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Slepian-Bangs formulas for parameterized density generator of elliptically symmetric distributions

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ABSTRACT

This paper mainly deals with an extension of the matrix Slepian-Bangs (SB) formula to elliptical symmetric (ES) distributions under the assumption that the arbitrary density generator depends on unknown parameters, aiming to rigorously quantify and understand the impact of this assumption on ES distributed parametric estimation models. This matrix SB formula is derived in a unified way within the framework of real (RES) and circular (C-CES) or noncircular (NC-CES) complex elliptically symmetric distributions, and then compared to the matrix SB formula obtained with fully known or completely unknown density generators. This new matrix SB formula involves a common structure to the existing one with a simple corrective coefficient. Closed-form expressions of this coefficient are given for Student's t and generalized Gaussian distributions and are each compared according to different knowledge of the density generator. This allows us to conclude that for an arbitrary parameterization, the Cramér-Rao bound (CRB) may be very sensitive to the knowledge of the density generator for super-Gaussian distributions contrary to sub-Gaussian distributions. Finally, we prove that for the parametrization with an unknown scale factor, the CRB for the estimation of the other parameters of the scatter matrix does not depend on the type of knowledge of the density generator. This latter result remains true for the specific noisy linear mixture data model where the parameter of interest is characterized by the range space of the mixing matrix.

1. Introduction

To assess the performance of many estimation algorithms, it is necessary to derive the Cramer-Rao bound (CRB), which is a lower bound on the variance of any unbiased estimator of the parameters of interest for the problem at hand. This bound relies on a parametric probabilistic model of the data which may be either exact or misspecified. Under the matched model assumption, the CRB is usually computed as the inverse of the Fisher information matrix (FIM). Fortunately a simple elementwise closed-form expression of this FIM, called Slepian-Bangs (SB) formula has been derived for the real Gaussian distribution in [1] and [2], in which both the expectation and the covariance are parameterized. Then this formula was extended to the circular complex Gaussian and non-circular Gaussian case in [3] and [4], respectively. However, in practice, the Gaussian assumption is not always adapted due to outliers. It is known from the literature that outliers can be modeled by elliptically symmetric (ES) distributions with heavy tails. The Gaussian-based SB formula was later extended to circular complex ES (C-CES) distributions (see e.g., [5–7]) in [8] and [9] and to noncircular complex ES (NC-CES) distributions [10]. We remind here the elliptically symmetric (ES) distributions encompass the Gaussian, the generalized Gaussian and all the compound Gaussian distributions, such as the Student's t and K-distributions, as special cases. Because of their great flexibility in modeling both heavy-tailed and light-tailed non-Gaussian distributed data, these distributions have been used in a variety of applications, in particular in the radar and array signal processing fields (see [7] and references therein).

In all above references on the derivations of SBs, the density generator is assumed to be perfectly known. Unlike this case, when considered as a nuisance parameter, an extension of SB formula was proposed in [11] in the context of semiparametric estimation for C-CES distributions. However, when the data model is misspecified by the parametric probabilistic model, the SB formula was generalized in [12] and [13] for the Gaussian model and then extended in [14] to C-CES distributions.

Given a particular ES distribution, its density generator might depend on some extra parameters (e.g., shape and scale parameters for the Student's t distribution) that are in general unknown. In this context, closed-form expressions of the FIM for the esti-

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mation of these parameters along with the symmetry center and scatter matrix for the circular Student's t and generalized Gaussian distribution have been derived in [15–17] where the trace of the scatter matrix is constrained. In this paper, we rather consider these extra parameters of the density generator as unknown nuisance parameters and we are interested in the FIM for the estimation of the parameters parameterizing the symmetry center and scatter matrix which is constrained to be equal to the covariance matrix. The derived SB formula is compared to that for which the density generator is fully known or completely unknown. This new SB formula involves a common structure to the existing one with a simple corrective coefficient. Closed-form expressions of this coefficient are given for Student's t and generalized Gaussian distributions and are each compared according to different knowledge of the density generator. This allows us to conclude that for an arbitrary parameterization, the CRB may be very sensitive to the type of knowledge of the density for super-Gaussian distributions contrary to sub-Gaussian distributions. Finally, we prove that if the symmetry center and the scatter matrix have no parameter in common with an unknown scale factor, the CRB for the estimation of the other parameters of the scatter matrix does not depend on the type of knowledge of the density generator. The same result is true for the specific noisy linear mixture data model where the parameter of interest is characterized by the range space of the mixing matrix.

The remainder of this paper is organized as follows. Section 2 recalls the real to complex representation of ES distributions useful to be able to deduce the SB formulas for complex data from those for real data. It also gives a brief reminder of Student's t and generalized Gaussian distributions under the constraint that the scatter matrix is equal to the covariance matrix, and of the classic and parametric SB formulas. The parameterized SB formula is derived in Section 3 for arbitrary density generators and then for Student's t and generalized Gaussian distributions, where comparisons are given for different parameterizations of the symmetric center and scatter matrix for three types of knowledge of the density generators. The specific model noisy linear mixture data model where the parameter of interest is characterized by the range space of the mixing matrix is covered in Section 4. Finally, the paper is concluded in Section 5.

The following notations are used throughout the paper. Matrices and vectors are represented by bold upper case and bold lower case characters, respectively. Vectors are by default in column orientation, while the superscripts T, H and $*$ stand for transpose, conjugate transpose and conjugate. $\mathrm{E}(.)$, $|.|$, $\mathrm{Re}(.)$ and $\mathrm{Im}(.)$ are the expectation, determinant, real and imaginary part operators respectively. $\mathbf{I}_N$ is the identity matrix of dimension N. $\mathrm{vec}(\cdot)$ is the "vectorization" operator that turns a matrix into a vector by stacking the columns of the matrix one below another which is used in conjunction with the Kronecker product $\mathbf{A}\otimes\mathbf{B}$ as the block matrix whose (i,j) block element is $a_{i,j}\mathbf{B}$. Finally, $\Gamma(x)$ and $B(x,y)$ are the usual gamma and beta functions and $x =_d y$ means that the r.v. x and y have the same distribution.

2. Preliminaries on elliptically symmetric distributions and Slepian-Bangs formulas

2.1. RES, C-CES and NC-CES distributions

Consider first the case of a N-dimensional RES distributed random variable (r.v.) $\mathbf{x}$ whose probability density function (p.d.f.) is of the form

$$p(\mathbf{x}) = |\boldsymbol{\Sigma}|^{-1/2} g_{r,N}[(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})], \tag{1}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are the symmetry center and the scatter matrix, respectively, and where the density generator[1] $g_{r,N}$: $\mathbb{R}^+ \mapsto \mathbb{R}^+$ satisfies[2] $\delta_{r,N} \stackrel{\text{def}}{=} \int_0^\infty t^{N/2-1} g_{r,N}(t)dt < \infty$. To derive the SB formula, we assume throughout this paper that the second-order moments of $\mathbf{x}$ are finite. To avoid the scale ambiguity problem between $\boldsymbol{\Sigma}$ and $g_{r,N}$, we here impose the constraint on $g_{r,N}$ such that $\boldsymbol{\Sigma} = \mathrm{E}[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T]$ rather than usual constraints on $\boldsymbol{\Sigma}$ that we cannot work on when it is parameterized. The r.v. $\mathbf{x}$ admits the following stochastic representation [18]

$$\mathbf{x} =_d \boldsymbol{\mu} + \sqrt{\mathcal{Q}_{r,N}}\boldsymbol{\Sigma}^{1/2}\mathbf{u}_{r,N}, \tag{2}$$

where $\mathcal{Q}_{r,N}$ and $\mathbf{u}_{r,N}$ are independent, $\mathbf{u}_{r,N}$ is uniformly distributed on the unit real N-sphere and $\mathcal{Q}_{r,N}$ has the p.d.f.

$$p(q) = \delta_{r,N}^{-1} q^{N/2-1} g_{r,N}(q). \tag{3}$$

An N-dimensional complex r.v. $\mathbf{x}$ is CES distributed if and only if the $2N$-dimensional r.v. $\bar{\mathbf{x}} \stackrel{\text{def}}{=} (\mathrm{Re}(\mathbf{x})^T, \mathrm{Im}(\mathbf{x})^T)^T$ is RES distributed [19]. Depending on whether $\boldsymbol{\Omega} \stackrel{\text{def}}{=} \mathrm{E}[(\mathbf{x}-\boldsymbol{\mu})(\mathbf{x}-\boldsymbol{\mu})^T] = \mathbf{0}$ or $\boldsymbol{\Omega} \neq \mathbf{0}$, $\mathbf{x}$ is C-CES or NC-CES distributed, respectively. Using the one-to one mapping $\bar{\mathbf{x}} \mapsto \widetilde{\mathbf{x}} \stackrel{\text{def}}{=} (\mathbf{x}^T, \mathbf{x}^H)^T = \sqrt{2}\mathbf{M}\bar{\mathbf{x}}$ where $\mathbf{M} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{I} & i\mathbf{I} \\ \mathbf{I} & -i\mathbf{I} \end{pmatrix}$ is unitary, we get $(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}})^T\bar{\boldsymbol{\Sigma}}^{-1}(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}}) = (\widetilde{\mathbf{x}}-\widetilde{\boldsymbol{\mu}})^H\widetilde{\boldsymbol{\Sigma}}^{-1}(\widetilde{\mathbf{x}}-\widetilde{\boldsymbol{\mu}})$ and $|\bar{\boldsymbol{\Sigma}}| = 2^{-2N}|\widetilde{\boldsymbol{\Sigma}}|$ where $\bar{\boldsymbol{\Sigma}} \stackrel{\text{def}}{=} \mathrm{E}[(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}})(\bar{\mathbf{x}}-\bar{\boldsymbol{\mu}})^T]$, $\widetilde{\boldsymbol{\Sigma}} \stackrel{\text{def}}{=} \mathrm{E}[(\widetilde{\mathbf{x}}-\widetilde{\boldsymbol{\mu}})(\widetilde{\mathbf{x}}-\widetilde{\boldsymbol{\mu}})^H] = \begin{pmatrix} \boldsymbol{\Sigma} & \boldsymbol{\Omega} \\ \boldsymbol{\Omega}^* & \boldsymbol{\Sigma}^* \end{pmatrix}$, $\bar{\boldsymbol{\mu}} \stackrel{\text{def}}{=} (\mathrm{Re}(\boldsymbol{\mu})^T, \mathrm{Im}(\boldsymbol{\mu})^T)^T$, $\widetilde{\boldsymbol{\mu}} \stackrel{\text{def}}{=} (\boldsymbol{\mu}^T, \boldsymbol{\mu}^H)^T$ and the p.d.f. (1) becomes

