{-\frac{1}{2}}\mathbf{S}_1\right)^{-1}. \tag{4.6}
$$

Remark 1: If the sources are Gaussian complex noncircular, the stochastic maximum likelihood estimator is a second-order algorithm based on $\mathbf{R}_T$ and $\mathbf{R}'_T$. Because it is asymptotically efficient, the closed-form expression (4.6), where the fourth-order terms $\mathbf{Q}_x$, $\mathbf{Q}'_x$, and $\mathbf{Q}''_x$ are canceled in $\mathbf{C}_s(\Theta)$, equals the Cramér–Rao bound on the DOA parameters alone in these conditions.

Remark 2: If the sources are complex circular up to the fourth order, $\mathbf{R}'_x = \mathbf{O}$, $\mathbf{Q}'_x = \mathbf{Q}''_x = \mathbf{O}$, and consequently, $\mathbf{C}_{r,r'} = \mathbf{O}$, and $\mathbf{C}'_{r'} = \mathbf{O}$. Therefore, $\mathbf{C}_s$ is block diagonal: $\mathbf{C}_s = \begin{pmatrix} \mathbf{C}_r & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \times & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \times \end{pmatrix}$. Consequently, the AMV of a second-order algorithm based on ($\mathbf{R}_T$, $\mathbf{R}'_T$) given by Theorem 5 reduces to

$$
\begin{aligned}
\mathbf{C}_\Theta &= \left(\left(\frac{d\mathbf{r}^H}{d\Theta}, \mathbf{0}^T\right)\begin{pmatrix} \mathbf{C}_r^{-1} & \mathbf{O} \\ \mathbf{O} & \times \end{pmatrix}\begin{pmatrix} \frac{d\mathbf{r}}{d\Theta} \\ \mathbf{0} \end{pmatrix}\right)^{-1} \\
&= \left(\frac{d\mathbf{r}^H}{d\Theta}\mathbf{C}_r^{-1}(\Theta)\frac{d\mathbf{r}}{d\Theta}\right)^{-1}
\end{aligned}
$$

which is the AMV given by a second-order algorithm based on $\mathbf{R}_T$ only.

V. Simulations

In this section, numerical comparisons and Monte Carlo simulations are made between the AMV estimator based on $\mathbf{R}_T$ only and the AMV estimator based on ($\mathbf{R}_T$, $\mathbf{R}'_T$). This will give an indication of the information contributed by the second covariance matrix. The sources emit equipowered unfiltered BPSK modulated signals. We consider a uniform linear array of $M = 6$ sensors separated by a half-wavelength for which $\mathbf{a}_k = (1, e^{i\theta_k}, \ldots, e^{i(M-1)\theta_k})^T$, where $\theta_k = \pi\sin(\alpha_k)$ with α_k, which is the DOAs relative to the normal of array broadside.

[5]If the K sources are independent, $\mathbf{Q}'_x$ and $\mathbf{Q}''_x$ are reduced to $\mathbf{Q}'_x = \sum_{k=1}^{K}\kappa'_{x_k}(\mathbf{e}_{K,k}\otimes\mathbf{e}_{K,k})(\mathbf{e}_{K,k}^T\otimes\mathbf{e}_{K,k}^T)$ and $\mathbf{Q}''_x = \sum_{k=1}^{K}\kappa''_{x_k}(\mathbf{e}_{\tilde{K},k}\otimes\mathbf{e}_{\tilde{K},k})(\mathbf{e}_{\tilde{K},k}^T\otimes\mathbf{e}_{\tilde{K},k}^T)$, respectively.

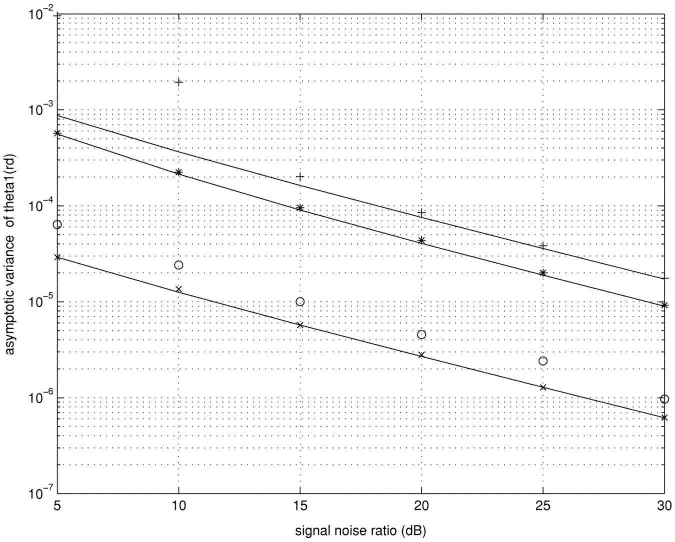

Fig. 1. Theoretical and empirical Var$(\theta_{1,T})$ given by the AMV estimator based on $\mathbf{R}_T$ only ($\star$), by the AMV estimator based on $(\mathbf{R}_T, \mathbf{R}'_T)$ ($\times$), by the MUSIC-like algorithm given in [5] (o), and by the standard MUSIC algorithm ($+$) versus the SNR.

