Asymptotic Normality of Sample Covariance Matrix for Mixed Spectra Time Series: Application to Sinusoidal Frequencies Estimation

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***Abstract*—This correspondence addresses the asymptotic normal distribution of the sample mean and the sample covariance matrix of mixed spectra time series containing a sum of sinusoids and a moving average (MA) process. Two central limit (CL) theorems are proved. As an application of this result, the asymptotic normal distribution of any sinusoidal frequencies estimator of such time series based on second-order statistics is deduced.**

***Index Terms*—Central limit (CL) theorem, covariance-based sinusoidal frequencies estimation, mixed spectra time series, sample covariance matrix.**

I. Introduction

There is considerable literature (e.g., [1]–[4]) concerning the asymptotic Gaussian distribution of the sample mean and the sample covariance matrix of the real-valued stationary processes x_t. Several situations have been considered, among them when x_t is a generalized linear process, satisfies mixing conditions, or is Gaussian with a power spectral density. However, few contributions have been devoted to the asymptotic distributions of the sample covariance matrix associated with mixed spectra time series. Subsequent to the revision of this manuscript, [5] was brought to our attention, where this problem is tackled with quite advanced statistical tools such as martingale theory.

We will be concerned with real- or complex-valued[1] processes of the type

$$x_t = m + \sum_{k=1}^{K} a_k \cos(2\pi f_k t + \phi_k) + v_t$$

$$\text{or} \quad x_t = m + \sum_{k=1}^{K} a_k e^{i\phi_k} e^{i2\pi f_k t} + v_t \quad (1.1)$$

with

$$v_t = \sum_{q=0}^{Q} b_q u_{t-q}. \quad (1.2)$$

Throughout this correspondence, $(u_t)_{t=1,\ldots,n}$ is a sequence of zero-mean independent and identically distributed (i.i.d.) random variables where $\mathrm{E}|u_t^4| < \infty$, with

$$c_u \stackrel{\text{def}}{=} \mathrm{E}(u_t^2), \quad \kappa_u \stackrel{\text{def}}{=} \mathrm{Cum}(u_t, u_t, u_t, u_t)$$

and

$$c_u \stackrel{\text{def}}{=} \mathrm{E}|u_t^2|, \quad c'_u \stackrel{\text{def}}{=} \mathrm{E}(u_t^2)$$
$$\kappa_u \stackrel{\text{def}}{=} \mathrm{Cum}(u_t, u_t^*, u_t, u_t^*)$$

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[1]Complex processes appear as complex envelope of bandpass real processes.

respectively in the real and the possibly noncircular[2] complex case. $(a_k)_{k=1,\ldots,K}$ and $(b_q)_{q=0,\ldots,Q}$ are unknown fixed real or complex numbers, respectively. m is an unknown fixed number and f_k are unknown fixed distinct real numbers in $]0, 1/2[$ for real-valued processes [resp., in] $-1/2, +1/2$ [for complex-valued processes]. For the phases ϕ_k, the model (1.1) can be interpreted in two different ways, leading to different statistical descriptions.

1) We can assume that ϕ_k are random variables uniformly distributed on $[0, 2\pi]$ and that $(\phi_k)_{k=1,\cdots,K}$ and u_t are mutually independent. In this case, x_t is a wide-sense-stationary process.
2) We can assume that ϕ_k are nonrandom unknown parameters[3] and so x_t is not a wide-sense-stationary process.

We are interested in the asymptotic distribution of the sample mean $m_n \stackrel{\text{def}}{=} \frac{1}{n}\sum_{t=1}^{n} x_t$ and of the sample covariance matrix

$$\boldsymbol{R}_n \stackrel{\text{def}}{=} \frac{1}{n}\sum_{t=1}^{n} (\boldsymbol{x}_t - \boldsymbol{m}_n)(\boldsymbol{x}_t - \boldsymbol{m}_n)^T$$

in the real case [resp.,

$$\boldsymbol{R}_n \stackrel{\text{def}}{=} \frac{1}{n}\sum_{t=1}^{n} (\boldsymbol{x}_t - \boldsymbol{m}_n)(\boldsymbol{x}_t - \boldsymbol{m}_n)^H$$

in the complex case], where

$$\boldsymbol{x}_t \stackrel{\text{def}}{=} (x_t, x_{t-1}, \ldots, x_{t-p+1})^T$$

and

$$\boldsymbol{m}_n \stackrel{\text{def}}{=} (m_n, \ldots, m_n)^T.$$

The asymptotic normality of the sample mean m_n and sample covariance matrix $\boldsymbol{R}_n$ is proved in Section II. As an application of this result, the asymptotic normal distribution of any sinusoidal frequencies estimator of such time series, based on second-order statistics, is derived in Section III.

II. Central Limit Theorems

For the convenience of the reader, the definition of the complex Gaussian distribution is recalled. A complex random $p \times 1$ vector $\boldsymbol{y}$ has a zero-mean complex Gaussian distribution specified by a $p \times p$ positive-definite matrix $\boldsymbol{\Sigma}_1$ and a $p \times p$ symmetric matrix $\boldsymbol{\Sigma}_2$ and denoted $\mathcal{N}(\boldsymbol{0}; \boldsymbol{\Sigma}_1, \boldsymbol{\Sigma}_2)$[4] if the $2p$-joint distribution of the real and imaginary part of $\boldsymbol{y}$ is $2p$-zero-mean real Gaussian, i.e., if for any complex $p \times 1$ vector $\boldsymbol{w}$, the real scalar $\boldsymbol{w}^H\boldsymbol{y} + (\boldsymbol{w}^H\boldsymbol{y})^H$ has a zero-mean real Gaussian distribution with variance

$$2\boldsymbol{w}^H\boldsymbol{\Sigma}_1\boldsymbol{w} + \boldsymbol{w}^H\boldsymbol{\Sigma}_2\boldsymbol{w}^* + \boldsymbol{w}^T\boldsymbol{\Sigma}_2^*\boldsymbol{w} \quad (2.1)$$

where $\mathrm{E}(\boldsymbol{y}\boldsymbol{y}^H) = \boldsymbol{\Sigma}_1$ and $\mathrm{E}(\boldsymbol{y}\boldsymbol{y}^T) = \boldsymbol{\Sigma}_2$.

Considering the sample mean, the following theorem is proved in the Appendix.

Theorem 1: $\sqrt{n}(m_n - m)$ converges in distribution to the zero-mean real [resp., complex] Gaussian distribution of variance c_m [resp., c_m, c'_m] in the real case [resp., in the complex case] irrespective of the phase model

$$\sqrt{n}(m_n - m) \xrightarrow{\mathcal{L}} \mathcal{N}(0, c_m) \qquad [\text{resp., } \mathcal{N}(0; c_m, c'_m)]. \quad (2.2)$$

[2]Here, circular refers to second-order circular (see, e.g., [6], which is sometimes called "proper" (see [7])).

[3]In this model, $f_k \neq 0$, otherwise, m would be a special case of a sinusoid.

[4]This notation was introduced in [6]. The matrix $\boldsymbol{\Sigma}_2$ is called relation matrix in [6] and pseudocovariance matrix in [7].

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Furthermore,

$$\lim_{n\to\infty} \mathrm{E}(m_n) = m \qquad \text{and} \qquad \lim_{n\to\infty} n\mathrm{Var}\,(m_n) = c_m \tag{2.3}$$

where c_m and c'_m are defined as

$$c_m = c_u \left|\sum_{q=0}^{Q} b_q\right|^2 \qquad \text{and} \qquad c'_m = c'_u \left(\sum_{q=0}^{Q} b_q\right)^2. \tag{2.4}$$

Then, considering the sample covariance matrix, if $\boldsymbol{R}$ denotes, respectively, $\mathrm{E}[(\boldsymbol{x}_t - \boldsymbol{m})(\boldsymbol{x}_t - \boldsymbol{m})^T]$ and $\mathrm{E}[(\boldsymbol{x}_t - \boldsymbol{m})(\boldsymbol{x}_t - \boldsymbol{m})^H]$ in the real and the complex case of the first statistical model with $\boldsymbol{m} \overset{\text{def}}{=} (m, \ldots, m)^T$, the following theorem is proved in the Appendix.

