Digital Signal Processing 131 (2022) 103767

Contents lists available at ScienceDirect

Digital Signal Processing

www.elsevier.com/locate/dsp

Performance of subspace-based algorithms associated with the sample sign covariance matrix

Habti Abeida [a], Jean-Pierre Delmas [b,*]

[a] Taif University, College of Engineering, Dept. of Electrical Engineering, Al-Haweiah, 21974, Saudi Arabia
[b] Samovar laboratory, Telecom SudParis, Institut Polytechnique de Paris, 91011 Evry Cedex, France

ARTICLE INFO

Article history:
Available online 7 October 2022

Keywords:
Subspace-based algorithms
Sample sign covariance matrix
Tyler's *M* estimator
Circular and non-circular CES distribution

ABSTRACT

Complex-valued data in statistical signal processing applications have many advantages over their real-valued counterparts. It allows us to use the complete statistical information of the signal thanks to its statistical property of non-circularity. This paper presents a general framework for developing asymptotic theoretical results on the distribution-free sample sign covariance matrix (SSCM) under circular complex-valued elliptically symmetric (C-CES) and non-circular CES (NC-CES) multidimensional distributed data. It extends some partial asymptotic results on SSCM derived for real elliptically symmetric (RES) distributed data. In particular closed-form expressions of the first and second-order of the SSCM are derived for arbitrary spectra of eigenvalues for C-CES and NC-CES distributed data which facilitates the derivation of numerous statistical properties. Then, the asymptotic distributions of associated projectors are deduced, which are applied in the study of asymptotic performance analysis of SSCM-based subspace algorithms, followed by a comparison to the asymptotic results derived using Tyler's *M* estimate. However, a more in-depth analytical analysis of the efficiency of the SSCM relative to Tyler's *M* estimate is performed, yielding that the performances of the SSCM and Tyler's *M* estimate are close for a high-dimensional data and not too small dimension of the principal component space. We conclude therefore that, although the SSCM is inefficient relative to Tyler's *M* estimate, it is of great interest from the point of view of its lower computational complexity for high-dimensional data. Finally, numerical results illustrating the theoretical analysis are presented through the direction-of-arrival (DOA) estimation CES data models.

1. Introduction

Covariance matrix (CM) estimation is a fundamental and long-standing problem facing the statistical signal processing community. Many algorithms for estimation and detection rely on accurate covariance estimators (see e.g., [1]). The problem is well solved under Gaussian distributed data for which the well-known sample covariance matrix (SCM) is the maximum likelihood ML CM estimator. It is, however, well known that the SCM estimator is non-robust, being highly sensitive to outliers and very inefficient for non-Gaussian heavy-tailed distributed data. Therefore, several robust alternatives have been proposed in the literature for real elliptically symmetric (RES) and complex elliptically symmetric (CES) distributed data. Among those, the M-estimates [2], Tyler's M-estimator [3,4] and its complex extension [5–7], Huber's M-estimator [8] and its complex extension [5], and the sample sign covariance matrix (SSCM), to the best of our knowledge, was introduced by [9] under the name of normalized sample covariance matrix in the signal processing community and by [10] in the statistics community.

This latter estimate is easy to compute, and was studied by several authors under various names, such as sign covariance matrix [11] and [12], spatial sign covariance matrix [13], [19], [20]. It was first studied in the RES framework in [11] with the sample spatial Kentall's tau covariance matrix and then in [12], they proved in particular that the expectation of the SSCM and SCM share the same eigenvectors with different eigenvalues with a one-to-one but rather complicated correspondence. The asymptotic distribution of projector estimates based on the SSCM was studied in [19] and [20] paying particular attention to the asymptotic relative efficiency of the SSCM to Tyler's M

* Corresponding author.
E-mail addresses: h.abeida@tu.edu.sa (H. Abeida), jean-pierre.delmas@it-sudparis.eu (J.-P. Delmas).

https://doi.org/10.1016/j.dsp.2022.103767

estimate. A one-dimensional integral representation of the eigenvalues of the expectation of the SSCM was provided in [21, Proposition 3], but requiring numerical approximations.

Regarding C-CES distributed data [18], closed-form expressions of the first and second-order moments of the SSCM have been given in the particular case of different eigenvalues in [22], that were partially completed in the case of a single multiple eigenvalue in [23], [24]. Recently, an approximation of the one-dimensional integral representation [21, Proposition 3] for the eigenvalues of the expectation of the SSCM has been given in [25], making possible an approximate bias correction to its eigenvalues leading a robust regularized SSCM based estimator. Note that this SSCM was mainly used in DOA estimation with heavy-tailed noise [13] and in radar clutter modeling [9], [14–17] in the framework of C-CES distributed data.

The focus of this paper is to refine and derive asymptotic normality and efficiency results of SSCM for underlying C-CES and NC-CES (also known as the generalized CES [26]) distributed data. In undertaking this, our main contribution is threefold: First, we present an asymptotic performance analysis of the SSCM by giving analytical closed-form expressions of the expectation and covariance of the SSCM by analytically solving one-dimensional integrals for arbitrary eigenvalue spectra of the associated SCM. Second, we deduce an asymptotic performance analysis of the associated projectors and then of subspace-based algorithms associated with the SSCM for arbitrary invariant subspace. And finally, this asymptotic performance based on SSCM is compared to that based on Tyler's M estimate, where an analytical analysis of the efficiency of the SSCM relative to Tyler's M estimate is studied. This allows us to conclude that although the SSCM is inefficient relative to Tyler's M estimate, whose performances are particularly close for a high-dimensional data and not too small dimension of the principal component space and, therefore, deduce that the SSCM estimate is of great interest from the point of view of their lower computational complexity for high-dimensional data.

The paper is organized as follows. Section 2 describes the second-order C-CES and NC-CES distributions, specifies different robust estimators of their scatter and extended scatter matrices, and introduces the problem formulation. The asymptotic distribution of the SSCM with closed-form expressions of the first and second-order moments is derived in Section 3 for arbitrary eigenvalue spectra of the scatter and extended scatter matrices. Then the asymptotic distribution of associated projectors and subspace-based algorithms are deduced, in Section 4, which are compared to those of Tyler's M estimate proving the efficiency of the SSCM relative to Tyler's M estimate. These results are applied to the factor models in Section 5 with a detailed analysis of the inefficiency of the SSCM relative to Tyler's M estimate illustrated with subspace-based DOA estimation in C-CES and NC-CES data models. Finally, the paper is concluded in Section 6.