$$p(\mathbf{x}) = |\widetilde{\boldsymbol{\Sigma}}|^{-1/2} g_{c,N}\left[\frac{1}{2}(\widetilde{\mathbf{x}}-\widetilde{\boldsymbol{\mu}})^H\widetilde{\boldsymbol{\Sigma}}^{-1}(\widetilde{\mathbf{x}}-\widetilde{\boldsymbol{\mu}})\right], \tag{4}$$

where $g_{c,N}(t) \stackrel{\text{def}}{=} 2^N g_{r,2N}(2t)$ which satisfies $\delta_{c,N} \stackrel{\text{def}}{=} \int_0^\infty t^{N-1} g_{c,N}(t)dt = \delta_{r,2N} = \frac{\Gamma(N)}{\pi^N}$. From the stochastic representation (2) where N is replaced by $2N$, we get

$$\mathbf{x} =_d \boldsymbol{\mu} + \sqrt{\mathcal{Q}_{c,N}}[(\widetilde{\boldsymbol{\Sigma}}^{1/2})_{1,1}\mathbf{u}_{c,N} + (\widetilde{\boldsymbol{\Sigma}}^{1/2})_{1,2}\mathbf{u}_{c,N}^*], \tag{5}$$

with $\widetilde{\boldsymbol{\Sigma}}^{1/2} \stackrel{\text{def}}{=} \begin{pmatrix} (\widetilde{\boldsymbol{\Sigma}}^{1/2})_{1,1} & (\widetilde{\boldsymbol{\Sigma}}^{1/2})_{1,2} \\ (\widetilde{\boldsymbol{\Sigma}}^{1/2})_{1,2}^* & (\widetilde{\boldsymbol{\Sigma}}^{1/2})_{1,1}^* \end{pmatrix}$ where closed-form expressions of $(\widetilde{\boldsymbol{\Sigma}}^{1/2})_{1,1}$ and $(\widetilde{\boldsymbol{\Sigma}}^{1/2})_{1,2}$ are given in [10] with $\widetilde{\boldsymbol{\Sigma}} = \widetilde{\boldsymbol{\Sigma}}^{1/2}(\widetilde{\boldsymbol{\Sigma}}^{1/2})^H$, $\mathcal{Q}_{c,N}$ and $\mathbf{u}_{c,N}$ are independent, $\mathbf{u}_{c,N}$ is uniformly distributed on the unit complex N-sphere and $\mathcal{Q}_{c,N} \stackrel{\text{def}}{=} \frac{1}{2}\mathcal{Q}_{r,2N}$ with p.d.f.

$$p(q) = \delta_{c,N}^{-1} q^{N-1} g_{c,N}(q). \tag{6}$$

In the particular case where $\boldsymbol{\Omega} = \mathbf{0}$, $\mathbf{x}$ is C-CES distributed, and (4) and (5) respectively reduce to

$$p(\mathbf{x}) = |\boldsymbol{\Sigma}|^{-1} g_{c,N}[(\mathbf{x}-\boldsymbol{\mu})^H\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})] \text{ and}$$
$$\mathbf{x} =_d \boldsymbol{\mu} + \sqrt{\mathcal{Q}_{c,N}}\boldsymbol{\Sigma}^{1/2}\mathbf{u}_{c,N}. \tag{7}$$

We consider from now on that $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are parameterized by a parameter $\boldsymbol{\alpha} \in \mathbb{R}^M$ that characterizes $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, but we omit this dependence $(\boldsymbol{\mu}(\boldsymbol{\alpha}), \boldsymbol{\Sigma}(\boldsymbol{\alpha}))$ to simplify the notations. We also assume that the density generators are either fully known, known up to unknown parameters $\boldsymbol{\beta} \in \mathbb{R}^L$, or completely unknown and interpreted as infinite-dimensional nuisance parameters. $\boldsymbol{\beta}$ acts as a nuisance parameter, while $\boldsymbol{\alpha}$ is a parameter of interest (time delay, direction of arrival, range, impulse response coefficients...) depending on the related problem.

[1] The readers interested in these density generator functions can refer to [7] which gives several examples.

[2] The usual normalizing constant being included in $g_{r,N}$ (1), $\delta_{r,N}$ depends here in fact only on N.

2.2. Constrained density generators of Student's t and generalized Gaussian distributions

We give here a brief reminder of the expressions of the density generators of Student's t and generalized Gaussian distributions (see [7] for details) under the constraint that the scatter matrices $\boldsymbol{\Sigma}$ are equal to the covariance matrices. We note that the Student's t distribution, which belongs to the subclass of compound Gaussian distributions, have gained popularity for modeling radar clutter [20] and the generalized Gaussian distributions have been used for modeling various images or features extracted from these images [21]. These distributions are used in the illustration of the parameterized SB formulas in Section 3.2. From the expressions of the unconstrained density generators [7], we deduce easily the following expressions that are reduced to a single parameter:

$$g^{\nu}_{c,N}(t) = \frac{2^N\Gamma(N+\frac{\nu}{2})}{\pi^N(\nu-2)^N\Gamma(\frac{\nu}{2})}\left(1+\frac{2t}{\nu-2}\right)^{-(N+\frac{\nu}{2})} \quad \text{and}$$

$$g^{s}_{c,N}(t) = \frac{s\Gamma(N)[\Gamma(\frac{N+1}{s})]^N}{\pi^N N^N[\Gamma(\frac{N}{s})]^{N+1}} e^{-\frac{t^s}{\left[\frac{N\Gamma(\frac{N}{s})}{\Gamma(\frac{N+1}{s})}\right]^s}}, \tag{8}$$

for Student's t distribution with $\nu > 2$ degrees of freedom and generalized Gaussian distributions with exponent $s > 0$, respectively. The expressions of the associated RES constrained density generators $g_{r,N}(t)$ is related to $g_{c,N}(t)$ by $g_{c,N}(t) = 2^N g_{r,2N}(2t)$.

2.3. Classic and semiparametric SB formulas

The purpose of this Subsection is to unify in a common structured matrix formula, the classic and semiparametric SB formulas relating to C-CES and NC-CES distributions from that of the RES distributions.

For RES distributed data, all the steps of the proof of the classic and semiparametric SB formula for C-CES distributions given in [8] and [11] (with [14, Appendix B]), respectively, apply by using the identity $\mathrm{E}[(\mathbf{y}^T\mathbf{A}\mathbf{y})(\mathbf{y}^T\mathbf{B}\mathbf{y})] = \mathrm{Tr}(\mathbf{A})\mathrm{Tr}(\mathbf{B}) + 2\mathrm{Tr}(\mathbf{A}\mathbf{B})$ for any symmetric $N\times N$ matrices $\mathbf{A}$ and $\mathbf{B}$, and N-dimensional zero-mean real-valued Gaussian distributed r.v. $\mathbf{y}$. This allows us to prove that the classic and semiparametric matrix SB formula for RES distributions have the following structure:

$$\mathrm{CRB}^{-1}(\boldsymbol{\alpha}) = a_1\frac{d\boldsymbol{\mu}^T}{d\boldsymbol{\alpha}^T}\boldsymbol{\Sigma}^{-1}\frac{d\boldsymbol{\mu}}{d\boldsymbol{\alpha}^T} + \left(\frac{d\mathrm{vec}(\boldsymbol{\Sigma})}{d\boldsymbol{\alpha}^T}\right)^T\Big(a_2(\boldsymbol{\Sigma}^{-T}\otimes\boldsymbol{\Sigma}^{-1}) + a_3\mathrm{vec}(\boldsymbol{\Sigma}^{-1})\mathrm{vec}^T(\boldsymbol{\Sigma}^{-1})\Big)\left(\frac{d\mathrm{vec}(\boldsymbol{\Sigma})}{d\boldsymbol{\alpha}^T}\right), \tag{9}$$

where $a_1 = \xi_{r,1,N}$, $a_2 = \frac{1}{2}\xi_{r,2,N}$ for both classic and semiparametric SB formulas and $a_3 \stackrel{\text{def}}{=} a_3^{\text{Clas}} = \frac{1}{4}(\xi_{r,2,N}-1)$ [resp., $a_3 \stackrel{\text{def}}{=} a_3^{\text{SePa}} = -\frac{\xi_{r,2,N}}{2N}$] for the classic [resp., semiparametric] SB formula with

$$\xi_{r,1,N} \stackrel{\text{def}}{=} \frac{\mathrm{E}\left[Q\phi^2_{r,N}(Q)\right]}{N} \quad \text{and} \quad \xi_{r,2,N} \stackrel{\text{def}}{=} \frac{\mathrm{E}\left[Q^2\phi^2_{r,N}(Q)\right]}{N(N+2)}, \tag{10}$$

where $Q \stackrel{\text{def}}{=} \mathcal{Q}_{r,N}$ and $\phi_{r,N}(t) \stackrel{\text{def}}{=} \frac{2}{g_{r,N}(t)}\frac{dg_{r,N}(t)}{dt}$.