Theorem 2: $\sqrt{n}\,(\mathrm{Vec}(\boldsymbol{R}_n) - \mathrm{Vec}(\boldsymbol{R}))$[5] converges in distribution to the zero-mean real [resp., complex] Gaussian distribution of covariance $\boldsymbol{C}_R$ [resp., $\boldsymbol{C}_R$, $\boldsymbol{C}_R\boldsymbol{K}$] in the real case [resp., in the complex case], irrespective of the phase model.

$$\sqrt{n}\,(\mathrm{Vec}(\boldsymbol{R}_n) - \mathrm{Vec}(\boldsymbol{R})) \xrightarrow{\mathcal{L}} \mathcal{N}(\boldsymbol{0}, \boldsymbol{C}_R) \quad [\text{resp., } \mathcal{N}(\boldsymbol{0}, \boldsymbol{C}_R, \boldsymbol{C}_R\boldsymbol{K})]. \tag{2.5}$$

Furthermore,

$$\lim_{n\to\infty} \mathrm{E}(\boldsymbol{R}_n) = \boldsymbol{R} \quad \text{and} \quad \lim_{n\to\infty} n\mathrm{Cov}\,(\mathrm{Vec}(\boldsymbol{R}_n)) = \boldsymbol{C}_R \tag{2.6}$$

where $\boldsymbol{C}_R$ is defined respectively in the real and the complex cases as

$$\begin{aligned}\boldsymbol{C}_R = &\int_{-1/2}^{+1/2} S_v^2(f) \\ &\cdot \left[\boldsymbol{e}(f)\boldsymbol{e}^H(f) \otimes_c \boldsymbol{e}(f)\boldsymbol{e}^H(f) + \boldsymbol{e}(f)\boldsymbol{e}^T(f) \otimes_c \boldsymbol{e}(f)\boldsymbol{e}^T(f)\right] df \\ &+ \kappa_u \mathrm{Vec}(\boldsymbol{B}\boldsymbol{B}^T)\mathrm{Vec}^T(\boldsymbol{B}\boldsymbol{B}^T) \\ &+ \frac{1}{2}\sum_{k=1}^{K} a_k^2 S_v(f_k) \left[\boldsymbol{e}(f_k)\boldsymbol{e}^H(f_k) \otimes_c \boldsymbol{e}(f_k)\boldsymbol{e}^H(f_k)\right. \\ &\qquad \left. + \boldsymbol{e}(-f_k)\boldsymbol{e}^H(-f_k) \otimes_c \boldsymbol{e}(-f_k)\boldsymbol{e}^H(-f_k)\right] \\ &+ \frac{1}{2}\sum_{k=1}^{K} a_k^2 S_v(f_k) \left[\boldsymbol{e}(f_k)\boldsymbol{e}^H(-f_k) \otimes_c \boldsymbol{e}(f_k)\boldsymbol{e}^H(-f_k)\right. \\ &\qquad \left. + \boldsymbol{e}(-f_k)\boldsymbol{e}^H(f_k) \otimes_c \boldsymbol{e}(-f_k)\boldsymbol{e}^H(f_k)\right]\end{aligned} \tag{2.7}$$

$$\begin{aligned}\boldsymbol{C}_R = &\int_{-1/2}^{+1/2} S_v^2(f) \left[\boldsymbol{e}(f)\boldsymbol{e}^H(f) \otimes_c \boldsymbol{e}(f)\boldsymbol{e}^H(f)\right] df \\ &+ \int_{-1/2}^{+1/2} S'^2_v(f) \left[\boldsymbol{e}(f)\boldsymbol{e}^T(f) \otimes_c \boldsymbol{e}(f)\boldsymbol{e}^T(f)\right] df \\ &+ 2\sum_{k=1}^{K} a_k^2 S_v(f_k)[\boldsymbol{e}(f_k)\boldsymbol{e}^H(f_k) \otimes_c \boldsymbol{e}(f_k)\boldsymbol{e}^H(f_k)] \\ &+ \kappa_u \mathrm{Vec}(\boldsymbol{B}\boldsymbol{B}^H)\mathrm{Vec}^H(\boldsymbol{B}\boldsymbol{B}^H)\end{aligned} \tag{2.8}$$

with

$$\boldsymbol{e}(f) \overset{\text{def}}{=} (1, e^{i2\pi f}, \ldots, e^{i2(p-1)\pi f})^H$$

where $\otimes_c$ denotes the complex Kronecker product $\boldsymbol{A} \otimes_c \boldsymbol{B}$, i.e., the block matrix, the (i, j) block element of which is $b_{i,j}^* \boldsymbol{A}$[6] and $\boldsymbol{K}$ is the vec-permutation matrix which transforms $\mathrm{Vec}\,(\boldsymbol{A})$ to $\mathrm{Vec}\,(\boldsymbol{A}^T)$ for any square matrix $\boldsymbol{A}$.

$$S_v(f) \overset{\text{def}}{=} c_u \left|\sum_{q=0}^{Q} b_q e^{-i2\pi fq}\right|^2$$

i.e., the spectral density of v_t and

$$S'_v(f) \overset{\text{def}}{=} |c'_u| \left|\left(\sum_{q=0}^{Q} b_q e^{-i2\pi fq}\right)\left(\sum_{q=0}^{Q} b_q e^{+i2\pi fq}\right)\right|.$$

$\boldsymbol{B}$ denotes the $p \times (p+Q)$ filtering matrix

$$\begin{pmatrix} b_0 & b_1 & \cdots & b_Q & & & \\ & \ddots & & & \ddots & & \\ & & & b_0 & b_1 & \cdots & b_Q \end{pmatrix}.$$

Remark 1: In (2.6), $\mathrm{Cov}\,(\mathrm{Vec}(\boldsymbol{R}_n))$ denotes

$$\mathrm{E}(\mathrm{Vec}(\boldsymbol{R}_n - \boldsymbol{R})\mathrm{Vec}^H(\boldsymbol{R}_n - \boldsymbol{R})).$$

We note that

$$\mathrm{Vec}^T(\boldsymbol{R}_n - \boldsymbol{R}) = \mathrm{Vec}^H(\boldsymbol{R}_n - \boldsymbol{R})\boldsymbol{K} \tag{2.9}$$

so

$$\mathrm{E}(\mathrm{Vec}(\boldsymbol{R}_n - \boldsymbol{R})\mathrm{Vec}^T(\boldsymbol{R}_n - \boldsymbol{R})) = \mathrm{E}(\mathrm{Vec}(\boldsymbol{R}_n - \boldsymbol{R})\mathrm{Vec}^H(\boldsymbol{R}_n - \boldsymbol{R}))\boldsymbol{K}.$$

Therefore, the noncircular complex Gaussian asymptotic distribution of $\boldsymbol{R}_n$ is characterized by $\boldsymbol{C}_R$ only.

Remark 2: We note that expression (2.7) of $\boldsymbol{C}_R$ obtained in the real case cannot be deduced from expression (2.8) in the complex case as suggested by the Euler relation applied to relation (1.1)

$$a_k \cos(2\pi f_k t + \phi_k) = \frac{a_k}{2} e^{i\phi_k} e^{i2\pi f_k t} + \frac{a_k}{2} e^{-i\phi_k} e^{-i2\pi f_k t}.$$

In fact, if $S'_v(f)$, $\boldsymbol{B}^H$, and $a_k^2[\boldsymbol{e}(f_k)\boldsymbol{e}^H(f_k) \otimes_c \boldsymbol{e}(f_k)\boldsymbol{e}^H(f_k)]$ are, respectively, replaced with $S_v(f)$, $\boldsymbol{B}^T$, and

$$\left(\frac{a_k}{2}\right)^2 [\boldsymbol{e}(f_k)\boldsymbol{e}^H(f_k) \otimes_c \boldsymbol{e}(f_k)\boldsymbol{e}^H(f_k)] + \left(\frac{a_k}{2}\right)^2 [\boldsymbol{e}(-f_k)\boldsymbol{e}^H(-f_k) \otimes_c \boldsymbol{e}(-f_k)\boldsymbol{e}^H(-f_k)]$$

an extra cross-term appears in (2.7). This surprising property is apparently contradictory, given the result that the asymptotic distribution of the sample covariance matrix does not depend on the phase model. In fact, the explanation comes from certain expressions proved in the complex case in the Appendix (see footnote 10) being valid only if no frequencies are opposite. The fact that ϕ_k and $-\phi_k$, associated with f_k and $-f_k$, are not independent is irrelevant.