The following notations are used throughout the paper. Matrices and vectors are represented by bold upper case and bold lower case characters, respectively. Vectors are by default in column orientation, while T, H, $*$ and $\#$ stand for transpose, conjugate transpose, conjugate and Moore Penrose inverse, respectively. $\mathbf{e}_k$ and $\widetilde{\mathbf{e}}_k$ denote the k-unit vector of dimension N and $2N$, respectively. $(\mathbf{a})_k$ and $(\mathbf{A})_{k,\ell}$ denotes the k and (k,ℓ)-th element of the vector $\mathbf{a}$ and the matrix $\mathbf{A}$, respectively. E(.), |.|, Diag(.), Re(.) and Im(.) are the expectation, determinant, diagonal, real and imaginary part operators respectively. $\mathbf{I}$ is the identity matrix and $\mathbf{J}$ is the exchange matrix $\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{pmatrix}$ of appropriate dimensions. vec$(\cdot)$ is the "vectorization" operator that turns a matrix into a vector by stacking the columns of the matrix one below another which is used in conjunction with the Kronecker product $\mathbf{A}\otimes\mathbf{B}$ as the block matrix whose (i,j) block element is $a_{i,j}\mathbf{B}$ and with the commutation matrix $\mathbf{K}$ of appropriate dimension such that $\text{vec}(\mathbf{C}^T)=\mathbf{K}\text{vec}(\mathbf{C})$. Finally, $\mathbb{1}$ is the indicator function, $\Gamma(x)$ is the Gamma function with $\Gamma(k)=(k-1)!$ for $k\in\mathbb{N}$, $B(k,\ell)$ is the Beta function with $B(k,\ell)=\frac{\Gamma(k)\Gamma(\ell)}{\Gamma(k+\ell)}$ and ${}_2F_1(a,b,c,x)$ is the Gauss hypergeometric functions with ${}_2F_1(a,b,c,x)=\frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}dt$ for $c>b>0$ and $|x|<1$. $x=_d y$ means that the r.v. x and y have the same distribution.

2. Data model and problem formulation

2.1. C-CES and NC-CES distributions

Let $(\mathbf{x}_t)_{t=1,\dots,T}$ be a set of T independent and identically zero-mean with finite second-order moments N-dimensional C-CES or NC-CES distributed data snapshots. Let us remind here that an N-dimensional complex r.v. $\mathbf{x}_t$ has a CES distribution if and only if the $2N$-dimensional real r.v. $\bar{\mathbf{x}}_t \stackrel{\text{def}}{=} (\text{Re}^T(\mathbf{x}_t), \text{Im}^T(\mathbf{x}_t))^T$ is RES distributed. Depending on whether $\text{E}(\mathbf{x}_t\mathbf{x}_t^T)=\mathbf{0}$ or $\text{E}(\mathbf{x}_t\mathbf{x}_t^T)\neq\mathbf{0}$, the associated complex distribution is said to be circular or non-circular. These associated p.d.f. are given by [18] and [26][27] for respectively C-CES and NC-CES distributions

$$p(\mathbf{x}_t)=c_{N,g}|\mathbf{\Sigma}_x|^{-1}g(\mathbf{x}_t^H\mathbf{\Sigma}_x^{-1}\mathbf{x}_t),\ [\text{resp., } c_{N,g}|\mathbf{\Sigma}_{\tilde{x}}|^{-1/2}g\left(\frac{1}{2}\tilde{\mathbf{x}}_t^H\mathbf{\Sigma}_{\tilde{x}}^{-1}\tilde{\mathbf{x}}_t\right)], \tag{1}$$

where $\widetilde{\mathbf{x}}_t \stackrel{\text{def}}{=} (\mathbf{x}_t^T, \mathbf{x}_t^H)^T$, $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_{\tilde{x}}$ are $N\times N$ [resp., $2N\times 2N$] Hermitian positive definite matrices respectively called scatter and extended scatter matrices. The density generator $g(.)$: $\mathbb{R}^+\mapsto\mathbb{R}^+$ which allows to describe heavier or lighter tailed distribution than the complex Gaussian distribution satisfies $\delta_{N,g} \stackrel{\text{def}}{=} \int_0^\infty t^{N-1}g(t)dt<\infty$ to ensure the integrability of $p(\mathbf{x}_t)$. $c_{N,g}$ is a normalizing constant given by $c_{N,g} \stackrel{\text{def}}{=} 2(s_N\delta_{N,g})^{-1}$ where $s_N \stackrel{\text{def}}{=} 2\pi^N/\Gamma(N)$ is the surface area of the unit complex N-sphere. We note that the so-called scale ambiguity usually present in the p.d.f. of $\mathbf{x}_t$ with the scatter and extended scatter matrices, is here removed thanks to the constraint on g: $\delta_{N+1,g}/\delta_{N,g}=N$ [18] which ensures that the scatter matrices are equal to the covariance matrices.

The r.v. $\mathbf{x}_t$ admits the following stochastic representation:

$$\mathbf{x}_t =_d \sqrt{\mathcal{Q}_t}\,\mathbf{\Sigma}_x^{1/2}\mathbf{u}_t,\quad \text{circular case [18]} \tag{2}$$

$$\widetilde{\mathbf{x}}_t =_d \sqrt{\mathcal{Q}_t}\,\mathbf{\Sigma}_{\tilde{x}}^{1/2}\widetilde{\mathbf{u}}_t,\quad \text{non-circular case [27]}, \tag{3}$$

where $\tilde{\mathbf{u}}_t \stackrel{\text{def}}{=} (\mathbf{u}_t^T, \mathbf{u}_t^H)^T$, $\mathcal{Q}_t$ and $\mathbf{u}_t$ are independent, and $\mathbf{u}_t$ is uniformly distributed on the unit complex N-sphere (denoted hereafter $U(\mathbb{C}S^N)$). Note that the CES distributions encompass the compound-Gaussian distributions whose r.v.s are also referred to as spherically invariant random vector (SIRP) in the engineering literature for modeling radar clutter (see e.g., [9]).

2.2. Robust estimators of the scatter matrix

When the distribution of $\mathbf{x}_t$ is known, the density generator g is fixed and the ML estimate of $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_{\tilde{x}}$ are solutions of the respective implicit equations:

$$\widehat{\mathbf{\Sigma}}_{x,\mathrm{ML}} = \frac{1}{T}\sum_{t=1}^{T}\phi(\mathbf{x}_t^H\widehat{\mathbf{\Sigma}}_x^{-1}\mathbf{x}_t)\mathbf{x}_t\mathbf{x}_t^H \ \text{ and } \ \widehat{\mathbf{\Sigma}}_{\tilde{x},\mathrm{ML}} = \frac{1}{T}\sum_{t=1}^{T}\phi(\frac{1}{2}\widetilde{\mathbf{x}}_t^H\widehat{\mathbf{\Sigma}}_{\tilde{x}}^{-1}\widetilde{\mathbf{x}}_t)\widetilde{\mathbf{x}}_t\widetilde{\mathbf{x}}_t^H \tag{4}$$

with $\phi(t) \stackrel{\text{def}}{=} -\frac{1}{g(t)}\frac{dg(t)}{dt}$ for respectively C-CES [18] and NC-CES [27] distributions.