These classic and semiparametric SB formulas allow us to directly deduce those of NC-CES distributed data obtained, thanks to the relationship between the representation of real and complex r.v.'s introduced in Subsection 2.1. These SB formulas are similarly structured where $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, $\frac{d\boldsymbol{\mu}^T}{d\boldsymbol{\alpha}^T}$, $\left(\frac{d\mathrm{vec}(\boldsymbol{\Sigma})}{d\boldsymbol{\alpha}^T}\right)^T$ and $\mathrm{vec}^T(\boldsymbol{\Sigma}^{-1})$ in (9) are replaced by $\widetilde{\boldsymbol{\mu}}$, $\widetilde{\boldsymbol{\Sigma}}$, $\frac{d\widetilde{\boldsymbol{\mu}}^H}{d\boldsymbol{\alpha}^T}$, $\left(\frac{d\mathrm{vec}(\widetilde{\boldsymbol{\Sigma}})}{d\boldsymbol{\alpha}^T}\right)^H$ and $\mathrm{vec}^H(\widetilde{\boldsymbol{\Sigma}}^{-1})$, respectively, where $a_1 = \xi_{c,1,N}$ and $a_2 = \frac{\xi_{c,2,N}}{2}$ for both classic and semiparametric SB formulas and $a_3 \stackrel{\text{def}}{=} a_3^{\text{Clas}} = \frac{1}{4}(\xi_{c,2,N}-1)$ [resp., $a_3 \stackrel{\text{def}}{=} a_3^{\text{SePa}} = -\frac{\xi_{c,2,N}}{2N}$] for the classic [resp., semiparametric] SB formula with

$$\xi_{c,1,N} \stackrel{\text{def}}{=} \frac{\mathrm{E}\left[Q\phi^2_{c,N}(Q)\right]}{N} \quad \text{and} \quad \xi_{c,2,N} \stackrel{\text{def}}{=} \frac{\mathrm{E}\left[Q^2\phi^2_{c,N}(Q)\right]}{N(N+1)}, \tag{11}$$

where $Q \stackrel{\text{def}}{=} \mathcal{Q}_{c,N}$ and $\phi_{c,N}(t) \stackrel{\text{def}}{=} \frac{1}{g_{c,N}(t)}\frac{dg_{c,N}(t)}{dt}$. On the other hand, the classic and semiparametric SB formulas for C-CES distributed data can be deduced directly by replacing $\widetilde{\boldsymbol{\Sigma}}$ by $\begin{pmatrix}\boldsymbol{\Sigma} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Sigma}^*\end{pmatrix}$, yielding the classic and semiparametric SB formulas proved in [8,9], and [11], respectively, which are also similarly structured where $\frac{d\boldsymbol{\mu}^T}{d\boldsymbol{\alpha}^T}\boldsymbol{\Sigma}^{-1}\frac{d\boldsymbol{\mu}}{d\boldsymbol{\alpha}^T}$, $\left(\frac{d\mathrm{vec}(\boldsymbol{\Sigma})}{d\boldsymbol{\alpha}^T}\right)^T$ and $\mathrm{vec}^T(\boldsymbol{\Sigma}^{-1})$ is replaced by $\mathrm{Re}\left(\frac{d\boldsymbol{\mu}^H}{d\boldsymbol{\alpha}^T}\boldsymbol{\Sigma}^{-1}\frac{d\boldsymbol{\mu}}{d\boldsymbol{\alpha}^T}\right)$, $\left(\frac{d\mathrm{vec}(\boldsymbol{\Sigma})}{d\boldsymbol{\alpha}^T}\right)^H$ and $\mathrm{vec}^H(\boldsymbol{\Sigma}^{-1})$ in (9) with $a_1 = 2\xi_{c,1,N}$ and $a_2 = \xi_{c,2,N}$ for both classic and semiparametric SB formulas and $a_3 \stackrel{\text{def}}{=} a_3^{\text{Clas}} = \xi_{c,2,N}-1$ [resp., $a_3 \stackrel{\text{def}}{=} a_3^{\text{SePa}} = -\frac{\xi_{c,2,N}}{N}$] for the classic [resp., semiparametric] SB formula.

Note that for real Gaussian distributions, $\phi_{r,N}(t) = -1$ and Q is χ^2_N distributed which give $\mathrm{E}(Q) = N$ and $\mathrm{E}(Q^2) = N(N+2)$, and thus from (10), $\xi_{r,1,N} = \xi_{r,2,N} = 1$ and $(a_1, a_2, a_3^{\text{Clas}}, a_3^{\text{SePa}}) = (1, \frac{1}{2}, 0, -\frac{1}{2N})$, which imply that (9) reduces to the well known elementwise classic SB formula $[\mathrm{FIM}(\boldsymbol{\alpha})]_{k,\ell} = \frac{d\boldsymbol{\mu}^T}{d\alpha_k}\boldsymbol{\Sigma}^{-1}\frac{d\boldsymbol{\mu}}{d\alpha_\ell} + \frac{1}{2}\mathrm{Tr}(\boldsymbol{\Sigma}^{-1}\frac{d\boldsymbol{\Sigma}}{d\alpha_k}\boldsymbol{\Sigma}^{-1}\frac{d\boldsymbol{\Sigma}}{d\alpha_\ell})$ [1],[2]. Similarly for complex circular and noncircular Gaussian distributions, we get $(a_1, a_2, a_3^{\text{Clas}}, a_3^{\text{SePa}}) = (2, 1, 0, -\frac{1}{N})$ and $(a_1, a_2, a_3^{\text{Clas}}, a_3^{\text{SePa}}) = (1, \frac{1}{2}, 0, -\frac{1}{2N})$, respectively.

3. Parameterized Slepian-Bangs formulas

3.1. Arbitrary density generator

In this Section the density generators $g_{r,N}$ and $g_{c,N}$ are assumed to be known up to unknown parameters $\boldsymbol{\beta} \in \mathbb{R}^L$ and here denoted by $g^{\beta}_{r,N}$ and $g^{\beta}_{c,N}$. Consequently the unknown parameter for the RES, C-CES and NC-CES distributions is $(\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T)^T \in \mathbb{R}^{M+L}$ where $\boldsymbol{\beta}$ is an unknown nuisance parameter. The following result is proved in the Appendix.

Result 1. For each RES, C-CES and NC-CES distribution, the classic, semiparametric and parameterized SB formula have the same structure (9) with identical coefficients a_1 and a_2 and differ only by their coefficients a_3 given in the parameterized SB formula by $a_3 \stackrel{\text{def}}{=} a_3^{\text{Par}} = a_3^{\text{Clas}} - a_4$ where $a_4 = \boldsymbol{\xi}^T_{r,3,N}\Xi^{-1}_{r,4,N}\boldsymbol{\xi}_{r,3,N}$ for RES distributions, with

$$\boldsymbol{\xi}_{r,3,N} \stackrel{\text{def}}{=} \frac{\mathrm{E}\left[Q\phi_{r,N}(Q)\boldsymbol{\phi}^{\beta}_{r,N}(Q)\right]}{N} \quad \text{and} \quad \Xi_{r,4,N} \stackrel{\text{def}}{=} \mathrm{E}\left[\boldsymbol{\phi}^{\beta}_{r,N}(Q)\boldsymbol{\phi}^{\beta\,T}_{r,N}(Q)\right], \tag{12}$$

where $\boldsymbol{\phi}^{\beta}_{r,N}(t) \stackrel{\text{def}}{=} \frac{1}{g^{\beta}_{r,N}(t)}\frac{\partial g^{\beta}_{r,N}(t)}{\partial\boldsymbol{\beta}} \in \mathbb{R}^L$ and $Q \stackrel{\text{def}}{=} \mathcal{Q}_{r,N}$, and similarly for C/NC-CES distributions by replacing r by c, T by H and the associated expression of a_3^{Clas} are given above.

Note that for Gaussian distributions for which the density generator $g_{r,N}(t) = \frac{1}{(2\pi)^{N/2}}\exp(-\frac{1}{2t})$ has no parameter, we get $a_3^{\text{Par}} = a_3^{\text{Clas}} = 0$.

In particular, if $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ have no parameters in common with $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ parameterized and characterized by $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$, respectively, the parameters $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are decoupled in the FIM (9). Consequently the CRB for the estimation of $\boldsymbol{\alpha}_1$ has the common expression

$$\mathrm{CRB}(\boldsymbol{\alpha}_1) = \left(a_1\frac{d\boldsymbol{\mu}^T}{d\boldsymbol{\alpha}_1^T}\boldsymbol{\Sigma}^{-1}\frac{d\boldsymbol{\mu}}{d\boldsymbol{\alpha}_1^T}\right)^{-1} \tag{13}$$

for fully known density generator, known up to parameters and completely unknown density generators, in contrast to the CRB for the estimation of $\boldsymbol{\alpha}_2$

$$\text{CRB}(\boldsymbol{\alpha}_2) = \left(\left(\frac{d\text{vec}(\boldsymbol{\Sigma})}{d\boldsymbol{\alpha}_2^T} \right)^T \left(a_2 (\boldsymbol{\Sigma}^{-T} \otimes \boldsymbol{\Sigma}^{-1}) + a_3 \text{vec}(\boldsymbol{\Sigma}^{-1}) \text{vec}^T(\boldsymbol{\Sigma}^{-1}) \right) \left(\frac{d\text{vec}(\boldsymbol{\Sigma})}{d\boldsymbol{\alpha}_2^T} \right) \right)^{-1} \quad (14)$$

which generally depends on the type of knowledge about the density generator through the term a_3. However, in the specific parameterization of the scatter matrix $\boldsymbol{\Sigma} = \alpha'_2 \boldsymbol{\Sigma}_0(\boldsymbol{\alpha}''_2)$ with unknown scaling factor α'_2 and $\boldsymbol{\alpha}_2 = (\alpha'_2, \boldsymbol{\alpha}''^T_2)^T$, the following result is proved in the Appendix:

Result 2. For each RES, C-CES and NC-CES distribution, where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ have no common parameters with the parameterization $\boldsymbol{\Sigma} = \alpha'_2 \boldsymbol{\Sigma}_0(\boldsymbol{\alpha}''_2)$ of the scatter matrix, the CRB for the estimation of $\boldsymbol{\alpha}''_2$ does not depend on α'_2, nor on the type of knowledge about the density generator and is given for RES distributions by the expression:

$$\text{CRB}(\boldsymbol{\alpha}''_2) = \frac{1}{a_2} \left(\left(\frac{d\text{vec}(\boldsymbol{\Sigma}_0)}{d\boldsymbol{\alpha}''^T_2} \right)^T \left((\boldsymbol{\Sigma}_0^{-T} \otimes \boldsymbol{\Sigma}_0^{-1}) - \frac{1}{N} \text{vec}(\boldsymbol{\Sigma}_0^{-1}) \text{vec}^T(\boldsymbol{\Sigma}_0^{-1}) \right) \left(\frac{d\text{vec}(\boldsymbol{\Sigma}_0)}{d\boldsymbol{\alpha}''^T_2} \right) \right)^{-1}. \quad (15)$$