Remark 3: We note that for a "very narrow band" (i.e., the bandwidth is very small with respect to the sampling frequency) moving average (MA) processes v_t, (2.7) and (2.8) are unbounded. For example, when v_t tends to be white in $[f_0 - b, f_0 + b]$ with finite fixed power c_v, (2.7) and (2.8) are not bounded with b, because $S_v(f)$ tends to $c_v/2b$ for $|f - f_0| \leq b$ and 0 elsewhere, which implies that $\boldsymbol{C}_R$ contains the terms $\int_{-1/2}^{+1/2} S_v^2(f)\,df$ which tends to $\frac{c_v^2}{2b}$.

Since the matrix $\boldsymbol{R}$ is Toeplitz, the "accuracy" of its sample covariance estimate $\boldsymbol{R}_n$, which is non-Toeplitz, should be improved by replacing it by its "Toeplitzed" estimate. This "Toeplitzation," also known as redundancy averaging in statistical signal and array processing applications [9], is carried out by averaging along the diagonals. The resulting estimate $\boldsymbol{R}_n^{\text{to}}$ is referred as the "Toeplitzed" estimated covariance matrix. Because this "Toeplitzation" operates a linear transform on $\boldsymbol{R}_n$, Theorem 2 extends as follows.

[5] $\mathrm{Vec}\,(.)$ is the "vectorization" operator that turns a matrix into a vector consisting of the columns of the matrix stacked one below another.

[6] This slightly unusual convention makes it easier to deal with complex matrices.

Corollary 1: $\mathrm{Vec}(\boldsymbol{R}_n)$ and $\mathrm{Vec}(\boldsymbol{R}_n^{\mathrm{to}})$ have the same asymptotic Gaussian distribution. It is characterized by the asymptotic distribution of the first column $\boldsymbol{r}_n$ of $\boldsymbol{R}_n$. In the real case [resp., in the complex case], we have

$$\sqrt{n}\,(\boldsymbol{r}_n-\boldsymbol{r})\xrightarrow{\mathcal{L}}\mathcal{N}(\mathbf{0},\,\boldsymbol{C}_r)\qquad[\text{resp., }\mathcal{N}(\mathbf{0};\boldsymbol{C}_r,\,\boldsymbol{C}'_r)]. \tag{2.10}$$

Furthermore,

$$\lim_{n\to\infty}\mathrm{E}(\boldsymbol{r}_n)=\boldsymbol{r}\quad\text{and}\quad\lim_{n\to\infty}n\mathrm{E}[(\boldsymbol{r}_n-\boldsymbol{r})(\boldsymbol{r}_n-\boldsymbol{r})^T]=\boldsymbol{C}_r \tag{2.11}$$

resp., $\lim_{n\to\infty}n\mathrm{E}[(\boldsymbol{r}_n-\boldsymbol{r})(\boldsymbol{r}_n-\boldsymbol{r})^H]=\boldsymbol{C}_r$

$$\lim_{n\to\infty}n\mathrm{E}[(\boldsymbol{r}_n-\boldsymbol{r})(\boldsymbol{r}_n-\boldsymbol{r})^T]=\boldsymbol{C}'_r] \tag{2.12}$$

where $\boldsymbol{C}_r$ [resp., $\boldsymbol{C}_r$, $\boldsymbol{C}'_r$] is defined in the real [resp., complex] case as

$$\begin{aligned}\boldsymbol{C}_r=&\int_{-1/2}^{+1/2}S_v^2(f)\left[\boldsymbol{e}(f)\boldsymbol{e}^H(f)+\boldsymbol{e}(f)\boldsymbol{e}^T(f)\right]df\\&+\kappa_u\boldsymbol{B}\boldsymbol{b}\boldsymbol{b}^T\boldsymbol{B}^T+\frac{1}{2}\sum_{k=1}^K a_k^2S_v(f_k)\\&\cdot\Big[\boldsymbol{e}(f_k)\boldsymbol{e}^H(f_k)+\boldsymbol{e}(-f_k)\boldsymbol{e}^H(-f_k)\\&\quad+\boldsymbol{e}(f_k)\boldsymbol{e}^H(-f_k)+\boldsymbol{e}(-f_k)\boldsymbol{e}^H(f_k)\Big]\end{aligned} \tag{2.13}$$

$$\begin{aligned}\boldsymbol{C}_r=&\int_{-1/2}^{+1/2}\left[S_v^2(f)\boldsymbol{e}(f)\boldsymbol{e}^H(f)+S'^2_v(f)\boldsymbol{e}(f)\boldsymbol{e}^T(f)\right]df\\&+\kappa_u\boldsymbol{B}\boldsymbol{b}\boldsymbol{b}^H\boldsymbol{B}^H+2\sum_{k=1}^K a_k^2S_v(f_k)\boldsymbol{e}(f_k)\boldsymbol{e}^H(f_k)\end{aligned} \tag{2.14}$$

$$\begin{aligned}\boldsymbol{C}'_r=&\int_{-1/2}^{+1/2}\left[S_v^2(f)\boldsymbol{e}(f)\boldsymbol{e}^T(f)+S'^2_v(f)\boldsymbol{e}(f)\boldsymbol{e}^H(f)\right]df\\&+\kappa_u\boldsymbol{B}\boldsymbol{b}^*\boldsymbol{b}^H\boldsymbol{B}^T+2\sum_{k=1}^K a_k^2S_v(f_k)\boldsymbol{e}(f_k)\boldsymbol{e}^T(f_k)\end{aligned} \tag{2.15}$$

with $\boldsymbol{b}$ is the $(p+Q)\times 1$ vector $(b_0,\,\ldots,\,b_Q,\,0,\,\ldots,\,0)^T$.

Remark 3: In the complex case, we note that contrary to $\boldsymbol{R}_n$ (see Remark 1), the asymptotic distribution of $\boldsymbol{r}_n$ is not characterized by $\boldsymbol{C}_r$ only.

Relation (2.11) reads componentwise with $\boldsymbol{r}_n\stackrel{\mathrm{def}}{=}(r_n^0,\,\ldots,\,r_n^{p-1})^T$ and

$$r_b^i\stackrel{\mathrm{def}}{=}\begin{cases}b_0b_i+\cdots+b_{Q-i}b_Q, & \text{for } 0\le i\le Q\\ 0, & \text{for } Q<i\le p-1\end{cases}$$

$$\begin{aligned}&\lim_{n\to\infty}n\mathrm{Cov}(r_n^i,\,r_n^j)\\&=2\int_{-1/2}^{+1/2}S_v^2(f)\cos(2\pi if)\cos(2\pi jf)\,df\\&\quad+2\sum_{k=1}^K a_k^2S_v^2(f_k)\cos(2\pi if_k)\cos(2\pi jf_k)+\kappa_ur_b^ir_b^j,\quad i,\,j\ge 0.\end{aligned}$$

This extends the property given by [10] and by [3, Theorem 9.4] where v_t is, respectively, a sequence of i.i.d. zero-mean random variables with $\mathrm{E}(v_t^4)<\infty$ or a sequence of i.i.d. Gaussian distributed zero-mean random variables.

III. Application to Estimation of Sinusoidal Frequencies

Theorem 2 allows us to derive the asymptotic performance of most covariance-based sinusoidal frequencies estimation algorithms. With this aim, we adopt a functional analysis approach which consists in recognizing that the whole process of constructing an estimate $\boldsymbol{f}_n$ of $\boldsymbol{f}\stackrel{\mathrm{def}}{=}(f_1,\,\ldots,\,f_K)^T$ is equivalent to defining a functional relationship linking this estimate $\boldsymbol{f}_n$ to the statistics $\boldsymbol{R}_n$ from which it is inferred. This functional dependence is denoted $\boldsymbol{f}_n=\mathrm{alg}(\boldsymbol{R}_n)$. Clearly, $\boldsymbol{f}=\mathrm{alg}(\boldsymbol{R})$ with $\boldsymbol{R}=\boldsymbol{E}(\boldsymbol{f})\boldsymbol{\Delta}\boldsymbol{E}^H(\boldsymbol{f})+c_u\boldsymbol{B}\boldsymbol{B}^H$,[7] where

$$\boldsymbol{E}(\boldsymbol{f})\stackrel{\mathrm{def}}{=}(\boldsymbol{e}(f_1),\,\ldots,\,\boldsymbol{e}(f_K))$$

and

$$\boldsymbol{\Delta}\stackrel{\mathrm{def}}{=}\mathrm{Diag}(a_1^2,\,\ldots,\,a_K^2).$$

So the different algorithms $\mathrm{alg}\,(.)$ constitute distinct extensions of the mapping $\boldsymbol{R}\stackrel{\mathrm{alg}}{\mapsto}\boldsymbol{f}$ generated by any unstructured Hermitian matrix $\boldsymbol{R}_n$. In the following, we consider "regular" algorithms. More specifically, we assume the following conditions.