But when the distribution of $\mathbf{x}_t$ is unknown, the simplest estimate of $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_{\tilde{x}}$ is the SCM given respectively by

$$\widehat{\mathbf{\Sigma}}_{x,\mathrm{SCM}} \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t^H \ \text{ and } \ \widehat{\mathbf{\Sigma}}_{\tilde{x},\mathrm{SCM}} \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^{T}\widetilde{\mathbf{x}}_t\widetilde{\mathbf{x}}_t^H. \tag{5}$$

Although ML estimate in the case of Gaussian distributions (solution of (4) for $\phi(t)=1$), this SCM is not robust and can perform poorly in comparison to M-estimators (solution of (4) where $\phi(t)$ may not be related to $g(t)$) in the CES framework or in the context of contaminated data. Despite this potential mismatch, M-estimators can ensure good performance accuracy on the whole CES family and can present robustness to contamination by outliers. Tyler's and Hubert's M-estimators are examples of such estimators. Tyler's M-estimators for C-CES [18] and NC-CES [27] distributions are solutions of the following equation:

$$\widehat{\mathbf{\Sigma}}_{x,\mathrm{Ty}} = \frac{N}{T}\sum_{t=1}^{T}\frac{\mathbf{x}_t\mathbf{x}_t^H}{\mathbf{x}_t^H\widehat{\mathbf{\Sigma}}_{x,T_y}^{-1}\mathbf{x}_t} \ \text{ and } \ \widehat{\mathbf{\Sigma}}_{\tilde{x},\mathrm{Ty}} = \frac{2N}{T}\sum_{t=1}^{T}\frac{\widetilde{\mathbf{x}}_t\widetilde{\mathbf{x}}_t^H}{\widetilde{\mathbf{x}}_t^H\widehat{\mathbf{\Sigma}}_{\tilde{x},\mathrm{Ty}}^{-1}\widetilde{\mathbf{x}}_t}. \tag{6}$$

In order to ensure the uniqueness of a consistent solution of (6), it suffices to impose the respective normalization conditions $\mathrm{Tr}(\mathbf{\Sigma}_x^{-1}\widehat{\mathbf{\Sigma}}_{x,\mathrm{Ty}}) = N$ and $\mathrm{Tr}(\mathbf{\Sigma}_{\tilde{x}}^{-1}\widehat{\mathbf{\Sigma}}_{\tilde{x},\mathrm{Ty}}) = 2N$. Interestingly, the distribution of the Tyler's M-estimator does not depend on the specific RES or CES distribution of the data. In the RES framework, this estimator is the ML for $\mathbf{\Sigma}_x$ when the data comes from the angular central Gaussian distribution [28]. This property was extended to the complex circular [5], [29] and non-circular case [27].

In the context of unknown distribution of $\mathbf{x}_t$, unlike Tyler's M-estimate, the SSCM are defined below by a closed-form expression simpler to calculate:

$$\widehat{\mathbf{\Sigma}}_{x,\mathrm{SSCM}} \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^{T}\mathbf{s}(\mathbf{x}_t)\mathbf{s}^H(\mathbf{x}_t) \ \text{ and } \ \widehat{\mathbf{\Sigma}}_{\tilde{x},\mathrm{SSCM}} \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^{T}\mathbf{s}_t(\widetilde{\mathbf{x}}_t)\mathbf{s}_t^H(\widetilde{\mathbf{x}}_t), \tag{7}$$

where $\mathbf{s}(\mathbf{x}_t) \stackrel{\text{def}}{=} \frac{\mathbf{x}_t}{\|\mathbf{x}_t\|}$ if $\mathbf{x}_t \neq \mathbf{0}$ and $\mathbf{s}(\mathbf{x}_t) \stackrel{\text{def}}{=} \mathbf{0}$ if $\mathbf{x}_t = \mathbf{0}$, are another possible robust estimate of $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_{\tilde{x}}$, respectively. Using the stochastic representations of the C-CES (2) and NC-CES (3) distributed data $\mathbf{x}_t$, we note that the distribution of $s(\mathbf{x}_t)$ is invariant under the distribution of $\mathcal{Q}_t$. We thus have the liberty of choosing any specific spherical distribution in $\mathbb{C}^N$ for $\mathbf{w}_t \stackrel{\text{def}}{=} \sqrt{\mathcal{Q}_t}\mathbf{u}_t$. Consequently, these SSCM have the same distribution as, respectively:

$$\widehat{\mathbf{\Sigma}}_{x,\mathrm{SSCM}} =_d \frac{1}{T}\sum_{t=1}^{T}\frac{\mathbf{\Sigma}_x^{1/2}\mathbf{u}_t\mathbf{u}_t^H\mathbf{\Sigma}_x^{1/2}}{\mathbf{u}_t^H\mathbf{\Sigma}_x\mathbf{u}_t} =_d \frac{1}{T}\sum_{t=1}^{T}\frac{\mathbf{\Sigma}_x^{1/2}\mathbf{w}_t\mathbf{w}_t^H\mathbf{\Sigma}_x^{1/2}}{\mathbf{w}_t^H\mathbf{\Sigma}_x\mathbf{w}_t} \tag{8}$$

and

$$\widehat{\mathbf{\Sigma}}_{\tilde{x},\mathrm{SSCM}} =_d \frac{1}{T}\sum_{t=1}^{T}\frac{\mathbf{\Sigma}_{\tilde{x}}^{1/2}\widetilde{\mathbf{u}}_t\widetilde{\mathbf{u}}_t^H\mathbf{\Sigma}_{\tilde{x}}^{1/2}}{\widetilde{\mathbf{u}}_t^H\mathbf{\Sigma}_{\tilde{x}}\widetilde{\mathbf{u}}_t} =_d \frac{1}{T}\sum_{t=1}^{T}\frac{\mathbf{\Sigma}_{\tilde{x}}^{1/2}\widetilde{\mathbf{w}}_t\widetilde{\mathbf{w}}_t^H\mathbf{\Sigma}_{\tilde{x}}^{1/2}}{\widetilde{\mathbf{w}}_t^H\mathbf{\Sigma}_{\tilde{x}}\widetilde{\mathbf{w}}_t}, \tag{9}$$

with $\widetilde{\mathbf{w}}_t \stackrel{\text{def}}{=} (\mathbf{w}_t^T, \mathbf{w}_t^H)^T$. We therefore see from (8) and (9) that the distribution of these SSCM does not depend on the CES distribution of the data, whereas this property was acquired by Tyler's M-estimate only asymptotically.