In other words, for this specific parametrization, the perfect knowledge or only the knowledge up to unknown extra parameter of the density generator does not reduce the CRB on $\boldsymbol{\alpha}''_2$. Similarly to (9), the CRB for C-CES distributions is deduced from (15), thanks to the relationship between the representation of real and complex r.v.'s, by replacing $\left(\frac{d\text{vec}(\boldsymbol{\Sigma}_0)}{d\boldsymbol{\alpha}''^T_2} \right)^T$ and $\text{vec}^T(\boldsymbol{\Sigma}_0^{-1})$ by $\left(\frac{d\text{vec}(\boldsymbol{\Sigma}_0)}{d\boldsymbol{\alpha}''^T_2} \right)^H$ and $\text{vec}^H(\boldsymbol{\Sigma}_0^{-1})$. For the NC-CES distributions, $\boldsymbol{\Sigma}_0$, $\left(\frac{d\text{vec}(\boldsymbol{\Sigma}_0)}{d\boldsymbol{\alpha}''^T_2} \right)^T$ and $\text{vec}^T(\boldsymbol{\Sigma}_0^{-1})$ must be replaced by $\widetilde{\boldsymbol{\Sigma}}_0$, $\left(\frac{d\text{vec}(\widetilde{\boldsymbol{\Sigma}}_0)}{d\boldsymbol{\alpha}''^T_2} \right)^H$ and $\text{vec}^H(\widetilde{\boldsymbol{\Sigma}}_0^{-1})$.

By contrast, the scale parameter α'_2, cannot be estimated in the absence of knowledge of the density generator due to the intrinsic ambiguity of the parametrization of the p.d.f. of the ES distributions, while the CRB on this parameter may depend on the knowledge of the density generator (with fully known or known up to unknown parameters). Moreover, unlike Result 2, the CRB for the estimation of $\boldsymbol{\alpha}''_2$ when the parameter α'_2 is known is given by (16) which depends on the type of knowledge of the density generator through the coefficient a_3

$$\text{CRB}(\boldsymbol{\alpha}''_2) = \left(\left(\frac{d\text{vec}(\boldsymbol{\Sigma}_0)}{d\boldsymbol{\alpha}''^T_2} \right)^T \left(a_2 (\boldsymbol{\Sigma}_0^{-T} \otimes \boldsymbol{\Sigma}_0^{-1}) + a_3 \text{vec}(\boldsymbol{\Sigma}_0^{-1}) \text{vec}^T(\boldsymbol{\Sigma}_0^{-1}) \right) \left(\frac{d\text{vec}(\boldsymbol{\Sigma}_0)}{d\boldsymbol{\alpha}''^T_2} \right) \right)^{-1}, \quad (16)$$

as illustrated by an example in Subsection 3.2.

It follows from Result 1 that for general parameterization of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, the comparison of the classical and semi-parametric SB formulas recalled in subsection 2.3 and the parameterized SB formula amounts to comparing the coefficient a_3 of the associated SB formulas. Naturally, more knowledge about the density generator results in a smaller CRB on parameter $\boldsymbol{\alpha}$, and we must therefore have the following inequalities on the coefficients a_3:

$$a_3^{\text{SePa}} \le a_3^{\text{Par}} \le a_3^{\text{Clas}} \text{ with } a_3^{\text{Par}} = a_3^{\text{Clas}} - a_4. \quad (17)$$

Note that the inequality $a_3^{\text{SePa}} \le a_3^{\text{Clas}}$ is equivalent for example for C-CES distributions to the inequality

$$-\frac{\xi_{c,2,N}}{N} \le \xi_{c,2,N} - 1 \Leftrightarrow \xi_{c,2,N} \ge \frac{N}{N+1}, \quad (18)$$

which is in fact strict. It follows directly from the Cauchy-Schwarz inequality $(\text{E}(XY))^2 \le \text{E}(X^2)\ \text{E}(Y^2)$ with $X = Q\phi_{c,N}(Q)$ (where $Q \stackrel{\text{def}}{=} \mathcal{Q}_{c,N}$) and $Y = 1$ with equality if and only if the r.v. $Q\phi_{c,N}(Q)$ is constant. Since this property is equivalent to $g_c(t) = t^a$ where a is constant, which cannot satisfy the condition $\int_0^\infty t^{N-1} g_{c,N}(t)dt < \infty$, and then the equality can not hold. To go further in the comparison of the coefficient a_3, we consider the following specific distributions.

3.2. Student's t and generalized Gaussian distributions

To illustrate Results 1 and 2, we consider the two commonly used Student's t distribution with $\nu > 2$ degrees of freedom[3] and generalized Gaussian distributions with exponent $s > 0$ reminded in Subsection 2.2. It is simple to prove that the coefficients $\xi_{r,1,N}$ and $\xi_{r,2,N}$, $\xi_{c,1,N}$ and $\xi_{c,2,N}$ are independent of the constraint on the density generators for arbitrary distributions. These have been calculated for complex Student's t or complex generalized Gaussian distributions by several authors (see e.g., [8],[9],[22]) and given respectively by:

$$\xi^S_{c,1,N} = \frac{\nu/2}{((\nu/2)-1)} \frac{(\nu/2)+N}{((\nu/2)+N+1)} \text{ and } \xi^S_{c,2,N} = \frac{(\nu/2)+N}{(\nu/2)+N+1}, \quad (19)$$

$$\xi^{CG}_{c,1,N} = \frac{\Gamma(2+\frac{N-1}{s})\Gamma(\frac{N+1}{s})}{(\Gamma(1+\frac{N}{s}))^2} \quad \text{and} \quad \xi^{CG}_{c,2,N} = \frac{N+s}{N+1}. \quad (20)$$

In contrast, the coefficients $\xi_{r,3,N}$ and $\xi_{r,4,N}$, $\xi_{c,3,N}$ and $\xi_{c,4,N}$, naturally, generally depend on the constraint imposed on the density generators which leads to a relation between the multidimensional parameters of the standard density generators (e.g., $\boldsymbol{\beta}$ reduces to the exponent s (8) for the generalized Gaussian distribution where the standard density generator is parameterized by exponent and scale [7]). The following result concerning Student's t and generalized Gaussian distribution is proved in the Appendix:

Result 3. For complex Student's t and generalized Gaussian distribution, the coefficients $\xi_{c,3,N}$ and $\xi_{c,4,N}$ are given respectively by

$$\xi^S_{c,3,N} = \frac{N+1}{2\left(\frac{\nu}{2}-1\right)\left(\frac{\nu}{2}+N\right)\left(\frac{\nu}{2}+N+1\right)}, \quad (21)$$

$$\xi^S_{c,4,N} = \sum_{\ell=0}^{N-1} \frac{1}{4(\ell+\frac{\nu}{2})^2} - \frac{N\left(\frac{\nu^2}{4}+N\left(\frac{\nu}{2}-2\right)-2\right)}{4\left(\frac{\nu}{2}-1\right)^2\left(\frac{\nu}{2}+N\right)\left(\frac{\nu}{2}+N+1\right)}, \quad (22)$$

and

$$\xi^{GG}_{c,3,N} = \frac{N+s+N(N+1)k_{N,s}}{Ns}, \quad (23)$$

[3] The constraint $\nu > 2$ ensures that the Student's t distribution has second-order finite moments

$$\xi^{GG}_{c,4,N} = \frac{2N+s}{Ns^2} + \frac{(N+1)s(2(N+s)+N(N+1)k_{N,s})k_{N,s} + N(N+s)\psi'\left(1+\frac{N}{s}\right)}{s^4}, \quad (24)$$

with $k_{N,s} \stackrel{\text{def}}{=} \psi\left(\frac{N}{s}\right) - \psi\left(\frac{N+1}{s}\right)$ where $\psi(x) \stackrel{\text{def}}{=} \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function and $\psi'(x) \stackrel{\text{def}}{=} \frac{d\psi(x)}{dx}$.

For the associated RES distributions, the coefficients $\xi_{r,1,N}$, $\xi_{r,2,N}$, $\xi_{r,3,N}$ and $\xi_{r,4,N}$ are related to $\xi_{c,1,N}$, $\xi_{c,2,N}$, $\xi_{c,3,N}$ and $\xi_{c,4,N}$ by the relations $\xi_{c,1,N} = 4\xi_{r,1,2N}$, $\xi_{c,2,N} = 4\xi_{r,2,2N}$, $\xi_{c,3,N} = 2\xi_{r,3,2N}$ and $\xi_{c,4,N} = \xi_{r,4,2N}$.

We see that $\xi_{c,2,N}$ given by (19) and (20) for respectively complex Student's t and generalized Gaussian distributions can be written as

$$\xi^{S}_{c,2,N} = \frac{N}{N+1}\left(\frac{1+\nu/2N}{1+\nu/(2(N+1))}\right) \text{ and } \xi^{CG}_{c,2,N} = \frac{N}{N+1}\left(1+\frac{s}{N}\right). \quad (25)$$

Consequently $\xi^{S}_{c,2,N}$ and $\xi^{CG}_{c,2,N}$ are very close to $\frac{N}{N+1}$ and therefore from (17) and (18), the coefficients a_3^{SePa}, a_3^{Par} and a_3^{Clas} are very close for $\nu/N \ll 1$ and $s/N \ll 1$, respectively. On the contrary, $\xi_{c,2,N} \approx 1$ and $\xi_{c,2,N} \approx s/(N+1)$ for respectively $\nu/N \gg 1$ and $s/N \gg 1$. We can deduce that for Student's t distributions which possess heavier tails than the Gaussian distribution, the knowledge of the density generator has a slight impact on the CRB for the estimation of parameters $\boldsymbol{\alpha}$. On the other hand, for the generalized Gaussian distribution, this impact is strong for $s/N \gg 1$, i.e., for much lighter tailed distributions than Gaussian distribution. To show the influence of the parameter s on the coefficient a_3, $a_3^{\text{Clas}} = \frac{s-1}{N+1}$ and $a_3^{\text{SePa}} = -\frac{N+s}{N(N+1)}$ are compared to a_3^{Par} obtained from tedious algebraic manipulation of (23) and (24) with the aid of symbolic algebra and calculus tools in the vicinity of $s = \infty$ which corresponds to a uniform distribution in an ellipsoid. We get

$$a_3^{\text{SePa}} < a_3^{\text{Par}} = \frac{N(\pi^2-6)-6}{\pi^2 N(N+1)} s(1+o(1)) < a_3^{\text{Clas}}. \quad (26)$$

This influence is illustrated in Fig. 1 which shows a large difference between these coefficients a_3.