1) The function $\mathrm{alg}\,(.)$ is differentiable in a neighborhood of $\boldsymbol{R}$, i.e., if $\boldsymbol{D}_{f,R}^{\mathrm{alg}}$[8] denotes the $K\times p^2$ matrix of this differential evaluated at point $\boldsymbol{R}$

$$\mathrm{alg}(\boldsymbol{R}+\delta\boldsymbol{R})=\boldsymbol{f}+\boldsymbol{D}_{f,R}^{\mathrm{alg}}\mathrm{Vec}(\delta\boldsymbol{R})+o(\delta\boldsymbol{R}). \tag{3.1}$$

2) For any $\boldsymbol{f}$, any positive-definite diagonal matrix $\boldsymbol{\Delta}$ and any c_u

$$\mathrm{alg}\left(\boldsymbol{E}(\boldsymbol{f})\boldsymbol{\Delta}\boldsymbol{E}^H(\boldsymbol{f})+c_u\boldsymbol{B}\boldsymbol{B}^H\right)=\boldsymbol{f}. \tag{3.2}$$

These two requirements are met for example by the high-resolution second-order frequency estimators such as MUSIC, weighted MUSIC, Min-Norm, TAM, and ESPRIT, which all assume that v_t is white. With (3.1) and (3.2), the following result is proved in the Appendix.

Theorem 3: $\sqrt{n}(\boldsymbol{f}_n-\boldsymbol{f})$ converges in distribution to the zero-mean Gaussian distribution of the covariance $\boldsymbol{C}_f$, which is invariant with respect to the distribution of the noise innovation

$$\sqrt{n}(\boldsymbol{f}_n-\boldsymbol{f})\xrightarrow{\mathcal{L}}\mathcal{N}(\mathbf{0},\,\boldsymbol{C}_f) \tag{3.3}$$

with

$$\begin{aligned}\boldsymbol{C}_f=\int_{-1/2}^{+1/2}&\boldsymbol{D}_{f,R}^{\mathrm{alg}}\Big[S_v^2(f)(\boldsymbol{e}(f)\boldsymbol{e}^H(f)\otimes_c\boldsymbol{e}(f)\boldsymbol{e}^H(f))\\&+S'^2_v(f)(\boldsymbol{e}(f)\boldsymbol{e}^T(f)\otimes_c\boldsymbol{e}(f)\boldsymbol{e}^T(f))\Big]\left(\boldsymbol{D}_{f,R}^{\mathrm{alg}}\right)^H df.\end{aligned} \tag{3.4}$$

So, although the asymptotic covariance of $\mathrm{Vec}\,(\boldsymbol{R}_n)$ is very sensitive to the distribution of u_t,[9] the asymptotic performance of most covariance-based sinusoidal frequency estimators is invariant with respect to the distribution of the noise innovation. However, this asymptotic covariance $\boldsymbol{C}_f$ stays unbounded for narrow-band noise of fixed power because of the term $\int_{-1/2}^{+1/2}S_v^2(f)\,df$ in (3.4). We note that Theorem 3 extends a result given in [13], where explicit expressions for the covariance of the estimation errors associated with MUSIC and ESPRIT methods are derived for complex circular Gaussian white noise.

In the special case, where the spectral density $S_v(f)$ is known up to a multiplicative constant, the whitening of the noise used classically in direction of arrival (DOA) estimation (see, e.g., [14]) can be used to

[7] We consider the complex case only, extension to the real case is straightforward.

[8] Expressions of $\boldsymbol{D}_{f,R}^{\mathrm{alg}}$ are ordinarily deduced from perturbation calculus (see, e.g., [11] for the standard MUSIC algorithm).

[9] $(n\mathrm{Cov}\,(\mathrm{Vec}\,(\boldsymbol{R}_n))$ is unbounded for the super-Gaussian case under the constraint of fixed power because of κ_u in (2.7) and (2.8).

advantage. In this approach, after v_t is whitened by a linear transformation applied to $\boldsymbol{x}_t$, any covariance-based DOA methods based on a calibrated array of generic steering vector in white noise (such as MUSIC, weighted MUSIC, Min-Norm,…) can be used. In these circumstances, it makes sense to study the influence of the spectrum of v_t and the selected linear transformation on the performance of this sinusoidal frequency estimator. Considering our functional analysis, Theorem 3 answers this question. The process v_t is whitened using the Cholesky decomposition $\boldsymbol{L}^H\boldsymbol{L}$ of $(\boldsymbol{B}\boldsymbol{B}^H)^{-1}$ (see, e.g., [15, relation 1.7.19]) and any unitary matrix $\boldsymbol{Q}$

$$(\boldsymbol{B}\boldsymbol{B}^H)^{-1} = \boldsymbol{L}'^H\boldsymbol{L}' \qquad \text{with} \qquad \boldsymbol{L}' \stackrel{\text{def}}{=} \boldsymbol{Q}\boldsymbol{L}$$

and the covariance matrix of $\boldsymbol{x}_t$ becomes

$$\boldsymbol{R}' = \boldsymbol{L}'\boldsymbol{E}(\boldsymbol{f})\boldsymbol{\Delta}\left(\boldsymbol{L}'\boldsymbol{E}(\boldsymbol{f})\right)^H + c_u\boldsymbol{I}.$$

If $\text{alg}\,(.)$ denotes an algorithm based on generic steering vector and white noise assumption, the sinusoidal frequencies are estimated through the following scheme:

$$\boldsymbol{R}_n \mapsto \boldsymbol{R}'_n \stackrel{\text{def}}{=} \boldsymbol{L}'\boldsymbol{R}_n\boldsymbol{L}'^H \stackrel{\text{alg}}{\mapsto} \boldsymbol{f}_n \quad \Rightarrow \quad \boldsymbol{R}_n \stackrel{\text{alg}'}{\mapsto} \boldsymbol{f}_n.$$

Applying the chain differential rule, Theorem 3 applies in this situation by replacing in (3.4), $\boldsymbol{D}^{\text{alg}}_{f,R}$ by $\boldsymbol{D}^{\text{al}'}_{f,R} = \boldsymbol{D}^{\text{alg}}_{f,R'}(\boldsymbol{L}' \otimes_c \boldsymbol{L}')$, because $\text{Vec}(\boldsymbol{R}'_n) = (\boldsymbol{L}' \otimes_c \boldsymbol{L}')\text{Vec}(\boldsymbol{R}_n)$.

Appendix
Proof of Theorems

The complex case is considered only as the same approach may be used for the real case.

A. Proof of Theorem 1

First of all, $\sqrt{n}(m_n - m)$ is decomposed as

$$\sqrt{n}(m_n - m) = \sum_{k=1}^{K} a_k e^{i\phi_k}\frac{1}{\sqrt{n}}\left(\sum_{t=1}^{n} e^{i2\pi f_k t}\right) + \frac{1}{\sqrt{n}}\sum_{t=1}^{n} v_t. \tag{A1}$$

Because

$$\left|\sum_{t=1}^{n} e^{i2\pi f_k t}\right| = \left|\frac{\sin \pi n f_k}{\sin \pi f_k}\right|$$

is bounded, the first term of (A1) converges almost surely to 0 when $n \to \infty$ whatever the phase model. Thus, we can consider the term $\frac{1}{\sqrt{n}}\sum_{t=1}^{n} v_t$ only, in the study of the convergence in distribution of $\sqrt{n}(m_n - m)$. Let

$$y_t \stackrel{\text{def}}{=} w^* v_t + w v_t^*.$$

y_t is a real stationary Q-dependent sequence of random variables with mean zero and correlation γ_k with

$$\gamma_k = \sum_{q=0}^{Q} c_u|w|^2(b^*_{k+q}b_q + b_{k+q}b^*_q) + c'_u w^{*2} b_{k+q}b_q + c'^*_u w^2 b^*_{k+q}b^*_q.$$

The conditions of [2, Theorem 6.4.2] are fulfilled. Thus, $\frac{1}{\sqrt{n}}\sum_{t=1}^{n} y_t$ converges in distribution to the zero-mean real Gaussian distribution of variance $c_y = \gamma_0 + 2\sum_{q=1}^{Q}\gamma_q$ and $\lim_{n\to\infty} n\text{Var}(\frac{1}{n}\sum_{t=1}^{n} y_t) = c_y$ with

$$c_y = 2|w|^2 c_u\left|\sum_{q=0}^{Q} b_q\right|^2 + w^{*2}c'_u\left(\sum_{q=0}^{Q} b_q\right)^2 + w^2 c'^*_u\left(\sum_{q=0}^{Q} b^*_q\right)^2.$$