2.3. Problem formulation

The spectral decomposition of the scatter matrix $\mathbf{\Sigma}_x$ is given by $\mathbf{\Sigma}_x = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H$ where $\mathbf{\Lambda} \stackrel{\text{def}}{=} \mathrm{Diag}(\lambda_1, .., \lambda_N)$, and $\mathbf{V} = (\mathbf{v}_1, ..., \mathbf{v}_N)$ where $(\mathbf{v}_k)_{k=1,...,N}$ are unit-norm eigenvectors associated with the eigenvalues $(\lambda_k)_{k=1,...,N}$. We assume here that $\lambda_1 \geq ... \geq \lambda_k \geq ... \geq \lambda_N > 0$ with J distinct eigenvalues $\lambda_{(1)} > ... > \lambda_{(j)} > ... > \lambda_{(J)} > 0$ with the respective multiplicities being $m_1, ..., m_j, ..., m_J$. We consider the eigenprojectors associated with the eigenvalue $\lambda_{(j)}$, i.e., the orthogonal projector $\mathbf{\Pi}_{(j)} = \sum_{k\in s_j}\mathbf{v}_k\mathbf{v}_k^H$ where s_j denotes the set of eigenvalues λ_k equal to $\lambda_{(j)}$, and paying attention to the derivation of the asymptotic distribution of these eigenprojectors estimated from the SSCM, and then compared it to that of eigenprojectors estimated from the SCM and Tyler's M-estimate.

Regarding the extended scatter matrix $\mathbf{\Sigma}_{\tilde{x}}$, there also exists a spectral decomposition[1] $\mathbf{\Sigma}_{\tilde{x}} = \widetilde{\mathbf{V}}\widetilde{\mathbf{\Lambda}}\widetilde{\mathbf{V}}^H$ where $\widetilde{\mathbf{\Lambda}} \stackrel{\text{def}}{=} \mathrm{Diag}(\widetilde{\lambda}_1, .., \widetilde{\lambda}_{2N})$, in which there exist unit-norm eigenvectors $(\widetilde{\mathbf{v}}_k)_{k=1,...,2N}$ associated with the eigenvalues $(\widetilde{\lambda}_k)_{k=1,..,2N}$ that are structured in the form $\widetilde{\mathbf{v}}_k = (\mathbf{v}_k^T, \mathbf{v}_k^H)^T$ [30]. We assume that $\widetilde{\lambda}_1 \geq ... \geq \widetilde{\lambda}_k \geq ... \geq \widetilde{\lambda}_{2N} > 0$. Similarly, we also consider the eigenprojectors $\widetilde{\mathbf{\Pi}}_{(j)} = \sum_{k\in s_j}\widetilde{\mathbf{v}}_k\widetilde{\mathbf{v}}_k^H$, $j = 1, ..., J$

[1] This spectral decomposition is different from the one proposed in [31] for which $\widetilde{\mathbf{V}}$ is widely unitary and $\widetilde{\mathbf{\Lambda}}$ is diagonal block instead.

and derive the asymptotic distribution of these eigenprojectors estimated from the SSCM which is compared to that of the eigenprojectors estimated from the SCM and Tyler's M-estimate.

The case of eigenvalues $\lambda_1 > ... > \lambda_P > \lambda_{P+1} = ... = \lambda_N > 0$ associated with $\mathbf{\Sigma}_x$ and $\widetilde{\lambda}_1 > ... > \widetilde{\lambda}_P > \widetilde{\lambda}_{P+1} = ... = \widetilde{\lambda}_{2N} > 0$ associated with $\mathbf{\Sigma}_{\tilde{x}}$ are important particular cases will be also considered.

3. Asymptotic distribution of the SSCM

For C-CES distributed data, it has been proved in [22, Th. 1] that $\mathrm{E}(\widehat{\mathbf{\Sigma}}_{x,\mathrm{SSCM}}) = \mathbf{V}\mathbf{\Delta}\mathbf{V}^H$ where $\mathbf{\Delta} = \mathrm{Diag}(\chi_1, .., \chi_N)$ with entries, χ_k, $k = 1, .., N$, are given by rather complicated expressions for distinct eigenvalues λ_k, $k = 1, .., N$. A general expression of χ_k was directly derived in [25, Th. 1, eq. (6)] from an integral representation of $\mathrm{E}\left(\frac{\mathbf{x}_t\mathbf{x}_t^H}{\mathbf{x}_t^H\mathbf{Z}\mathbf{x}_t}\right)$ where $\mathbf{Z}$ is a positive definite $N \times N$ matrix for arbitrary eigenvalues. Here, we deduce the expressions of χ_k from those of RES distributed data given by [21, proposition 3] and extend them to NC-CES distributed data. Using the stochastic representations (8) and (9) of the SSCM, the following theorem is proved in Appendix A:

Theorem 1. *The first-order moments of the SSCM associated with C-CES and NC-CES distributed data are, respectively, given by*

$$\mathrm{E}(\widehat{\mathbf{\Sigma}}_{x,\mathrm{SSCM}}) = \mathbf{V}\mathbf{\Delta}\mathbf{V}^H \ \text{ and } \ \mathrm{E}(\widehat{\mathbf{\Sigma}}_{\tilde{x},\mathrm{SSCM}}) = \widetilde{\mathbf{V}}\widetilde{\mathbf{\Delta}}\widetilde{\mathbf{V}}^H, \tag{10}$$

where $\mathbf{\Delta} = \mathrm{Diag}(\chi_1, .., \chi_N)$ *with* $\chi_k = \lambda_k \mathrm{E}\left(\frac{|(\mathbf{u}_t)_k|^2}{\mathbf{u}_t^H\mathbf{\Lambda}\mathbf{u}_t}\right)$, $k = 1, ..., N$, *and* $\widetilde{\mathbf{\Delta}} = \mathrm{Diag}(\widetilde{\chi}_1, .., \widetilde{\chi}_{2N})$ *with* $\widetilde{\chi}_k = \widetilde{\lambda}_k \mathrm{E}\left(\frac{|(\bar{\mathbf{u}}_t)_k|^2}{\bar{\mathbf{u}}_t^T\widetilde{\mathbf{\Lambda}}\bar{\mathbf{u}}_t}\right)$, $k = 1, ..., 2N$, *with* $\bar{\mathbf{u}}_t \stackrel{\mathrm{def}}{=} (\mathrm{Re}^T(\mathbf{u}_t), \mathrm{Im}^T(\mathbf{u}_t))^T$, *and* χ_k *and* $\widetilde{\chi}_k$ *have the same multiplicity order than the eigenvalues* λ_k *and* $\widetilde{\lambda}_k$, *respectively, with*

$$\chi_{(j)} = \lambda_{(j)} \int_0^\infty \frac{1}{(1+\lambda_{(j)}x)^{m_j+1}\prod_{1\le k\ne j\le J}(1+\lambda_{(k)}x)^{m_k}}dx, \tag{11}$$

$$\widetilde{\chi}_{(j)} = \frac{\widetilde{\lambda}_{(j)}}{2} \int_0^\infty \frac{1}{(1+\widetilde{\lambda}_{(j)}x)^{\frac{m_j}{2}+1}\prod_{1\le k\ne j\le J}(1+\widetilde{\lambda}_{(k)}x)^{\frac{m_k}{2}}}dx. \tag{12}$$