We can therefore conjecture that the knowledge of the density generator brings little information on the parameter $\boldsymbol{\alpha}$ for the heavy-tailed distributions unlike for lighter-tailed distributions.

To illustrate the impact of the knowledge of the scale factor α'_2 on the estimation of the parameter α''_2 of $\boldsymbol{\Sigma}_0$ for the modeling of Result 2, we assume here that generalized Gaussian distributed data are modeled as a stationary, zero-mean autoregressive process of first order with one lag correlation $\alpha''_2 = \rho$ which gives $[\boldsymbol{\Sigma}]_{k,\ell} = [\boldsymbol{\Sigma}_0(\rho)]_{k,\ell} = \rho^{|k-\ell|}$ (with $\alpha'_2 = 1$). Fig. 2 shows a large difference between the CRB for the estimation of ρ according to the knowledge available on the density generator. This observed behavior is consistent with that of Fig. 1 as explained at the end of section III-A, thanks to the sensitivity of the coefficient a_3 in (16) to the knowledge available on the density generator.

To reinforce the behavior of the CRB on the parameter of interest ρ under different knowledge on the density generator, we compare in Fig. 3, the CRB on ρ to the mean square error (MSE) (estimated by 2000 runs) of the associated maximum likelihood (ML) estimator derived from T independent snapshots $\mathbf{x}_t, t = 1, \ldots, T$ identically distributed as in the scenario of Fig. 2. More precisely, when α'_2 is assumed to be unknown, the semiparametric, parameterized, and classic CRB given by (15) which are equal are compared to the MSE of the joint ML estimates $\hat{\rho}$ where $(\hat{\alpha}'_2, \hat{\rho}, \hat{s}) = \arg\max_{\alpha'_2,\rho,s} \sum_{t=1}^T \log p(\mathbf{x}_t)$ and to the MSE of the ML estimate $\hat{\rho}$ obtained from $(\hat{\alpha}'_2, \hat{\rho}) = \arg\max_{\alpha'_2,\rho} \sum_{t=1}^T \log p(\mathbf{x}_t)$ (given s), where the density generator $g^s_{c,N}(t)$ is given by (8). Similarly, when α'_2 is assumed to be known, our parameterized CRB and the classic CRB given by (16) are respectively compared to the MSE of the joint ML estimate $\hat{\rho}$ where $(\hat{\rho}, \hat{s}) = \arg\max_{\rho,s} \sum_{t=1}^T \log p(\mathbf{x}_t)$ and to the MSE of the ML estimate $\hat{\rho} = \arg\max_{\rho} \sum_{t=1}^T \log p(\mathbf{x}_t)$ (given s), respectively. Note that these different maximizations are derived form simple numerical optimizations because, to the best of our knowledge, there is no ML algorithm in the literature, despite the sub-optimal approaches proposed in [16], [17] for the Student's t distribution.Fig. 3 confirms clearly the efficiency of the ML as these CRBs are very close to the associated MSE with the ML estimates for $T = 500$ snapshots.

4. Noisy linear mixture data model

We consider here the following model[4]

$$\mathbf{x}_t = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}_t + \mathbf{n}_t \in \mathbb{R}^{N'} \ (\text{or } \mathbb{C}^{N'}), \quad t = 1, \ldots, T \quad (27)$$

where the real-valued parameter of interest $\boldsymbol{\theta}$ is characterized by the range space of the full column matrix $\mathbf{A}(\boldsymbol{\theta})$. Two assumptions have been commonly used for the signals $\mathbf{s}_t$ and $\mathbf{n}_t$.

In the conditional or deterministic model, $(\mathbf{s}_t)_{t=1,.,T}$ are conditioned from an independent zero-mean process (as it was explained in [25]) and are considered as deterministic nuisance parameters. $\mathbf{n}_t$, $t = 1, \ldots, T$ are zero-mean, independent RES (or C-CES) distributed with scatter matrix $\sigma_n^2 \mathbf{I}_{N'}$. In this case $\mathbf{x} \stackrel{\text{def}}{=} (\mathbf{x}_1^T, .., \mathbf{x}_T^T)^T \in \mathbb{R}^N$ (or $\mathbb{C}^N$) where $N = TN'$ is RES (or C-CES) distributed with $\boldsymbol{\mu} = ((\mathbf{A}(\boldsymbol{\theta})\mathbf{s}_1)^T, \ldots, (\mathbf{A}(\boldsymbol{\theta})\mathbf{s}_T)^T)^T$ and $\boldsymbol{\Sigma} = \sigma_n^2 \mathbf{I}_N$ with $\boldsymbol{\alpha} = (\boldsymbol{\theta}^T, \boldsymbol{\rho}^T, \sigma_n^2)^T$ with $\boldsymbol{\rho} \stackrel{\text{def}}{=} (\text{Re}^T(\mathbf{s}_1), \text{Im}^T(\mathbf{s}_1), .., \text{Re}^T(\mathbf{s}_T), \text{Im}^T(\mathbf{s}_T))^T$. This model extends also to rectilinear CN-CES[5] distributed data. By slightly modifying the end of the proof given in [25] in the estimation framework of DOA, we obtain the following result.

Result 4. For each RES, C-CES or NC-CES noisy linear mixture distributed conditional model, the CRB for the estimation of $\boldsymbol{\theta}$ is given by a common expression for fully known, known up to parameters and completely unknown density generators. We get, for example, for C-CES distributed data the following expression:

$$\text{CRB}(\boldsymbol{\theta}) = \frac{\sigma_n^2}{2T\xi_{c,1,N}} \left[\text{Re}\left(\frac{d\mathbf{a}^H(\theta)}{d\boldsymbol{\theta}} (\mathbf{R}^T_{s,T} \otimes \boldsymbol{\Pi}^{\perp}_{\mathbf{A}(\theta)}) \frac{d\mathbf{a}(\theta)}{d\boldsymbol{\theta}} \right) \right]^{-1}, \quad (28)$$

where $\mathbf{R}_{s,T} \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^T \mathbf{x}_t\mathbf{x}_t^H$, $\mathbf{a}(\theta) \stackrel{\text{def}}{=} \text{vec}(\mathbf{A}(\theta))$ and $\boldsymbol{\Pi}^{\perp}_{\mathbf{A}(\theta)} \stackrel{\text{def}}{=} \mathbf{I} - \mathbf{A}(\theta)[\mathbf{A}^H(\theta)\mathbf{A}(\theta)]^{-1}\mathbf{A}^H(\theta)$ is the ortho-complement of the projection matrix on the columns of $\mathbf{A}(\theta)$.

In the unconditional or stochastic model, both $\mathbf{s}_t$ and $\mathbf{n}_t$ are assumed zero-mean random, not correlated with each other such that $\mathbf{x}_t$, $t = 1, \ldots, T$ are zero-mean, independent RES (or C-CES) distributed[6] with $\boldsymbol{\mu} = \mathbf{0}$ where the scatter matrix has the following

[4] This model encompasses many far or near-field, narrow or wide-band DOA models with scalar or vector-sensors for an arbitrary number of parameters per source and many other models as the bandlimited SISO, SIMO [23] and MIMO [24] channel models.

[5] This model can be applied for DOA estimation modeling with rectilinear or strictly second-order sources and for SIMO channels estimation modeling with BPSK or MSK symbols [26] where $\boldsymbol{\theta}$ represents both the localization parameters (azimuth, elevation, range) and the phase of the sources, and the real and imaginary parts of channel impulse response coefficients, respectively.

[6] We note that $\mathbf{s}_t$ and $\mathbf{n}_t$ cannot be both elliptical symmetric distributed as the family of elliptical symmetric distributions is not closed under summation except for the Gaussian distribution. But fixing both the structure (29) of $\boldsymbol{\Sigma}$ and the elliptical symmetric distribution of $\mathbf{x}_t$ can be considered as good approximations thanks to the flexibility of the family of the elliptical symmetric distributions. Furthermore, this family of distributions offers robustness to outliers and heavy tailed samples.

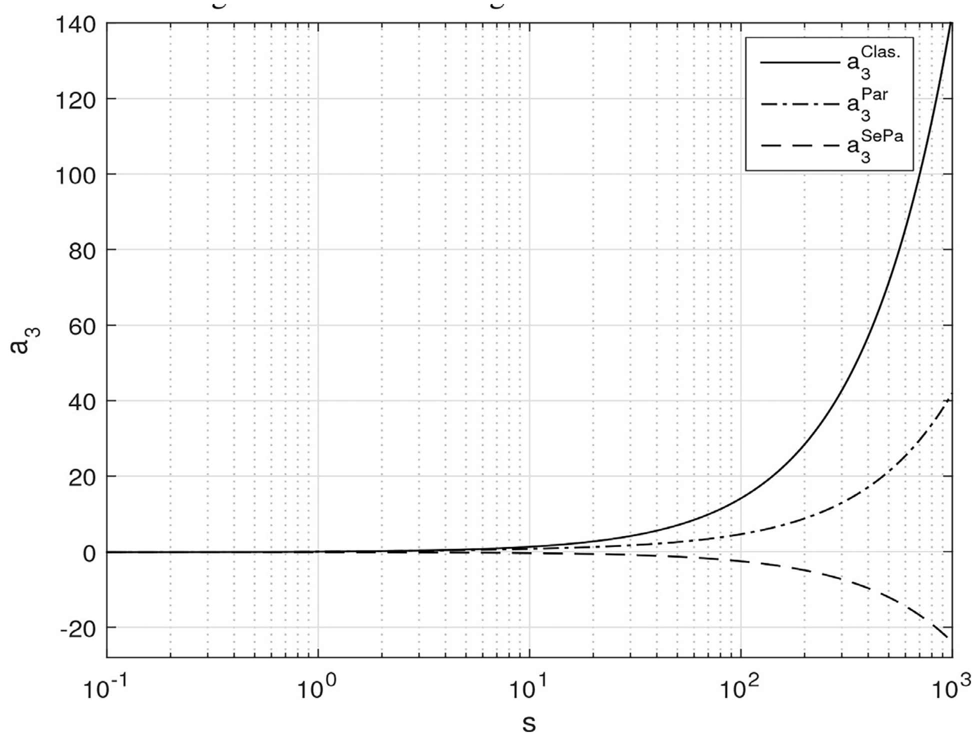

Fig. 1. Coefficients a_3^{SePa}, a_3^{Par} and a_3^{Clas} versus s for $N = 6$.