The theorem follows thanks to (2.1). □

B. Proof of Theorem 2

Because $\frac{1}{n}\sum_{t=1}^{n} v_t$ converges in probability to 0 (see, e.g., [2, Proposition 6.3.10]), $\boldsymbol{m}_n$ converges in probability to $\boldsymbol{m}$. Using a classical result (e.g., deduced from [2, Propositions 6.3.4 and 6.3.7]), we can deduce that studying the asymptotic distribution of $\boldsymbol{R}_n$, boils downs to studying the asymptotic distribution of

$$\boldsymbol{R}'_n \stackrel{\text{def}}{=} \frac{1}{n}\sum_{t=1}^{n}(\boldsymbol{x}_t - \boldsymbol{m})(\boldsymbol{x}_t - \boldsymbol{m})^H.$$

Then, using $\text{Vec}(\boldsymbol{a}\boldsymbol{b}^H) = \boldsymbol{a} \otimes_c \boldsymbol{b}$, $\text{Vec}(\boldsymbol{R}'_n - \boldsymbol{R})$ is decomposed as

$$\text{Vec}(\boldsymbol{R}'_n - \boldsymbol{R}) = \frac{1}{n}\sum_{t=1}^{n}(\boldsymbol{z}^1_t + = \boldsymbol{z}^2_t)$$

with

$$\boldsymbol{z}^1_t \stackrel{\text{def}}{=} \boldsymbol{c}_t \otimes_c \boldsymbol{v}_t + \boldsymbol{v}_t \otimes_c \boldsymbol{c}_t + \boldsymbol{v}_t \otimes_c \boldsymbol{v}_t - \text{Vec}(\boldsymbol{R}_v)$$

and

$$\boldsymbol{z}^2_t \stackrel{\text{def}}{=} \sum_{1\le k\ne k'\le K} a_k a_{k'} e^{i(\phi_k - \phi_{k'})} e^{i2\pi(f_k - f_{k'})t}\boldsymbol{e}(f_k) \otimes_c \boldsymbol{e}(f_{k'})$$

where

$$\boldsymbol{c}_t \stackrel{\text{def}}{=} \sum_{k=1}^{K} a_k e^{i\phi_k} e^{i2\pi f_k t}\boldsymbol{e}(f_k)$$

$$\boldsymbol{v}_t \stackrel{\text{def}}{=} (v_t, v_{t-1}, \ldots, v_{t-p+1})^T$$

and

$$\boldsymbol{R}_v \stackrel{\text{def}}{=} \text{E}(\boldsymbol{v}_t\boldsymbol{v}_t^H).$$

Because

$$\left\|\sqrt{n}\frac{1}{n}\sum_{t=1}^{n}\boldsymbol{z}^2_t\right\| \le \frac{1}{\sqrt{n}}\sum_{1\le k\ne k'\le K} a_k a_{k'}\left|\frac{\sin(\pi n(f_k - f_{k'})}{\sin(\pi(f_k - f_{k'})}\right| \|\boldsymbol{e}(f_k) \otimes_c \boldsymbol{e}(f_{k'})\|$$

$\sqrt{n}\frac{1}{n}\sum_{t=1}^{n}\boldsymbol{z}^2_t$ converges almost surely to $\boldsymbol{0}$ when $n \to \infty$. Thus, we can consider the term $\frac{\sqrt{n}}{n}\sum_{t=1}^{n}\boldsymbol{z}^1_t$ alone, in the study of the convergence in distribution of $\sqrt{n}\text{Vec}(\boldsymbol{R}'_n - \boldsymbol{R})$. To prove the convergence in distribution of $\sqrt{n}\frac{1}{n}\sum_{t=1}^{n}\boldsymbol{z}^1_t$ to a zero-mean noncircular complex Gaussian distribution, we consider the associated scalar real random variable $y_t \stackrel{\text{def}}{=} \boldsymbol{w}^H\boldsymbol{z}^1_t + (\boldsymbol{w}^H\boldsymbol{z}^1_t)^H$ (see definition in Section II). The conditional distribution of $(y_t)_{t=1,\ldots,n}$ given the phases $(\phi_k)_{k=1,\ldots,K}$ is a zero-mean real distribution of L-dependent (with $L = p + Q$) but not strictly stationary random variables because y_t are not identically distributed. As such, the conditions of [2, Theorem 6.4.2] are no longer fully satisfied.

To prove Theorem 2, we continue to make use of [2, Theorem 6.4.2] by some modifications of its proof. Following this proof, we must consider first, the limit of $n\mathrm{Var}(\frac{1}{n}\sum_{t=1}^{n} y_t/\phi)$ when $n \to \infty$. Thanks to (2.9), this expression may be written as

$$2\boldsymbol{w}^H \left[\frac{1}{n}\sum_{1\le s,\,t\le n} \mathrm{E}(\boldsymbol{z}_s^1 \boldsymbol{z}_t^{1H}/\phi)\right]\boldsymbol{w} + \boldsymbol{w}^H\left[\frac{1}{n}\sum_{1\le s,\,t\le n} \mathrm{E}(\boldsymbol{z}_s^1 \boldsymbol{z}_t^{1H}/\phi)\right]\boldsymbol{K}\boldsymbol{w}^* + \boldsymbol{w}^T\left[\frac{1}{n}\sum_{1\le s,\,t\le n} \mathrm{E}(\boldsymbol{z}_s^1 \boldsymbol{z}_t^{1H}/\phi)\right]^* \boldsymbol{K}\boldsymbol{w} \quad \text{(A2)}$$

where

$$\frac{1}{n}\sum_{1\le s,\,t\le n} \mathrm{E}(\boldsymbol{z}_s^1 \boldsymbol{z}_t^{1H}/\phi) = \boldsymbol{T}_a + \boldsymbol{T}_b + \boldsymbol{T}_c$$

with

$$\boldsymbol{T}_a \stackrel{\text{def}}{=} \frac{1}{n}\sum_{1\le s,\,t\le n} \mathrm{E}((\boldsymbol{v}_s \otimes_c \boldsymbol{c}_s)(\boldsymbol{c}_t^H \otimes_c \boldsymbol{v}_t^H)/\phi) + \mathrm{E}((\boldsymbol{c}_s \otimes_c \boldsymbol{v}_s)(\boldsymbol{v}_t^H \otimes_c \boldsymbol{c}_t^H)/\phi)$$

$$\boldsymbol{T}_b \stackrel{\text{def}}{=} \frac{1}{n}\sum_{1\le s,\,t\le n} \mathrm{E}((\boldsymbol{v}_s \otimes_c \boldsymbol{c}_s)(\boldsymbol{v}_t^H \otimes_c \boldsymbol{c}_t^H)/\phi) + \mathrm{E}((\boldsymbol{c}_s \otimes_c \boldsymbol{v}_s)(\boldsymbol{c}_t^H \otimes_c \boldsymbol{v}_t^H)/\phi)$$

$$\boldsymbol{T}_c \stackrel{\text{def}}{=} \frac{1}{n}\sum_{1\le s,\,t\le n} \mathrm{E}((\boldsymbol{v}_s \otimes_c \boldsymbol{v}_s - \mathrm{Vec}(\boldsymbol{R}_v)(\boldsymbol{v}_t \otimes_c \boldsymbol{v}_t - \mathrm{Vec}(\boldsymbol{R}_v))^H.$$