This theorem shows that $\mathrm{E}(\widehat{\mathbf{\Sigma}}_{x,\mathrm{S}})$ [resp., $\mathrm{E}(\widehat{\mathbf{\Sigma}}_{\tilde{x},\mathrm{S}})$] and $\mathbf{\Sigma}_x$ [resp., $\mathbf{\Sigma}_{\tilde{x}}$] share the same eigenvectors with possible different eigenvalues. From definition (7) of the SSCM, the central limit theorem[2] applies and consequently:

$$\sqrt{T}(\mathrm{vec}(\widehat{\mathbf{\Sigma}}_{x,\mathrm{SSCM}}) - \mathrm{vec}(\mathbf{V}\mathbf{\Delta}\mathbf{V}^H)) \to_d \mathcal{N}(\mathbf{0}; \mathbf{R}_{\widehat{\Sigma}_{\mathrm{SSCM}}}, \mathbf{R}_{\widehat{\Sigma}_{\mathrm{SSCM}}}\mathbf{K}_{N^2}) \tag{13}$$

$$\sqrt{T}(\mathrm{vec}(\widehat{\mathbf{\Sigma}}_{\tilde{x},\mathrm{SSCM}}) - \mathrm{vec}(\widetilde{\mathbf{V}}\widetilde{\mathbf{\Delta}}\widetilde{\mathbf{V}}^H)) \to_d \mathcal{N}(\mathbf{0}; \mathbf{R}_{\widehat{\widetilde{\Sigma}}_{\mathrm{SSCM}}}, \mathbf{R}_{\widehat{\widetilde{\Sigma}}_{\mathrm{SSCM}}}\mathbf{K}_{(2N)^2}), \tag{14}$$

where the covariance matrices of these asymptotic distributions are given by the following theorem proved in Appendix A:

Theorem 2. *The second-order moments of the SSCM associated with C-CES and NC-CES distributed data are respectively given by*

$$\mathbf{R}_{\widehat{\Sigma}_{\mathrm{SSCM}}} = \sum_{1\le k\ne \ell\le N} \gamma_{k,\ell}(\mathbf{v}_\ell^* \otimes \mathbf{v}_k)(\mathbf{v}_\ell^T \otimes \mathbf{v}_k^H) + \sum_{1\le k,\ell\le N}(\gamma_{k,\ell} - \chi_k\chi_\ell)(\mathbf{v}_k^* \otimes \mathbf{v}_k)(\mathbf{v}_\ell^T \otimes \mathbf{v}_\ell^H) \tag{15}$$

$$\begin{aligned}\mathbf{R}_{\widehat{\widetilde{\Sigma}}_{\mathrm{SSCM}}} = &\sum_{1\le k\ne \ell\le 2N} \widetilde{\gamma}_{k,\ell}(\widetilde{\mathbf{v}}_\ell^* \otimes \widetilde{\mathbf{v}}_k)(\widetilde{\mathbf{v}}_\ell^T \otimes \widetilde{\mathbf{v}}_k^H) + \sum_{1\le k,\ell\le 2N}(\widetilde{\gamma}_{k,\ell} - \widetilde{\chi}_k\widetilde{\chi}_\ell)(\widetilde{\mathbf{v}}_k^* \otimes \widetilde{\mathbf{v}}_k)(\widetilde{\mathbf{v}}_\ell^T \otimes \widetilde{\mathbf{v}}_\ell^H)\\ &+ \sum_{1\le k\ne \ell\le 2N} \widetilde{\gamma}_{k,\ell}(\widetilde{\mathbf{v}}_\ell^* \otimes \widetilde{\mathbf{v}}_k)(\widetilde{\mathbf{v}}_k^T \otimes \widetilde{\mathbf{v}}_\ell^H),\end{aligned} \tag{16}$$

where $\chi_k \stackrel{\mathrm{def}}{=} \chi_{(j)}$ *and* $\widetilde{\chi}_k \stackrel{\mathrm{def}}{=} \widetilde{\chi}_{(j)}$ *for* $k \in s_j$ *are given in* (11) *and* (12), *respectively, and where* $\gamma_{k,\ell} = \lambda_k\lambda_\ell \mathrm{E}\left(\frac{|(\mathbf{u}_t)_k|^2|(\mathbf{u}_t)_\ell|^2}{(\mathbf{u}_t^H\mathbf{\Lambda}\mathbf{u}_t)^2}\right)$, $k = 1, ..., N$, $\ell = 1, ..., N$ *and* $\widetilde{\gamma}_{k,\ell} = \widetilde{\lambda}_k\widetilde{\lambda}_\ell \mathrm{E}\left(\frac{(\bar{\mathbf{u}}_t)_k^2(\bar{\mathbf{u}}_t)_\ell^2}{(\bar{\mathbf{u}}_t^T\widetilde{\mathbf{\Lambda}}\bar{\mathbf{u}}_t)^2}\right)$, $k = 1, ..., 2N$, $\ell = 1, ..., 2N$ *are given by*

$$\gamma_{k,\ell} = 2\lambda_{(j)}^2 \int_0^\infty \frac{x}{(1+\lambda_{(j)}x)^{m_j+2}\prod_{1\le k\ne j\le J}(1+\lambda_{(k)}x)^{m_k}}dx, \quad k \in s_j \text{ and } \ell \in s_j, \tag{17}$$

$$\gamma_{k,\ell} = \lambda_{(i)}\lambda_{(j)} \int_0^\infty \frac{x}{(1+\lambda_{(i)}x)^{m_i+1}(1+\lambda_{(j)}x)^{m_j+1}\prod_{\substack{1\le n\ne i\le J\\ 1\le n\ne j\le J}}(1+\lambda_{(n)}x)^{m_n}}dx, \quad k \in s_i, \ell \in s_j \text{ with } s_i \ne s_j \tag{18}$$

$$\widetilde{\gamma}_{k,\ell} = \frac{3\widetilde{\lambda}_{(j)}^2}{4} \int_0^\infty \frac{x}{(1+\lambda_{(j)}x)^{\frac{m_j}{2}+2}\prod_{1\le n\ne j\le J}(1+\lambda_{(n)}x)^{\frac{m_n}{2}}}dx, \quad k \in s_j \text{ and } \ell \in s_j, \tag{19}$$

[2] $\mathcal{N}(\mathbf{m}; \mathbf{R}, \mathbf{C})$ denotes the complex Gaussian distribution with mean $\mathbf{m}$, covariance $\mathbf{R}$ and complementary covariance $\mathbf{C}$.