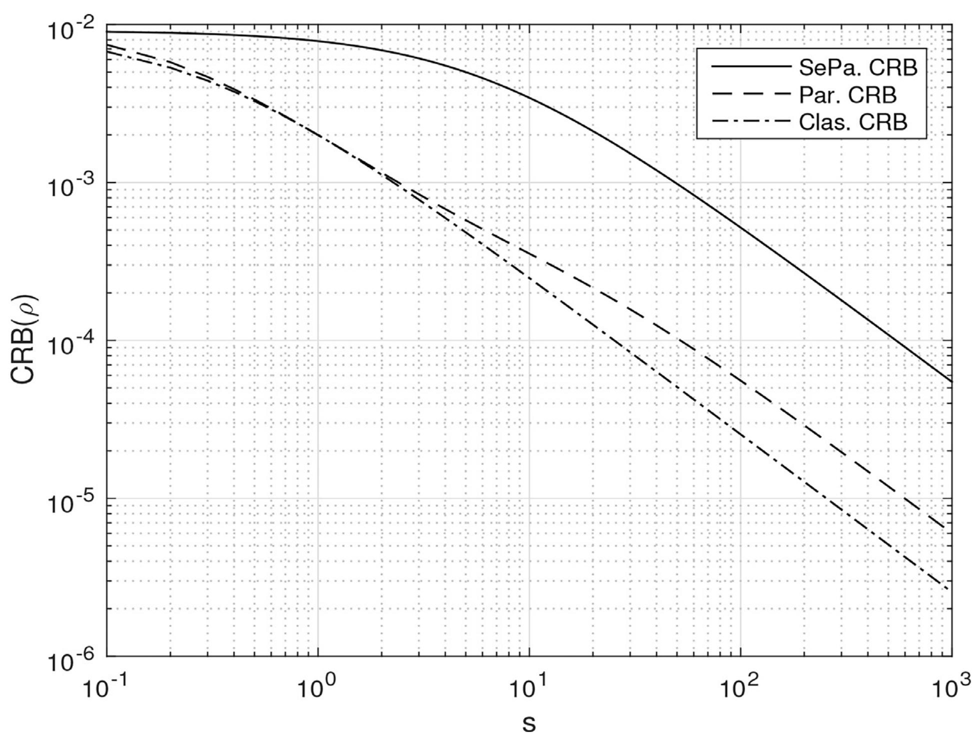

Fig. 2. CRB(ρ) versus s for $\rho = 0.9$ and $N = 6$.

structure

$$\mathbf{\Sigma} = \mathbf{A}(\boldsymbol{\theta})\mathbf{R}_s\mathbf{A}^H(\boldsymbol{\theta}) + \sigma_n^2\mathbf{I}_N, \quad (29)$$

where $\mathbf{R}_s$ is real-valued positive definite symmetric (or Hermitian). Here too $\boldsymbol{\alpha} = (\boldsymbol{\theta}^T, \boldsymbol{\rho}^T, \sigma_n^2)^T$ but $\boldsymbol{\rho}$ collects the entries of $\mathbf{R}_s$. This model also extends to NC-CES distributed data with $\widetilde{\mathbf{\Sigma}} = \widetilde{\mathbf{A}}(\boldsymbol{\theta})\mathbf{R}_r\widetilde{\mathbf{A}}^H(\boldsymbol{\theta}) + \sigma_n^2\mathbf{I}_{2N}$, where here $\boldsymbol{\theta}$ is only characterized by the range space of the full column matrix $\widetilde{\mathbf{A}}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \begin{pmatrix}\mathbf{A}(\boldsymbol{\theta})\\ \mathbf{A}^*(\boldsymbol{\theta})\end{pmatrix}$ and $\mathbf{R}_r$ is real-valued positive definite symmetric. For this model, the proof given in [10] that the CRB on the DOA parameter $\boldsymbol{\theta}$ is proportional to the CRB for Gaussian distributed data in the context of fully

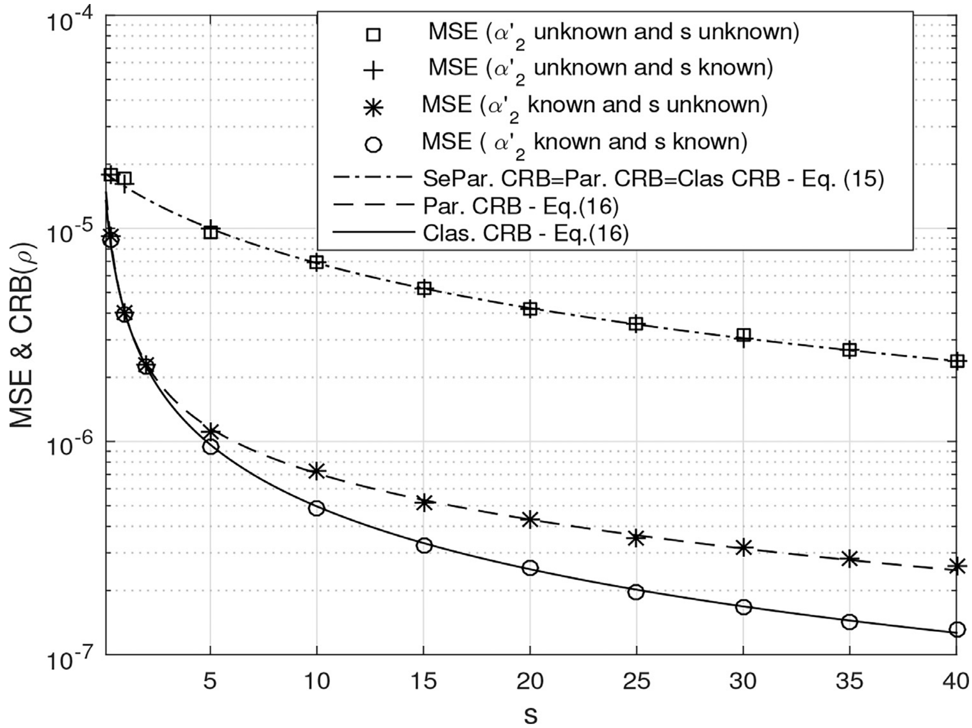

Fig. 3. CRB(ρ) and MSE(ρ) versus s for $\rho = 0.9$ and $N = 6$.

known density generator, directly extends to known up to parameters and completely unknown density generators because the proof is based on the structure (9) with $\boldsymbol{\mu} = \mathbf{0}$, irrelevant the value of the coefficient a_3. Thus we obtain the following result.

Result 5. For each RES, C-CES or NC-CES noisy linear mixture distributed unconditional model, the CRB for the estimation of $\boldsymbol{\theta}$ has a common expression for fully known, known up to parameters and completely unknown density generators. For example for CES distributed data, we get:

$$\mathrm{CRB}(\boldsymbol{\theta}) = \frac{\sigma_n^2}{2T\xi_{c,2,N}}\left[\mathrm{Re}\left(\frac{d\mathbf{a}^H(\theta)}{d\boldsymbol{\theta}}(\mathbf{H}^T \otimes \boldsymbol{\Pi}^{\perp}_{\mathbf{A}(\theta)})\frac{d\mathbf{a}(\theta)}{d\boldsymbol{\theta}}\right)\right]^{-1}, \quad (30)$$

where $\mathbf{H} \stackrel{\text{def}}{=} \mathbf{R}_s\mathbf{A}^H(\theta)\boldsymbol{\Sigma}^{-1}\mathbf{A}(\theta)\mathbf{R}_s$. Note that (30) reduces to $\mathrm{CRB}(\boldsymbol{\theta}) = \frac{\sigma_n^2}{2T\xi_{c,2,N}}\left[\mathrm{Re}\left((\mathbf{D}_\theta^H\boldsymbol{\Pi}^{\perp}_{\mathbf{A}(\theta)}\mathbf{D}_\theta)\odot\mathbf{H}^T\right)\right]^{-1}$ for DOA modeling with one parameter per source where $\mathbf{A}(\theta) \stackrel{\text{def}}{=} [\mathbf{a}_1, \ldots, \mathbf{a}_K]$ where $(\mathbf{a}_k)_{k=1,\ldots,K}$ are the steering vectors parameterized by the DOA θ_k with $\boldsymbol{\theta} \stackrel{\text{def}}{=} (\theta_1, \ldots, \theta_K)^T$, and $\mathbf{D}_\theta \stackrel{\text{def}}{=} [\frac{d\mathbf{a}_1}{d\theta_1}, \ldots, \frac{d\mathbf{a}_K}{d\theta_K}]$ for K sources. This last expression of CRB was given in [11] and [27] as semiparametric CRB without noticing that it was equal to the classic CRB.

In other words from Results 3 and 4, the fully knowledge or the functional knowledge (unknown parameter) does not provide any additional information about the parameter $\boldsymbol{\theta}$ unlike arbitrary parameter $\boldsymbol{\alpha}$.