Thanks to the following property of the vec-permutation matrix $\boldsymbol{K}$

$$(\boldsymbol{a}^H \otimes_c \boldsymbol{b}^H)\boldsymbol{K} = \boldsymbol{b}^T \otimes_c \boldsymbol{a}^T \quad \text{(A3)}$$

$$E((\boldsymbol{v}_s \otimes_c \boldsymbol{c}_s)(\boldsymbol{c}_t^H \otimes_c \boldsymbol{v}_t^H)/\phi) + E((\boldsymbol{c}_s \otimes_c \boldsymbol{v}_s)(\boldsymbol{v}_t^H \otimes_c \boldsymbol{c}_t^H)/\phi) = [(E(\boldsymbol{v}_s\boldsymbol{v}_t^T) \otimes_c \boldsymbol{c}_s\boldsymbol{c}_t^T) + (\boldsymbol{c}_s\boldsymbol{c}_t^T \otimes_c E(\boldsymbol{v}_s\boldsymbol{v}_t^T))]\boldsymbol{K}$$

thus the first term of $\boldsymbol{T}_a$ becomes

$$\sum_{l=-p-Q+1}^{p+Q-1} \boldsymbol{R}'_v(l) \otimes_c \cdot \left(\sum_{1\le k,\,k'\le K} a_k a_{k'} e^{i(\phi_k+\phi_{k'})} e^{-i2\pi f_{k'} l} \cdot \left[\frac{1}{n}\sum_{s\in S_{n,l}} e^{i2\pi(f_k+f_{k'})s}\right] \boldsymbol{e}(f_k)\boldsymbol{e}^T(f_{k'})\right)\boldsymbol{K}$$

with $\boldsymbol{R}'_v(l) \stackrel{\text{def}}{=} E(\boldsymbol{v}_s\boldsymbol{v}_{s-l}^T)$ and $S_{n,l}$ is the set $\{s,\ 1\le s\le n-l$ for $l\ge 0$ or $-l+1\le s\le n$ for $l\le 0\}$. Because [10]

$$\lim_{n\to\infty}\left|\frac{1}{n}\sum_{s\in S_{n,l}} e^{i2\pi(f_k+f_{k'})s}\right| = 0$$

the first term of $\boldsymbol{T}_a$ and, therefore, the term $\boldsymbol{T}_a$ tends to $\boldsymbol{O}$ when $n\to\infty$. As

$$E((\boldsymbol{v}_s \otimes_c \boldsymbol{c}_s)(\boldsymbol{v}_t^H \otimes_c \boldsymbol{c}_t^H)/\phi) + E((\boldsymbol{c}_s \otimes_c \boldsymbol{v}_s)(\boldsymbol{c}_t^H \otimes_c \boldsymbol{v}_t^H)/\phi) = (E(\boldsymbol{v}_s\boldsymbol{v}_t^H) \otimes_c \boldsymbol{c}_s\boldsymbol{c}_t^H) + (\boldsymbol{c}_s\boldsymbol{c}_t^H \otimes_c E(\boldsymbol{v}_s\boldsymbol{v}_t^H))$$

[10]Except for the specific case where two frequencies f_k are opposite.

the first term of $\boldsymbol{T}_b$ becomes:

$$\sum_{l=-p-Q+1}^{p+Q-1} \boldsymbol{R}_v(l) \otimes_c \cdot \left(\sum_{1\le k,\,k'\le K} a_k a_{k'} e^{i(\phi_k-\phi_{k'})} e^{i2\pi f_{k'} l} \cdot \left[\frac{1}{n}\sum_{s\in S_{n,l}} e^{i2\pi(f_k-f_{k'})s}\right] \boldsymbol{e}(f_k)\boldsymbol{e}^H(f_{k'})\right)$$

with $\boldsymbol{R}_v(l) \stackrel{\text{def}}{=} E(\boldsymbol{v}_s\boldsymbol{v}_{s-l}^H)$. Because

$$\lim_{n\to\infty}\left|\frac{1}{n}\sum_{s\in S_{n,l}} e^{i2\pi(f_k-f_{k'})s}\right| = 0$$

for $f_k \ne f_{k'}$ and 1 for $f_k = f_{k'}$, the first term of $\boldsymbol{T}_b$ tends to

$$\sum_{1\le k\le K} a_k^2 \left(\sum_{l=-p-Q+1}^{p+Q-1} \boldsymbol{R}_v(l)\right) e^{-i2\pi f_k l} \otimes_c \boldsymbol{e}(f_k)\boldsymbol{e}^H(f_k)$$

when $n\to\infty$. With

$$\left(\sum_{l=-p-Q+1}^{p+Q-1} \boldsymbol{R}_v(l)\right) e^{-i2\pi f_k l} = S_v(f_k)\boldsymbol{e}(f_k)\boldsymbol{e}^H(f_k)$$

the term $\boldsymbol{T}_b$ tends to

$$2\sum_{1\le k\le K} a_k^2 S_v(f_k)[\boldsymbol{e}(f_k)\boldsymbol{e}^H(f_k) \otimes_c \boldsymbol{e}(f_k)\boldsymbol{e}^H(f_k)]. \quad \text{(A4)}$$

Then, because $\boldsymbol{v}_t = \boldsymbol{B}\boldsymbol{u}_t$ with $\boldsymbol{u}_t \stackrel{\text{def}}{=} (u_t, u_{t-1,\ldots,u_{t-p-Q+1}})^T$, the term $\boldsymbol{T}_c$ becomes

$$(\boldsymbol{B} \otimes_c \boldsymbol{B}) n\mathrm{Cov}(\mathrm{Vec}(\boldsymbol{R}_n^u))(\boldsymbol{B}^H \otimes_c \boldsymbol{B}^H) \quad \text{(A5)}$$

with $\boldsymbol{R}_n^u \stackrel{\text{def}}{=} \frac{1}{n}\sum_{t=1}^n \boldsymbol{u}_t\boldsymbol{u}_t^H$.

$$[n\mathrm{Cov}(\mathrm{Vec}(\boldsymbol{R}_n^u))]_{(j-1)(p+Q)+i,\,(l-1)(p+Q)+k}$$

then becomes

$$\frac{1}{n}\sum_{1\le s,\,t\le n} E(u_{s-i+1}u_{s-j+1}^* u_{t-k+1}^* u_{t-l+1}) - E(u_{s-i+1}u_{s-j+1}^*)E(u_{t-k+1}^* u_{t-l+1}). \quad \text{(A.6)}$$

By the definition of $\mathrm{Cum}(u_{s-i+1},\ u_{s-j+1}^*,\ u_{t-k+1}^*,\ u_{t-l+1})$, (A6) is decomposed as

$$E(u_{s-i+1}u_{t-k+1}^*)E(u_{s-j+1}u_{t-k+1}^*) + E(u_{s-i+1}u_{t-l+1})E(u_{s-j+1}^* u_{t-k+1}^*) + \mathrm{Cum}(u_{s-i+1},\ u_{s-j+1}^*,\ u_{t-k+1}^*,\ u_{t-l+1}).$$

These three terms are, respectively, equal to

$$\begin{cases} c_u^2, & \text{for } s-t=i-k=j-l \\ 0, & \text{elsewhere} \end{cases}$$

$$\begin{cases} |c'_u|^2, & \text{for } s-t=i-l=j-k \\ 0, & \text{elsewhere} \end{cases}$$

$$\begin{cases} \kappa_u, & \text{for } i=j,\ k=l \text{ and } s-t=i-k \\ 0, & \text{elsewhere.} \end{cases}$$

Consequently,

$$\lim_{n\to\infty}[n\mathrm{Cov}(\mathrm{Vec}(\boldsymbol{R}_n^u))]_{(j-1)(p+Q)+i,\,(l-1)(p+Q)+k}$$

is defined and decomposed as

$$\begin{cases} c_u^2, & \text{for } i-k=j-l \\ 0. & \text{elsewhere} \end{cases} + \begin{cases} |c'_u|^2, & \text{for } i-l=j-k \\ 0, & \text{elsewhere} \end{cases} + \begin{cases} \kappa_u, & \text{for } i=j,\, k=l \\ 0, & \text{elsewhere} \end{cases}$$

whose associated matrix is

$$c_u^2 \int_{-1/2}^{+1/2} [\boldsymbol{e}(f)\boldsymbol{e}^H(f) \otimes_c \boldsymbol{e}(f)\boldsymbol{e}^H(f)]\, df + |c'_u|^2 \int_{-1/2}^{+1/2} [\boldsymbol{e}(f)\boldsymbol{e}^H(f) \otimes_c \boldsymbol{e}(f)\boldsymbol{e}^H(f)]\boldsymbol{K}\, df + \kappa_u \mathrm{Vec}(\boldsymbol{I})\mathrm{Vec}^T(\boldsymbol{I}).$$