$$\widetilde{\gamma}_{k,\ell}=\frac{\widetilde{\lambda}_{(i)}\widetilde{\lambda}_{(j)}}{4}\int_0^\infty \frac{x}{(1+\lambda_{(i)}x)^{\frac{m_i}{2}+1}(1+\lambda_{(j)}x)^{\frac{m_j}{2}+1}\prod_{\substack{1\leq n\neq i\leq J\\ 1\leq n\neq j\leq J}}(1+\lambda_{(n)}x)^{\frac{m_n}{2}}}dx,\ k\in s_i,\ \ell\in s_j \text{ with } s_i\neq s_j. \tag{20}$$

We note, because the SSCM and Tyler's M estimator are distribution-free within CES distributions, that the asymptotic distributions of SSCM given in Theorems 1 and 2 for C (and NC)-CES distributed data and of Tyler's M estimate given in [4] and [27] for RES and C (and NC)-CES respectively, do not require that the second or fourth-order moments of the CES distributed data to be finite. In these cases $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_{\tilde{x}}$ denote respectively the scatter and extended scatter matrices defined up to a multiplicative constant.

Unlike the asymptotic distributions of the SCM and ML estimate which are defined only for C-CES and NC-CES distributed data with finite fourth-order moments. However, the covariance matrices $\mathbf{R}_{\widehat{\Sigma}_{\rm SCM}}$, $\mathbf{R}_{\widehat{\Sigma}_{\rm ML}}$, $\mathbf{R}_{\widehat{\Sigma}_{\rm Ty}}$, $\mathbf{R}_{\widehat{\Sigma}_{\rm SSCM}}$ and $\mathbf{R}_{\widehat{\widetilde{\Sigma}}_{\rm SCM}}$, $\mathbf{R}_{\widehat{\widetilde{\Sigma}}_{\rm ML}}$, $\mathbf{R}_{\widehat{\widetilde{\Sigma}}_{\rm Ty}}$, $\mathbf{R}_{\widehat{\widetilde{\Sigma}}_{\rm SSCM}}$ of the asymptotic distributions of $\widehat{\mathbf{\Sigma}}_{x,\rm SCM}$, $\widehat{\mathbf{\Sigma}}_{x,\rm ML}$, $\widehat{\mathbf{\Sigma}}_{x,\rm Ty}$, $\widehat{\mathbf{\Sigma}}_{x,\rm SSM}$ and of $\widehat{\mathbf{\Sigma}}_{\tilde{x},\rm SCM}$, $\widehat{\mathbf{\Sigma}}_{\tilde{x},\rm ML}$, $\widehat{\mathbf{\Sigma}}_{\tilde{x},\rm Ty}$, $\widehat{\mathbf{\Sigma}}_{\tilde{x},\rm SSCM}$ share respectively the same structure:

$$\sum_{1\leq k\neq\ell\leq N} a_{k,\ell}(\mathbf{v}_\ell^*\otimes\mathbf{v}_k)(\mathbf{v}_\ell^T\otimes\mathbf{v}_k^H)+\sum_{1\leq k,\ell\leq N} b_{k,\ell}(\mathbf{v}_k^*\otimes\mathbf{v}_k)(\mathbf{v}_\ell^T\otimes\mathbf{v}_\ell^H) \tag{21}$$

and

$$\sum_{1\leq k\neq\ell\leq 2N}\widetilde{a}_{k,\ell}(\widetilde{\mathbf{v}}_\ell^*\otimes\widetilde{\mathbf{v}}_k)(\widetilde{\mathbf{v}}_\ell^T\otimes\widetilde{\mathbf{v}}_k^H)+\sum_{1\leq k,\ell\leq 2N}\widetilde{b}_{k,\ell}(\widetilde{\mathbf{v}}_k^*\otimes\widetilde{\mathbf{v}}_k)(\widetilde{\mathbf{v}}_\ell^T\otimes\widetilde{\mathbf{v}}_\ell^H)+\sum_{1\leq k\neq\ell\leq 2N}\widetilde{a}_{k,\ell}(\widetilde{\mathbf{v}}_\ell^*\otimes\widetilde{\mathbf{v}}_k)(\widetilde{\mathbf{v}}_k^T\otimes\widetilde{\mathbf{v}}_\ell^H), \tag{22}$$

with $a_{k,\ell}=a_{\ell,k}$, $b_{k,\ell}=b_{\ell,k}$, $\widetilde{a}_{k,\ell}=\widetilde{a}_{\ell,k}$ and $\widetilde{b}_{k,\ell}=\widetilde{b}_{k,\ell}$.

Moreover in the particular case of C-CES distributed data $\mathbf{\Sigma}_x=\lambda\mathbf{I}$, we have $J=1$ and $m_1=N$ in (11) and (17) which respectively give $\chi_k=\frac{1}{N}$ and $\gamma_{k,\ell}=\frac{2}{N(N+1)}$, $k,\ell=1,...,N$, and consequently the covariance of the asymptotic distributions of the SSCM and Tyler's M-estimate are proportional:

$$\mathbf{R}_{\widehat{\Sigma}_{\rm SSCM}}=\frac{1}{\lambda^2(N+1)^2}\mathbf{R}_{\widehat{\Sigma}_{\rm Ty}}=\frac{1}{N(N+1)}\mathbf{I}-\frac{1}{N^2(N+1)}\mathrm{vec}(\mathbf{I})\mathrm{vec}^T(\mathbf{I}). \tag{23}$$

Similarly, this property of proportionality is preserved for $\widetilde{\mathbf{x}}_t\stackrel{\rm def}{=}(\mathbf{x}_t^T,\mathbf{x}_t^H)^T$ with $\mathbf{x}_t$ is C-CES distributed data and $\mathbf{\Sigma}_{\tilde{x}}=\lambda\mathbf{I}$, and we have

$$\mathbf{R}_{\widehat{\widetilde{\Sigma}}_{\rm SSCM}}=\frac{1}{4\lambda^2(N+1)^2}\mathbf{R}_{\widehat{\widetilde{\Sigma}}_{\rm Ty}}=\frac{1}{4N(N+1)}[\mathbf{I}+\mathbf{K}(\mathbf{J}\otimes\mathbf{J})]-\frac{1}{4N^2(N+1)}\mathrm{vec}(\mathbf{I})\mathrm{vec}^T(\mathbf{I}). \tag{24}$$