5. Conclusion

This paper rigorously quantifies the impact of the arbitrary density generators depending on unknown parameters of ES distributed parametric estimation models, by deriving an extension of the SB formula of the FIM for known elliptical symmetric distributions. This SB formula was derived in a unified way within the framework of RES, C-CES and NC-CES distributed data. It was then compared to the SB formula obtained with fully known or completely unknown density generators for different types of the symmetry center and scatter matrix, in particular for the specific noisy linear mixture data model where the parameter of interest is characterized by the range space of the mixing matrix. This allows us to conclude, contrary to commonly known results, that for an arbitrary parameterization, the CRB may be very sensitive to the type knowledge of the density generator for super-Gaussian distributions contrary to sub-Gaussian distributions. These results make it possible to know the situations where it is advantageous or not to use all the information available on the ES distributed data to construct efficient estimators.

Declaration of Competing Interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Jean-Pierre Delmas reports was provided by TELECOM SudParis.

CRediT authorship contribution statement

Habti Abeida: Conceptualization, Methodology, Formal analysis, Writing – review & editing, Visualization. **Jean-Pierre Delmas:** Conceptualization, Methodology, Formal analysis, Writing – review & editing, Visualization.

Data Availability

Data will be made available on request.

Appendix

Proof of Result 1.. It is well known that the CRB for the estimation of $(\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T)^T$ is the inverse of the FIM and thus given from the matrix inversion lemma by $\text{CRB}(\boldsymbol{\alpha}) = \left(\mathbf{I}_\alpha - \mathbf{I}_{\alpha,\beta}\mathbf{I}_\beta^{-1}\mathbf{I}_{\alpha,\beta}^T\right)^{-1}$, where $\begin{pmatrix}\mathbf{I}_\alpha & \mathbf{I}_{\alpha,\beta}\\ \mathbf{I}_{\alpha,\beta}^T & \mathbf{I}_\beta\end{pmatrix}$ is the FIM for $(\boldsymbol{\alpha}^T, \boldsymbol{\beta}^T)^T$ and $\mathbf{I}_\alpha$ is given by the structured matrix (9) for the RES distribution. Following the derivation of $\mathbf{I}_\alpha$, it is straightforward to get $(\mathbf{I}_{\alpha,\beta})_{k,\ell} = -\frac{1}{N}\text{E}[Q\phi_{r,N}(Q)\phi_{r,N}^{\beta_\ell}(Q)]\text{Tr}(\frac{d\boldsymbol{\Sigma}}{d\alpha_k}\boldsymbol{\Sigma}^{-1})$ where $\phi_{r,N}^{\beta_\ell}(t) \overset{\text{def}}{=} \frac{1}{g_{r,N}^\beta(t)}\frac{dg_{r,N}^\beta(t)}{d\beta_\ell}$, $Q \overset{\text{def}}{=} \mathcal{Q}_{r,N}$, which gives the $M \times L$ matrix $\mathbf{I}_{\alpha,\beta} = \left(\frac{d\text{vec}(\boldsymbol{\Sigma})}{d\boldsymbol{\alpha}^T}\right)^T \text{vec}(\boldsymbol{\Sigma}^{-1})\boldsymbol{\xi}_{r,3,N}^T$ with $\boldsymbol{\xi}_{r,3,N} \overset{\text{def}}{=} \frac{\text{E}[Q\phi_{r,N}(Q)\boldsymbol{\phi}_{r,N}^\beta(Q)]}{N}$ where $\boldsymbol{\phi}_{r,N}^\beta(t) \overset{\text{def}}{=} \frac{1}{g_{r,N}^\beta(t)}\frac{\partial g_{r,N}^\beta(t)}{\partial \boldsymbol{\beta}}$. From (1), we get by definition of the FIM for $\boldsymbol{\beta}$, $\mathbf{I}_\beta = \text{E}[\boldsymbol{\phi}_{r,N}^\beta(Q)\boldsymbol{\phi}_{r,N}^{\beta\,T}(Q)] \overset{\text{def}}{=} \Xi_{r,4,N}$. Gathering the expressions of $\mathbf{I}_\alpha$, $\mathbf{I}_\beta$, and $\mathbf{I}_{\alpha,\beta}$ we prove that the CRB for the estimation of $\boldsymbol{\alpha}$ has always the structure (9) with $a_1 = 4\xi_{r,1,N}$, $a_2 = 2\xi_{r,2,N}$ are the coefficients given for both classic and semiparametric SB and $a_3 = a_3^{\text{Clas}} - a_4$ with $a_4 = \boldsymbol{\xi}_{r,3,N}^T\Xi_{r,4,N}^{-1}\boldsymbol{\xi}_{r,3,N}$ where $\xi_{r,1,N}$ and $\xi_{r,2,N}$ are given by (10).

Using again the real to complex representation introduced in Subsection 2.1, it is straightforward to prove that the parameterized SB formulas for NC-CES and C-CES distributions have also the structure (9) where a_1 and a_2 are both equal to those of the classic and semiparametric SB formulas and where a_3 is given by $a_3^{\text{Clas}} - a_4$ with $a_4 = \boldsymbol{\xi}_{c,3,N}^T\Xi_{c,4,N}^{-1}\boldsymbol{\xi}_{c,3,N}$ with $\boldsymbol{\xi}_{c,3,N} \overset{\text{def}}{=} \frac{\text{E}[Q\phi_{c,N}(Q)\boldsymbol{\phi}_{c,N}^\beta(Q)]}{N}$ and $\Xi_{c,4,N} \overset{\text{def}}{=} \text{E}[\boldsymbol{\phi}_{c,N}^\beta(Q)\boldsymbol{\phi}_{c,N}^{\beta\,T}(Q)]$ where $\boldsymbol{\phi}_{c,N}^\beta(t) \overset{\text{def}}{=} \frac{1}{g_{c,N}^\beta(t)}\frac{\partial g_{c,N}^\beta(t)}{\partial \boldsymbol{\beta}}$ and $Q \overset{\text{def}}{=} \mathcal{Q}_{c,N}$. □

Proof of Result 2.. It follows from Result 1 that the FIM associated with the parameter $\boldsymbol{\alpha}_2 = (\alpha_2', \boldsymbol{\alpha}''^T_2)^T$ parameterizing the scatter matrix $\boldsymbol{\Sigma} = \alpha_2'\boldsymbol{\Sigma}_0(\boldsymbol{\alpha}_2'')$, can be written for RES distributed data in the following partitioned matrix form

$$\text{FIM}(\boldsymbol{\alpha}_2) = \frac{1}{\alpha'^2_2}\begin{pmatrix} a & \mathbf{b}^T \\ \mathbf{b} & \mathbf{C}\end{pmatrix} \tag{A.31}$$

with $a = N(a_2 + Na_3)$, $\mathbf{b} = \alpha'_2(a_2 + Na_3)\left(\frac{d\text{vec}(\boldsymbol{\Sigma}_0)}{d\boldsymbol{\alpha}_2''^T}\right)^T(\boldsymbol{\Sigma}_0^{-T}\otimes\boldsymbol{\Sigma}_0^{-1})\text{vec}(\boldsymbol{\Sigma}_0)$ and $\mathbf{C} = \alpha'^2_2\left(\frac{d\text{vec}(\boldsymbol{\Sigma}_0)}{d\boldsymbol{\alpha}_2''^T}\right)^T\left(a_2(\boldsymbol{\Sigma}_0^{-T}\otimes\boldsymbol{\Sigma}_0^{-1}) + a_3\text{vec}(\boldsymbol{\Sigma}_0^{-1})\text{vec}^T(\boldsymbol{\Sigma}_0^{-1})\right)\left(\frac{d\text{vec}(\boldsymbol{\Sigma}_0)}{d\boldsymbol{\alpha}_2''^T}\right)$. We note that for the semiparametric SB formula, $a_2 = 2\xi_{r,2,N}$ and $a_3 = -\frac{2\xi_{r,2,N}}{N}$ implies $a_2 + Na_3 = 0$ and consequently $\text{FIM}(\boldsymbol{\alpha}_2) = \frac{1}{\alpha'^2_2}\begin{pmatrix}0 & \mathbf{0}^T\\ \mathbf{0} & \mathbf{C}\end{pmatrix}$. Thus, $\text{CRB}(\boldsymbol{\alpha}_2'') = \alpha'^2_2\mathbf{C}^{-1}$ which gives (15). For the classic and parametric SB formulas, $a_2 + Na_3 \neq 0$ and the inverse of the partitioned FIM (A.31) allows us to derive $\text{CRB}(\boldsymbol{\alpha}_2'') = \alpha'^2_2[\mathbf{C} - \mathbf{b}a^{-1}\mathbf{b}^T]^{-1}$ which also gives (15) not containing the coefficient a_3. □

Proof of Result 3.. *Complex Student's t−distribution*

From (8), we get after simple algebraic manipulation

$$\phi_{c,N}(t) \overset{\text{def}}{=} \frac{1}{g_{c,N}^\nu(t)}\frac{dg_{c,N}^\nu(t)}{dt} = -\frac{2N+\nu}{\nu'+2t}, \tag{A.32}$$

$$\phi_{c,N}^\nu(t) \overset{\text{def}}{=} \frac{1}{g_{c,N}^\nu(t)}\frac{\partial g_{c,N}^\nu(t)}{\partial \nu} = k'_{N,\nu} - \frac{1}{2}\log\left(1+\frac{2t}{\nu'}\right) - \frac{N\nu' - \nu t}{\nu'(\nu'+2t)}, \tag{A.33}$$

with $\nu' \overset{\text{def}}{=} \nu - 2$ and $k'_{N,\nu} \overset{\text{def}}{=} \frac{1}{2}\left(\psi\left(N+\frac{\nu}{2}\right) - \psi\left(\frac{\nu}{2}\right)\right)$.