Then

$$(\boldsymbol{B}\otimes_c \boldsymbol{B})\left[\int_{-1/2}^{+1/2}[\boldsymbol{e}(f)\boldsymbol{e}^H(f)\otimes_c \boldsymbol{e}(f)\boldsymbol{e}^H(f)]\, df\right](\boldsymbol{B}^H\otimes_c \boldsymbol{B}^H) = \int_{-1/2}^{+1/2}(\boldsymbol{B}\boldsymbol{e}(f)\boldsymbol{e}^H(f)\boldsymbol{B}^H)\otimes_c(\boldsymbol{B}\boldsymbol{e}(f)\boldsymbol{e}^H(f)\boldsymbol{B}^H)\, df,$$

$$(\boldsymbol{B}\otimes_c \boldsymbol{B})\left[\int_{-1/2}^{+1/2}[\boldsymbol{e}(f)\boldsymbol{e}^H(f)\otimes_c \boldsymbol{e}(f)\boldsymbol{e}^H(f)]\boldsymbol{K}\, df\right](\boldsymbol{B}^H\otimes_c \boldsymbol{B}^H) = \int_{-1/2}^{+1/2}(\boldsymbol{B}\boldsymbol{e}(f)\boldsymbol{e}^H(f)\boldsymbol{B}^T)\otimes_c(\boldsymbol{B}\boldsymbol{e}(f)\boldsymbol{e}^H(f)\boldsymbol{B}^T)\, df$$

thanks to (A3) with

$$c_u \boldsymbol{B}\boldsymbol{e}(f)\boldsymbol{e}^H(f)\boldsymbol{B}^H = S_v(f)\boldsymbol{e}(f)\boldsymbol{e}^H(f)$$

and

$$|c'_u|\boldsymbol{B}\boldsymbol{e}(f)\boldsymbol{e}^H(f)\boldsymbol{B}^T = S'_v(f)\boldsymbol{e}(f)\boldsymbol{e}^H(f).$$

And then

$$\kappa_u(\boldsymbol{B}\otimes_c \boldsymbol{B})\mathrm{Vec}(\boldsymbol{I})\mathrm{Vec}^T(\boldsymbol{I})(\boldsymbol{B}^H\otimes_c \boldsymbol{B}^H) = \kappa_u \mathrm{Vec}(\boldsymbol{B}\boldsymbol{B}^H)\mathrm{Vec}^H(\boldsymbol{B}\boldsymbol{B}^H)$$

thanks to $\mathrm{Vec}(\boldsymbol{ABC}) = (\boldsymbol{A}\otimes_c \boldsymbol{C}^H)\mathrm{Vec}(\boldsymbol{B})$. Putting together the limits of term $\boldsymbol{T}_b$ and of term $\boldsymbol{T}_c$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{1\le s,\, t\le n} E(\boldsymbol{z}_s^1 \boldsymbol{z}_t^{1H}/\phi)$$

is defined and does not depend on the phases ϕ_k. Thanks to (A2) and (2.1), we get (2.8) irrespective of the phase model.

Then, following the proof of [2, Theorem 6.4.2], the application of the classical central limit (CL) theorem (e.g., [2, Theorem 6.4.1]) to the sum

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{r} y_{t,k}$$

with $y_{t,k} \stackrel{\text{def}}{=} y_{(t-1)k+1} + \cdots + y_{tk-L}$, where $r \stackrel{\text{def}}{=} \lfloor n/k \rfloor$ and k fixed with $k > L$ for $n \to \infty$ is not valid because $(y_{t,k})_{t=1,\ldots,r}$ are zero-mean independent but not identically distributed random variables. We replace the classical CL theorem by the Lyapunov theorem (see, e.g., [8, p. 371]) by verifying the following Lyapunov's condition [8, Relation (27.16)] with $\delta = 2$:

$$\lim_{r\to\infty} \frac{\sum_{t=1}^{r} E(y_{t,k}^{2+2}/\phi)}{\left(\sqrt{\sum_{t=1}^{r} E(y_{t,k}^2/\phi)}\right)^{2+2}} = 0. \qquad \text{(A7)}$$

As

$$\frac{1}{n}\sum_{t=1}^{r} E(y_{t,k}^2/\phi) = n\mathrm{Var}\left(\frac{1}{n}\sum_{t=1}^{r} y_{t,k}/\phi\right)$$

we get after similar manipulations as previously that

$$\sum_{t=1}^{r} E(y_{t,k}^2/\phi) \propto n, \qquad \text{when } n \to \infty.$$

On the other hand, $\frac{1}{n}\sum_{t=1}^{r} E(y_{t,k}^4/\phi)$ can be decomposed as the sum of terms

$$c_{a,b,c,d}(\boldsymbol{w})\frac{1}{n}\sum_{t=1}^{r}\left(\sum_{1\le l,\, l',\, l'',\, l'''\le k-L} E\big((\boldsymbol{z}_{(t-1)k+l}^1)_a^{s_a}(\boldsymbol{z}_{(t-1)k+l'}^1)_b^{s_b} \cdot (\boldsymbol{z}_{(t-1)k+l''}^1)_c^{s_c}(\boldsymbol{z}_{(t-1)k+l'''}^1)_d^{s_d}/\phi\big)\right) \qquad \text{(A.8)}$$

for $0 \le a,\, b,\, c,\, d \le p^2$, where $c_{a,b,c,d}(\boldsymbol{w})$ is an appropriate function of $\boldsymbol{w}$ and where $(\boldsymbol{y}_{(t-1)k+l}^1)_i^{s_i}$ denotes the ith component of $\boldsymbol{y}_{(t-1)k+l}^1$ which is conjugated for certain indexes. Because an examination of the term (A8) shows that it has a limit when $n \to \infty$,

$$\sum_{t=1}^{r} E(y_{t,k}^4/\phi) \propto n, \qquad \text{when } n \to \infty.$$

So Lyapounov's condition (A7) with $\delta = 2$ is proved. Therefore, the conditional real scalar random variable

$$\frac{\sum_{t=1}^{r} y_{t,k}}{\sqrt{\sum_{t=1}^{r} E(y_{t,k}^2/\phi)}}$$

converges in distribution to the zero-mean, unit variance real Gaussian distribution when $r \to \infty$.

Finally, incorporating the other elements of the proof of [2, Theorem 6.4.2], $\sqrt{n}\frac{1}{n}\sum_{t=1}^{n} y_t$ converges in distribution to the zero-mean, Gaussian distribution of variance

$$2\boldsymbol{w}^H \boldsymbol{C}_R \boldsymbol{w} + \boldsymbol{w}^H \boldsymbol{C}_R \boldsymbol{K}\boldsymbol{w}^* + \boldsymbol{w}^T \boldsymbol{C}_R^* \boldsymbol{K}\boldsymbol{w}$$

with $\boldsymbol{C}_R$ given by (2.8). And by application of the Cramer–Wold theorem [8, Theorem 29.4], the complex random vector $\sqrt{n}\frac{1}{n}\sum_{t=1}^{n}\boldsymbol{z}_t^1$ converges in distribution to the zero-mean complex noncircular Gaussian distribution $\mathcal{N}(\boldsymbol{0}; \boldsymbol{C}_R,\, \boldsymbol{C}_R\boldsymbol{K})$. □

C. Proof of Corollary 1

Thanks to the "Toeplitzation" projection matrix $\boldsymbol{T}_o$, $\mathrm{Vec}(\boldsymbol{R}_n^{\mathrm{to}}) = \boldsymbol{T}_o\mathrm{Vec}(\boldsymbol{R}_n)$. Therefore, Theorem 2 extends to $\boldsymbol{R}_n^{\mathrm{to}}$ with the asymptotic covariance matrix $\boldsymbol{C}_R^{\mathrm{to}} = \boldsymbol{T}_o\boldsymbol{C}_R\boldsymbol{T}_o$. Because

$$\boldsymbol{e}(f_k)\boldsymbol{e}^+(f_l)\otimes_c \boldsymbol{e}(f_k)\boldsymbol{e}^+(f_l) = (\boldsymbol{e}(f_k)\otimes_c \boldsymbol{e}(f_k))(\boldsymbol{e}^+(f_l)\otimes_c \boldsymbol{e}^+(f_l)), \qquad \text{with } + \stackrel{\text{def}}{=} T \text{ or } H$$

and $\boldsymbol{e}(f_i)\otimes_c \boldsymbol{e}(f_i) = \mathrm{Vec}(\boldsymbol{e}(f_i)\boldsymbol{e}^H(f_i))$ with $\boldsymbol{e}(f_i)\boldsymbol{e}^H(f_i)$ is a Toeplitz matrix

$$\boldsymbol{T}_o(\boldsymbol{e}(f_k)\boldsymbol{e}^+(f_l)\otimes_c \boldsymbol{e}(f_k)\boldsymbol{e}^+(f_l))\boldsymbol{T}_o = \boldsymbol{e}(f_k)\boldsymbol{e}^+(f_l)\otimes_c \boldsymbol{e}(f_k)\boldsymbol{e}^+(f_l).$$