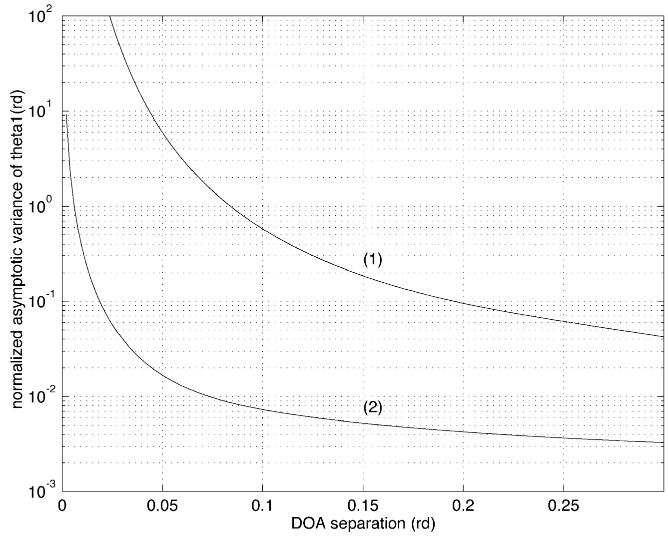

Fig. 2. Theoretical normalized asymptotic variance of $\theta_{1,T}$ ($[\mathbf{C}_{\Theta_1}]_{1,1}$) given by the AMV estimator based on ($\mathbf{R}_T$ only (1) and the AMV estimator based on $\mathbf{R}_T, \mathbf{R}'_T$) (2) versus the DOA separation.

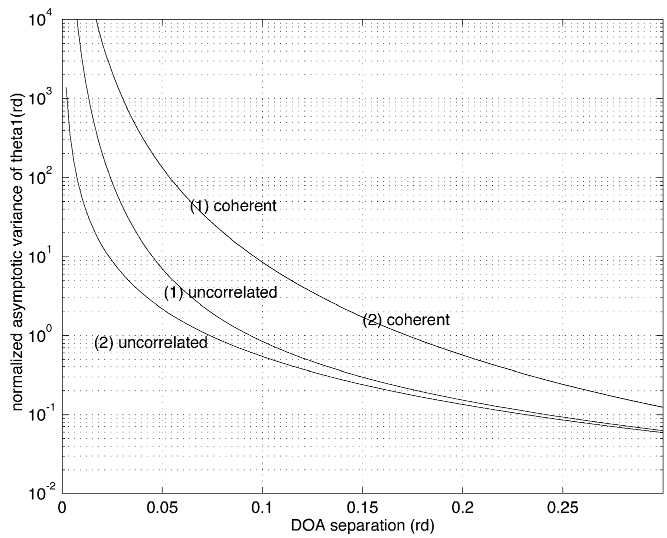

Fig. 3. Theoretical normalized asymptotic variance of $\theta_{1,T}$ ($[\mathbf{C}_{\Theta_1}]_{1,1}$) given by the AMV estimator based on $\mathbf{R}_T$ only (1) and the AMV estimator based on $(\mathbf{R}_T, \mathbf{R}'_T)$ (2) for uncorrelated or coherent sources versus the DOA separation.

In the first experiment, the two sources are independent, and matrices $\mathbf{R}_x$ and $\mathbf{R}'_x$ are parameterized by their diagonal terms. Fig. 1 exhibits the theoretical and empirical (averaged on 1000 independent Monte Carlo runs) Var$(\theta_{1,T})$ given by

- the AMV estimator based on $\mathbf{R}_T$ only;
- the AMV estimator based on $(\mathbf{R}_T, \mathbf{R}'_T)$;
- the MUSIC-like algorithm introduced in [5][6]
- the standard MUSIC algorithm;

versus the signal-to-noise ratio (SNR) for $\theta_2 - \theta_1 = 0.2$ rd and $T = 500$. This figure shows a good agreement between the theoretical and empirical curves, and we notice that the AMV estimator based on $(\mathbf{R}_T, \mathbf{R}'_T)$ outperforms the AMV estimator based on $\mathbf{R}_T$ only, for all values of the SNR. Naturally, the AMV estimators based on $\mathbf{R}_T$ only and $(\mathbf{R}_T, \mathbf{R}'_T)$ perform better than the MUSIC algorithms based on, respectively, $\mathbf{R}_T$ only and $(\mathbf{R}_T, \mathbf{R}'_T)$. Fig. 2 exhibits the theoretical normalized asymptotic variance $[\mathbf{C}_{\Theta_1}]_{1,1}$ given by the AMV estimator based on $\mathbf{R}_T$ only and the AMV estimator based on $(\mathbf{R}_T, \mathbf{R}'_T)$ versus the DOA separation for an SNR of 10 dB. The AMV estimator based on $(\mathbf{R}_T, \mathbf{R}'_T)$ clearly outperforms the AMV estimator based on $\mathbf{R}_T$ only, and the difference is particularly prominent when the sources are very close.