4. Asymptotic distribution of subspace projectors

4.1. Asymptotic distribution of SSCM subspace projectors

Now we consider the orthogonal projector $\widehat{\mathbf{\Pi}}_{{\rm SSCM},(j)}=\sum_{k\in s_j}\widehat{\mathbf{v}}_k\widehat{\mathbf{v}}_k^H$ derived from the spectral decomposition of the SSCM $\widehat{\mathbf{\Sigma}}_{x,\rm SSCM}$ whose eigenvalues are $\widehat{\lambda}_1>...>\widehat{\lambda}_k>...>\widehat{\lambda}_N>0$ and associated orthonormal eigenvectors $\widehat{\mathbf{v}}_1,...,\widehat{\mathbf{v}}_k,...,\widehat{\mathbf{v}}_N$. Then, using the standard perturbation result associated with the mapping $\widehat{\mathbf{\Sigma}}_{x,\rm SSCM}=\mathrm{E}(\widehat{\mathbf{\Sigma}}_{x,\rm SSCM})+\delta(\mathbf{\Sigma}_{x,\rm SSCM})=\mathbf{V\Delta V}^H+\delta(\mathbf{\Sigma}_{x,\rm SSSM})\mapsto\widehat{\mathbf{\Pi}}_{{\rm SSCM},(j)}=\mathbf{\Pi}_{(j)}+\delta(\mathbf{\Pi}_{{\rm SSCM},(j)})$ for orthogonal projectors [32] (see also the operator approach in [33]) applied to the eigenprojector $\mathbf{\Pi}_{(j)}$ of $\mathbf{\Sigma}_x$ which is the same of $\mathbf{V\Delta V}^H$:

$$\delta(\mathbf{\Pi}_{{\rm SSCM},(j)})=-\mathbf{\Pi}_{(j)}\delta(\mathbf{\Sigma}_{x,\rm SSCM})\mathbf{S}_j^\#-\mathbf{S}_j^\#\delta(\mathbf{\Sigma}_{x,\rm SSCM})\mathbf{\Pi}_{(j)}+o(\delta(\mathbf{\Sigma}_{x,\rm SSCM})), \tag{25}$$

where $\mathbf{\Sigma}_{(j)}\stackrel{\rm def}{=}\mathbf{V\Delta V}^H-\chi_{(j)}\mathbf{I}_N=\sum_{k\notin s_j}(\chi_k-\chi_{(j)})\mathbf{v}_k\mathbf{v}_k^H$, the asymptotic behaviors of $\widehat{\mathbf{\Pi}}_{{\rm SSCM},(j)}$ and $\widehat{\mathbf{\Sigma}}_{x,\rm SSCM}$ are directly related. The standard theorem of continuity (see e.g., [34, p. 122]) on regular functions of asymptotically Gaussian statistics applies: $\sqrt{T}(\mathrm{vec}(\widehat{\mathbf{\Pi}}_{{\rm SSCM},(j)}-\mathrm{vec}(\mathbf{\Pi}_{(j)})\to_d\mathcal{N}(\mathbf{0};\mathbf{R}_{\widehat{\Pi}_{{\rm SSCM},(j)}},\mathbf{R}_{\widehat{\Pi}_{{\rm SSCM},(j)}}\mathbf{K}_{N^2})$ with

$$\mathbf{R}_{\widehat{\Pi}_{{\rm SSCM},(j)}}=[(\mathbf{\Sigma}_{(j)}^{\#^*}\otimes\mathbf{\Pi}_{(j)})+(\mathbf{\Pi}_{(j)}^*\otimes\mathbf{\Sigma}_{(j)}^\#)]\mathbf{R}_{\widehat{\Sigma}_{\rm SSCM}}[(\mathbf{\Sigma}_{(j)}^{\#^*}\otimes\mathbf{\Pi}_{(j)})+(\mathbf{\Pi}_{(j)}^*\otimes\mathbf{\Sigma}_{(j)}^\#)]. \tag{26}$$

Then plugging (15) into (26) and using the same steps to derive the result associated with the orthogonal projector $\widehat{\widetilde{\mathbf{\Pi}}}_{{\rm SSCM},(j)}=\sum_{k\in s_j}\widehat{\widetilde{\mathbf{v}}}_k\widehat{\widetilde{\mathbf{v}}}_k^H$ derived from the spectral decomposition of the extended SSCM $\widehat{\mathbf{\Sigma}}_{\tilde{x},\rm SSCM}$, the following theorem is proved in Appendix A after simple algebraic manipulations:

Theorem 3. *The covariance of the asymptotic distribution of the eigenprojectors* $\widehat{\mathbf{\Pi}}_{{\rm SSCM},(j)}$ *and* $\widehat{\widetilde{\mathbf{\Pi}}}_{{\rm SSCM},(j)}$ *are, respectively, given by*

$$\mathbf{R}_{\widehat{\Pi}_{{\rm SSCM},(j)}}=(\mathbf{U}_{{\rm SSCM},(j)}^*\otimes\mathbf{\Pi}_{(j)})+(\mathbf{\Pi}_{(j)}^*\otimes\mathbf{U}_{{\rm SSCM},(j)}), \tag{27}$$

$$\mathbf{R}_{\widehat{\widetilde{\Pi}}_{{\rm SSCM},(j)}}=[\mathbf{I}+\mathbf{K}(\mathbf{J}\otimes\mathbf{J})][(\widetilde{\mathbf{U}}_{{\rm SSCM},(j)}^*\otimes\widetilde{\mathbf{\Pi}}_{(j)})+(\widetilde{\mathbf{\Pi}}_{(j)}^*\otimes\widetilde{\mathbf{U}}_{{\rm SSCM},(j)})] \tag{28}$$

with

$$\mathbf{U}_{{\rm SSCM},(j)}\stackrel{\rm def}{=}\sum_{k\notin s_j}\frac{\gamma_{k,(j)}}{(\chi_k-\chi_{(j)})^2}\mathbf{v}_k\mathbf{v}_k^H \text{ and } \widetilde{\mathbf{U}}_{{\rm SSCM},(j)}\stackrel{\rm def}{=}\sum_{k\notin s_j}\frac{\widetilde{\gamma}_{k,(j)}}{(\widetilde{\chi}_k-\widetilde{\chi}_{(j)})^2}\widetilde{\mathbf{v}}_k\widetilde{\mathbf{v}}_k^H. \tag{29}$$