Then, because $\boldsymbol{B}\boldsymbol{B}^+$ is also a Toeplitz matrix, the relation $\boldsymbol{T}_o\boldsymbol{C}_R\boldsymbol{T}_o = \boldsymbol{C}_R$ is proved. Therefore, the "Toeplitzation" does not improve the covariance estimate and the expressions of $\boldsymbol{C}_r$ and $\boldsymbol{C}'_r$ are given by the blocks $(1, 1)$ of $\boldsymbol{C}_R$ and $\boldsymbol{C}'_R$, respectively. □

D. Proof of Theorem 3

From regularity condition (3.1), the asymptotic behaviors of $\boldsymbol{f}_n$ and $\boldsymbol{R}_n$ are directly related. The standard result on regular functions of asymptotically normal statistics (see, e.g., [12, Theorem, p. 22]) applies. So (3.3) with $\boldsymbol{C}_f = \boldsymbol{D}_{f,R}^{\text{alg}} \boldsymbol{C}_R (\boldsymbol{D}_{f,R}^{\text{alg}})^H$. Furthermore, this closed-form expression simplifies if (1) and (2) are taken into account

$$\begin{aligned} \boldsymbol{f} &= \text{alg}(\boldsymbol{E}(\boldsymbol{f})(\boldsymbol{\Delta}+\delta\boldsymbol{\Delta})\boldsymbol{E}^H(\boldsymbol{f})+(c_u+\delta c_u)\boldsymbol{B}\boldsymbol{B}^H) \\ &= \boldsymbol{f}+\boldsymbol{D}_{f,R}^{\text{alg}}\text{Vec}(\boldsymbol{E}(\boldsymbol{f})\delta\boldsymbol{\Delta}\boldsymbol{E}^H(\boldsymbol{f})+\delta c_u\boldsymbol{B}\boldsymbol{B}^H)+o(\delta\boldsymbol{\Delta})+o(\delta c_u) \\ &= \boldsymbol{f}+\boldsymbol{D}_{f,R}^{\text{alg}}\left(\sum_{k=1}^{K}\delta a_k^2(\boldsymbol{e}(f_k)\otimes_c \boldsymbol{e}^H(f_k))+\delta c_u\text{Vec}(\boldsymbol{B}\boldsymbol{B}^H)\right) \\ &\quad +o(\delta\boldsymbol{\Delta})+o(\delta c_u) \end{aligned} \tag{A9}$$

where

$$\text{Vec}(\boldsymbol{e}(f_k)\boldsymbol{e}^H(f_k)) = \boldsymbol{e}(f_k)\otimes_c \boldsymbol{e}^H(f_k)$$

is used in the third equality. Therefore, the following constraints upon $\boldsymbol{D}_{f,R}^{\text{alg}}$ hold:

$$\boldsymbol{D}_{f,R}^{\text{alg}}[\boldsymbol{e}(f_k)\otimes_c \boldsymbol{e}^H(f_k)] = \boldsymbol{0}, \qquad k=1,\ldots,K \quad \text{and} \quad \boldsymbol{D}_{f,R}^{\text{alg}}\text{Vec}(\boldsymbol{B}\boldsymbol{B}^H) = \boldsymbol{0} \tag{A10}$$

and using (2.8), the proof follows. □

REFERENCES

[1] E. J. Hannan, *Multiple Time Series*. New York: Wiley, 1970.
[2] P. J. Brockwell and R. A. Davis, *Time Series, Theory and Methods*, 2nd ed. New York: Springer-Verlag, 1991.
[3] B. Porat, *Digital Processing of Random Signals, Theory and Methods*. Englewood Cliffs, NJ: Prentice-Hall, 1993.
[4] M. Rosenblatt, *Stationary Sequences and Random Fields*. Basel, Switzerland: Birkhäuser, 1985.
[5] T. H. Li, B. Kedem, and S. Yakowitz, "Asymptotic normality of sample autocovariances with an application in frequency estimation," *Stochastic Processes Their Applic.*, vol. 52, pp. 329–349, 1994.
[6] B. Picinbono, "Second-order complex random vectors and normal distributions," *IEEE Trans. Signal Processing*, vol. 44, pp. 2637–2640, Oct. 1996.
[7] F. D. Neeser and J. L. Massey, "Proper complex random processes with applications to information theory," *IEEE Trans. Inform. Theory*, vol. 39, pp. 1293–1302, July 1993.
[8] P. Billingsley, *Probability and Measure*, 2nd ed. New York: Wiley, 1986.
[9] A. Gorokhov, Y. Abramovich, and J. F. Böhme, "Unified analysis of DOA estimation algorithms for covariance matrix transforms," *Signal Processing*, vol. 55, pp. 107–115, 1996.
[10] P. Stoica, T. Söderströom, and F. Ti, "Overdetermined Yule–Walker estimation of the frequencies of multiple sinusoids: Accuracy aspects," *Signal Processing*, vol. 16, pp. 155–174, 1989.
[11] J. F. Cardoso and E. Moulines, "Asymptotic performance analysis of direction-finding algorithms based on fourth-order cumulants," *IEEE Trans. Signal Processing*, vol. 43, pp. 214–224, Jan. 1995.
[12] R. J. Serfling, *Approximation Theorems of Mathematical Statistics*. New York: Wiley, 1980.
[13] P. Stoica and T. Söderström, "Statistical analysis of MUSIC and subspace rotation estimates of sinusoidal frequencies," *IEEE Trans. Signal Processing*, vol. 39, pp. 1836–1847, Aug. 1991.
[14] G. Bienvenu, "Influence of the spatial coherence of the background noise on high resolution passive methods," in *Proc. ICASSP*, Washington, DC, Apr. 1979, pp. 306–309.
[15] S. Orfanidis, *Optimum Signal Processing, an Introduction*. New York: Macmillan, 1988.

Generalizing Carathéodory's Uniqueness of Harmonic Parameterization to N Dimensions

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***Abstract*—Consider a sum of F exponentials in N dimensions, and let I_n be the number of equispaced samples taken along the nth dimension. It is shown that if the frequencies or decays along every dimension are distinct and $\sum_{n=1}^{N} I_n \geq 2F + (N-1)$, then the parameterization in terms of frequencies, decays, amplitudes, and phases is unique. The result can be viewed as generalizing a classic result of Carathéodory to N dimensions. The proof relies on a recent result regarding the uniqueness of low-rank decomposition of N-way arrays.**

***Index Terms*—Multidimensional harmonic retrieval, multiway analysis, PARAllel FACtor (PARAFAC) analysis, spectral analysis, uniqueness.**

I. INTRODUCTION

The problem of harmonic retrieval and, more generally, exponential retrieval permeates the applied sciences and engineering. Although one-dimensional (1-D) exponential retrieval is most common (e.g., see [17] and references therein), the multidimensional case appears in a variety of important applications like joint azimuth, elevation, delay, and Doppler estimation in antenna array processing for communications [3]–[6], synthetic aperture radar (e.g., [7], [10] and references therein), and also certain signal separation problems in chemistry.

A wide variety of nonparametric and parametric techniques have been developed for the harmonic retrieval problem in one or more dimensions. Underpinning technique and practice of harmonic retrieval is the issue of identifiability, i.e., uniqueness of model parameterization. Owing to the work of Carathéodory [1] and later Pisarenko [11], this issue is well understood for the case of 1-D harmonics. In the case of multidimensional harmonics (and, more generally, exponentials), one can apply the 1-D result separately in each dimension, but this has two serious drawbacks. First, this approach does not reap the benefits of the rich multidimensional structure, leading to uniqueness conditions that are unnecessarily strict. Second, the association problem (i.e., whether the "pairing" of frequencies along different dimensions is unique) remains.

The uniqueness problem is hard for harmonics in two or higher dimensions. Only partial results are known for the two–dimensional (2-D) case [8], [10]. For example, [10] considers one possible formulation of the 2-D harmonic retrieval problem wherein the frequencies are assumed to occur at the intersections of certain unknown grid lines in the 2-D frequency domain, and provides sufficient conditions for identifiability. In the case of a single realization of the 2-D harmonic mixture, the conditions in [10] require that one has sufficiently many samples in each dimension for the 1-D result of Carathéodory to kick in.

The contribution of this correspondence is a general uniqueness result for N-dimensional exponential mixtures that is valid for any N and

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