In the second experiment, we select a scenario where the second covariance matrix contributes almost no additional information beyond the information in the first covariance matrix. We consider two spatially correlated waveforms including coherence. The matrices $\mathbf{R}_x$ and $\mathbf{R}'_x$ are parameterized by the real and imaginary parts of their entries (i.e., by $\Re([\mathbf{R}_x]_{2,1})$, $\Im([\mathbf{R}_x]_{2,1})$, $\Re([\mathbf{R}'_x]_{2,1})$, $\Im([\mathbf{R}'_x]_{2,1})$, and $([\mathbf{R}_x]_{i,i}, \Re([\mathbf{R}'_x]_{i,i}), \Im([\mathbf{R}'_x]_{i,i}))_{i=1,2}$). We suppose that the signals consist of two equipowered multipaths issued from the DOAs θ_1 and θ_2. Referenced on the first sensor and from the DOA θ_1, we have equivalently $x_{t,1} = \tilde{x}_{t,1}$ and $x_{t,2} = \cos(\alpha)\tilde{x}_{t,1} + \sin(\alpha)\tilde{x}_{t,2}$ with $\mathbf{R}_{\tilde{x}} = \sigma_1^2\mathbf{I}_2$ and $\mathbf{R}'_{\tilde{x}} = \sigma_1^2\begin{pmatrix} e^{i\phi_1} & 0 \\ 0 & e^{i\phi_2}\end{pmatrix}$. Consequently

$$\mathbf{R}_x = \sigma_1^2\begin{pmatrix} 1 & \cos(\alpha) \\ \cos(\alpha) & 1 \end{pmatrix} \quad \text{and}$$

$$\mathbf{R}'_x = \sigma_1^2\begin{pmatrix} e^{2i\phi_1} & \cos(\alpha)e^{2i\phi_1} \\ \cos(\alpha)e^{2i\phi_1} & \cos^2(\alpha)e^{2i\phi_2} + \sin^2(\alpha)e^{2i\phi_2} \end{pmatrix}.$$

Fig. 3 exhibits the theoretical normalized asymptotic variance $[\mathbf{C}_{\Theta_1}]_{1,1}$ given by the AMV estimator based on $\mathbf{R}_T$ only and the AMV estimator based on $(\mathbf{R}_T, \mathbf{R}'_T)$ versus the DOA separation for uncorrelated $(\alpha = (\pi/2))$ and coherent $(\alpha = 0)$ sources for a SNR of 10 dB. We see that the AMV estimators based on $\mathbf{R}_T$ and on $(\mathbf{R}_T, \mathbf{R}'_T)$ have the same performance with coherent signals, whereas the AMV estimator based on $(\mathbf{R}_T, \mathbf{R}'_T)$ slightly outperforms the AMV estimator based on $\mathbf{R}_T$ for uncorrelated

[6]Because no performance study is available in the literature, only the empirical Var$(\theta_{1,T})$ is plotted for this algorithm.

sources. Compared with Fig. 1, we see the crucial role of the parameterization of $\mathbf{R}_x$ and $\mathbf{R}'_x$. If the sources are known to be uncorrelated, we must parameterize these matrices by their diagonal only to benefit from the second covariance matrix.

VI. Conclusion

This paper has introduced asymptotically minimum variance algorithms in the class of algorithms based on second-order statistics for estimating DOA parameters of possibly spatially correlated even coherent narrowband noncircular sources impinging on arbitrary array structures. The performance of the proposed algorithms were evaluated by closed-form expressions of the asymptotic covariance of the DOA estimates that can be used as a lower bound to assess the performance of any suboptimal second-order algorithms. These asymptotic covariances were numerically compared with those obtained by AMV algorithms based on the first covariance matrix only. We have then realized that the expected benefits due to the noncircular property mainly happen for uncorrelated sources and, furthermore, if the parameterization takes this information into account. Naturally, these conclusions must be mitigated because a thorough comparison between these two AMV algorithms would need a large quantity of scenarios (various geometry arrays, number of sources, noncircularity, correlation, and SNR).

An issue that was not addressed in this paper is the sufficient conditions that guarantee the identifiability of the DOA parameters from the two covariance matrices for noncircular signals. This crucial question is not trivial, and it is, in fact, application specific since it depends on the structure of the array, the spatial covariance, and the type of noncircularity of the sources. A study to deal with this issue is underway.

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