Hence with $Q = \mathcal{Q}_{c,N}$

$$\begin{aligned}\xi_{c,3,N} &\overset{\text{def}}{=} \frac{\text{E}[Q\phi_{c,N}(Q)\phi_{c,N}^\nu(Q)]}{N}\\ &= -\frac{(2N+\nu)}{N}\left[(k'_{N,\nu}-N)\text{E}\left(\frac{Q}{\nu'+2Q}\right) + \frac{\nu}{\nu'}\text{E}\left(\frac{Q^2}{\nu'+2Q}\right) - \frac{1}{2}\text{E}\left(\frac{Q}{\nu'+2Q}\log\left(1+\frac{2Q}{\nu'}\right)\right)\right],\end{aligned} \tag{A.34}$$

$$\begin{aligned}\xi_{c,4,N} &\overset{\text{def}}{=} \text{E}[\phi_{c,N}^{\nu\ 2}(Q)]\\ &= k'^2_{N,\nu} + \frac{1}{4}\text{E}\left(\log^2\left(1+\frac{2Q}{\nu'}\right)\right) + \frac{1}{\nu'^2}\text{E}\left(\frac{(N\nu'-\nu Q)^2}{(\nu'+2Q)^2}\right)\\ &- k'_{N,\nu}\left[\text{E}\left(\log\left(1+\frac{2Q}{\nu'}\right)\right) + \frac{2}{\nu'}\text{E}\left(\frac{N\nu'-\nu Q}{\nu'+2Q}\right)\right] + \frac{1}{\nu'}\text{E}\left(\frac{(N\nu'-\nu Q)}{(\nu'+2Q)}\log\left(1+\frac{2Q}{\nu'}\right)\right).\end{aligned} \tag{A.35}$$

Then, observing that $Q =_d NF_{2N,\nu}$ where Q is associated with Student's t distribution without constraint on $\boldsymbol{\Sigma}$ and $F_{\ell,q}$ denotes the F-distribution with ℓ and q degrees of freedom [7], we have here $Q =_d \frac{\nu-2}{\nu}NF_{2N,\nu}$. with p.d.f.

$$p(q) = \frac{1}{((\nu/2)-1)^N B(N,\nu/2)}q^{N-1}\left(1+\frac{2q}{\nu-2}\right)^{-(N+\nu/2)} \text{for } q \geq 0 \text{ and } 0 \text{ for } q < 0. \tag{A.36}$$

Using $\int_0^{+\infty} \frac{t^{p-1}}{(1+\frac{2}{s}t)^{p+q}}dt = \left(\frac{s}{2}\right)^p B(p,q)$, $p>0$, $q>0$, $s>0$, and thanks to [28,29] $\int_0^1 u^{x-1}(1-u)^{y-1}\log^p(u)\log^q(1-u)du = B_{p,q}(x,y) \stackrel{\text{def}}{=} \frac{\partial^{p+q}}{\partial x^p \partial x^q}B(x,y)$ for integers $p,q>0$ and $q+x, p+y>0$, and particularly $B_{1,0}(x,y) = B_{0,1}(y,x) = B(x,y)(\psi_(x) - \psi(x+y))$, we get the following expressions of the different expectations included in (A.34) and (A.35)

$$
\begin{aligned}
\mathrm{E}\Big(\frac{1}{\nu'+2Q}\Big) &= \frac{\nu}{\nu'(2N+\nu)},\\
\mathrm{E}\Big(\frac{1}{(\nu'+2Q)^2}\Big) &= \frac{\nu(\nu+2)}{\nu'^2(2N+\nu)(2N+\nu+2)},\\
\mathrm{E}\Big(\frac{Q}{\nu'+2Q}\Big) &= \frac{N}{2N+\nu},\\
\mathrm{E}\Big(\frac{Q^2}{\nu'+2Q}\Big) &= \frac{N(N+1)}{2N+\nu},\\
\mathrm{E}\Big(\frac{Q}{\nu'+2Q}\Big) &= \frac{N\nu}{\nu'(2N+\nu)(2N+\nu+2)},\\
\mathrm{E}\Big(\frac{Q^2}{(\nu'+2Q)^2}\Big) &= \frac{N(N+1)}{(2N+\nu)(2N+\nu+2)},\\
\mathrm{E}\Big(\log\Big(1+\frac{2Q}{\nu'}\Big)\Big) &= \psi\Big(1-N-\frac{\nu}{2}\Big) - \psi\Big(1-\frac{\nu}{2}\Big),\\
\mathrm{E}\Big(\log^2\Big(1+\frac{2Q}{\nu'}\Big)\Big) &= \psi'\Big(\frac{\nu}{2}\Big) - \psi'\Big(\frac{N}{2}+\frac{\nu}{2}\Big) + \Big(\psi\Big(\frac{\nu}{2}\Big) - \psi\Big(N+\frac{\nu}{2}\Big)\Big)^2,\\
\mathrm{E}\Big(\frac{Q}{\nu'+2Q}\log\Big(1+\frac{2Q}{\nu'}\Big)\Big) &= \frac{N\big(\psi\big(1+N+\frac{\nu}{2}\big) - \psi\big(\frac{\nu}{2}\big)\big)}{2N+\nu},\\
\mathrm{E}\Big(\frac{1}{\nu'+2Q}\log\Big(1+\frac{2Q}{\nu'}\Big)\Big) &= \frac{1}{\nu'}\mathrm{E}\Big(\log\Big(1+\frac{2Q}{\nu'}\Big)\Big) - \frac{2}{\nu'}\mathrm{E}\Big(\frac{Q}{\nu'+2Q}\log\Big(1+\frac{2Q}{\nu'}\Big)\Big).
\end{aligned}
$$

Plugging these expressions of the expectations in (A.34) and (A.35) and using $\psi'\big(\frac{\nu}{2}\big) - \psi'\big(N+\frac{\nu}{2}\big) = \sum_{\ell=0}^{N-1}\frac{1}{(\ell+\frac{\nu}{2})^2}$ allows us to obtain after some tedious algebraic manipulations the values (21) of $\xi_{c,3,N}$ and (22) of $\xi_{c,4,N}$. □

Complex generalized Gaussian distribution. From (8), $g^s_{c,N}(t)$ is given by

$$
g^s_{c,N}(t) = c_{N,s}e^{-t^s/b} \text{ with } b \stackrel{\text{def}}{=} \left[\frac{N\Gamma\big(\frac{N}{s}\big)}{\Gamma\big(\frac{N+1}{s}\big)}\right]^s \text{ and } c_{N,s} \stackrel{\text{def}}{=} \frac{s\Gamma(N)b^{-N/s}}{\pi^N\Gamma\big(\frac{N}{s}\big)}, \tag{A.37}
$$

yielding after some algebraic manipulation

$$
\phi_{c,N}(t) \stackrel{\text{def}}{=} \frac{1}{g^s_{c,N}(t)}\frac{dg^s_{c,N}(t)}{dt} = -\frac{s}{b}t^{s-1}, \tag{A.38}
$$

$$
\phi^s_{c,N}(t) \stackrel{\text{def}}{=} \frac{1}{g^s_{c,N}(t)}\frac{\partial g^s_{c,N}(t)}{\partial s} = \alpha_{N,s} + (\gamma_{N,s} - \beta_{N,s})t^s - \frac{1}{b}t^s\log(t), \tag{A.39}
$$

with $\alpha_{N,s} \stackrel{\text{def}}{=} \frac{1}{s^2}(s+N(N+1)k_{N,s})$, $\beta_{N,s} \stackrel{\text{def}}{=} \frac{Nk_{N,s}-\psi\left(\frac{N+1}{s}\right)}{bs}$ and $\gamma_{N,s} \stackrel{\text{def}}{=} \frac{\log(b)}{bs}$ and where $k_{N,s} \stackrel{\text{def}}{=} \psi\big(\frac{N}{s}\big) - \psi\big(\frac{N+1}{s}\big)$ where $\psi(x) \stackrel{\text{def}}{=} \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function.

Hence with $Q = \mathcal{Q}_{c,N}$

$$
\begin{aligned}
\xi_{c,3,N} &\stackrel{\text{def}}{=} \frac{\mathrm{E}[Q\phi_{c,N}(Q)\phi^s_{c,N}(Q)]}{N}\\
&= -\frac{s}{Nb}\Big[\alpha_{N,s}\mathrm{E}(Q^s) + (\gamma_{N,s} - \beta_{N,s})\mathrm{E}(Q^{2s}) - \frac{1}{b}\mathrm{E}(Q^{2s}\log(Q))\Big]
\end{aligned} \tag{A.40}
$$

$$\xi_{c,4,N} \overset{\text{def}}{=} \mathrm{E}[\phi^{s}_{c,N}{}^{2}(Q)]$$
$$= \alpha^2_{N,s} + (\gamma_{N,s} - \beta_{N,s})^2 \mathrm{E}(Q^{2s}) + \frac{1}{b^2}\mathrm{E}(Q^{2s}\log^2(Q))$$
$$+2\alpha_{N,s}(\gamma_{N,s} - \beta_{N,s})\mathrm{E}(Q^s) - \frac{2\alpha_{N,s}}{b}\mathrm{E}(Q^s\log(Q))$$
$$+\frac{2(\beta_{N,s} - \gamma_{N,s})}{b}\mathrm{E}(Q^{2s}\log(Q)). \quad (A.41)$$

Then observing that $Q =_d \mathcal{G}^{1/s}$ where $\mathcal{G} \sim \mathrm{Gam}(N/s, b)$ [7], the following equalities [15, Eqs. (a.14)-(a.17)]

$$\mathrm{E}(Q^s) = \frac{Nb}{s}$$
$$\mathrm{E}(Q^{2s}) = \frac{Nb^2(N+s)}{s^2}$$
$$\mathrm{E}(Q^s\log(Q)) = \frac{Nb}{s^2}A_{N,s}$$
$$\mathrm{E}(Q^{2s}\log(Q)) = \frac{Nb^2(N+s)}{s^3}\left(A_{N,s} + \frac{s}{N+s}\right)$$
$$\mathrm{E}(Q^{2s}\log^2(Q)) = \frac{Nb^2(N+s)}{s^4}\left(A^2_{N,s} + \frac{2s}{N+s}A_{N,s} + \psi'\left(\frac{N+s}{s}\right)\right),$$

where $A_{N,s} \overset{\text{def}}{=} \log(b) + \psi(\frac{N+s}{s})$ used in (A.40) and (A.41), allows us to obtain after some algebraic manipulations the values (23) of $\xi_{c,3,N}$ and (24) of $\xi_{c,4,N}$. $\square$

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