We note first that this theorem extends [20, rels (3.12), (3.13)] dedicated to RES distributed data. On the other hand, these expressions have a similar structure than the ones derived for the noise projector associated with the SCM and extended SCM, respectively [30]. Furthermore, we note that the covariance matrices of the Gaussian asymptotic distribution of subspace projectors built from the SCM, ML and Tyler's M-estimate are similarly structured. They are respectively given for C-CES and NC-CES data by [27]:

$$\mathbf{R}_{\widehat{\Pi}_{(j)}} = (\mathbf{U}^*_{(j)} \otimes \mathbf{\Pi}_{(j)}) + (\mathbf{\Pi}^*_{(j)} \otimes \mathbf{U}_{(j)}), \tag{30}$$

$$\mathbf{R}_{\widehat{\widetilde{\Pi}}_{(j)}} = [\mathbf{I} + \mathbf{K}(\mathbf{J} \otimes \mathbf{J})][(\widetilde{\mathbf{U}}^*_{(j)} \otimes \widetilde{\mathbf{\Pi}}_{(j)}) + (\widetilde{\mathbf{\Pi}}^*_{(j)} \otimes \widetilde{\mathbf{U}}_{(j)})], \tag{31}$$

where

$$\mathbf{U}_{(j)} = \vartheta \sum_{k \notin s_j} \frac{\lambda_k \lambda_{(j)}}{(\lambda_k - \lambda_{(j)})^2} \mathbf{v}_k \mathbf{v}_k^H \ \text{ and } \ \widetilde{\mathbf{U}}_{(j)} = \vartheta \sum_{k \notin s_j} \frac{\widetilde{\lambda}_k \widetilde{\lambda}_{(j)}}{(\widetilde{\lambda}_k - \widetilde{\lambda}_{(j)})^2} \widetilde{\mathbf{v}}_k \widetilde{\mathbf{v}}_k^H, \tag{32}$$

with

$$\vartheta = \frac{\mathrm{E}(\mathcal{Q}_t^2)}{N(N+1)} \ \text{ for SCM estimate,} \tag{33}$$

$$= \frac{\mathrm{E}[\phi^2(\mathcal{Q}_t)\mathcal{Q}_t^2]}{N(N+1)(1+[N(N+1)]^{-1}\mathrm{E}[\phi'(\mathcal{Q}_t)\mathcal{Q}_t^2])^2} = \frac{N(N+1)}{\mathrm{E}[\phi^2(\mathcal{Q}_t)\mathcal{Q}_t^2]} \ \text{for ML estimate,} \tag{34}$$

$$= \frac{N+1}{N} \ \text{ for Tyler's } M \text{ estimate,} \tag{35}$$

with $\phi'(u) \stackrel{\text{def}}{=} \frac{d\phi(u)}{du}$. We note that $\vartheta = 1$ for Gaussian distributed data for which the ML estimate is reduced to the SCM estimate.

4.2. Asymptotic inadmissibility of subspace projectors

For circular complex angular central Gaussian distributed data, Tyler's M estimate is the ML of $\mathbf{\Sigma}_x$ and thus by the invariance property of the ML, its associated orthogonal projectors $\widehat{\mathbf{\Pi}}_{\mathrm{Ty},(j)}$ is the ML estimate of $\mathbf{\Pi}_{(j)}$. Thanks to the free distribution property of the asymptotic distribution of Tyler's M estimate in the C-CES family, where is added the circular complex angular central Gaussian distribution [4], the covariance matrix $\mathbf{R}_{\widehat{\Pi}_{\mathrm{Ty},(j)}}$ is less than or equal to the covariance of the asymptotic distribution of any other asymptotically unbiased estimator of $\mathbf{\Pi}_{(j)}$ for arbitrary C-CES distributed data. We get in particular for the SSCM estimate:

$$\mathbf{R}_{\widehat{\Pi}_{\mathrm{Ty},(j)}} \le \mathbf{R}_{\widehat{\Pi}_{\mathrm{SSCM},(j)}}. \tag{36}$$

In the case of non-circular distributed data, it is straightforward to prove using the associated r.v. $\tilde{\mathbf{x}}_t$ that non-circular Tyler's M estimate (6) is the ML of $\mathbf{\Sigma}_{\tilde{x}}$ for the distribution of p.d.f. $p(\mathbf{x}_t) = s_N^{-1}|\mathbf{\Sigma}_{\tilde{x}}^{-1/2}|(\widetilde{\mathbf{x}}_t\mathbf{\Sigma}_{\tilde{x}}^{-1}\widetilde{\mathbf{x}}_t)^{-N}$ with respect to the $U(\mathbb{C}S^N)$ distribution. Now by following similar arguments to those in the proof of (36), we also obtain:

$$\mathbf{R}_{\widehat{\widetilde{\Pi}}_{\mathrm{Ty},(j)}} \le \mathbf{R}_{\widehat{\widetilde{\Pi}}_{\mathrm{SSCM},(j)}}. \tag{37}$$

Furthermore from (23) and (24), we deduce that when $\mathbf{\Sigma}_x \to \lambda\mathbf{I}$ and $\mathbf{\Sigma}_{\tilde{x}} \to \lambda\mathbf{I}$, the inequalities (36) and (37) approach equalities, respectively. These inequalities show that the estimator $\widehat{\mathbf{\Pi}}_{\mathrm{Ty},(j)}$ asymptotically dominates the estimator $\widehat{\mathbf{\Pi}}_{\mathrm{SSCM},(j)}$ for arbitrary parameter $\mathbf{\Pi}_{(j)}$ in the sense of the mean squared error. This property of asymptotic inadmissibility of the projector associated with the SSCM proved firstly for RES distributed data in [19] and [20], is thus extended to the arbitrary C-CES and NC-CES distributed data.

5. Application to factor models

5.1. Asymptotic distribution of the noise subspace

We consider now the case of low-rank plus identity scatter matrices $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_{\tilde{x}}$ that are commonly used in signal processing to account for low dimensional signal of interest embedded in spatial white noise.

$$\mathbf{\Sigma}_x = \mathbf{\Sigma}_s + \lambda\mathbf{I} \ \text{ and } \ \mathbf{\Sigma}_{\tilde{x}} = \mathbf{\Sigma}_{\tilde{s}} + \lambda\mathbf{I}, \tag{38}$$

where $\mathbf{\Sigma}_s$ and $\mathbf{\Sigma}_{\tilde{s}}$ have rank P with $P < N$ and $P < 2N$, respectively. In this case, Theorem 3 applies where the subspace (j) is the so-called noise subspace and the covariances of the asymptotic distribution of the associated subspace are denoted by $\mathbf{R}_{\widehat{\Pi}_{\mathrm{SSCM}}}$ and $\mathbf{R}_{\widehat{\widetilde{\Pi}}_{\mathrm{SSCM}}}$.

In the particular case where $\lambda_1 > \ldots > \lambda_P > \underbrace{\lambda_{P+1} = \ldots = \lambda_N}_{\lambda} > 0$, it is proved in Appendix A that the one-dimensional integrals (11), (17) and (18) can be reduced to closed-form expressions without integrals except for Gauss hypergeometric functions that extend the complicated expressions given in [22, Th. 1] for distinct eigenvalues (i.e., for $P + 1 = N$). These expressions are reported in Appendix A for the ease of the readers.

While for NC-CES distributed data, the one-dimensional integrals (12), (19) and (20) cannot be reduced to closed-form expressions for any eigenvalues except the case $\widetilde{\lambda}_1 = \ldots = \widetilde{\lambda}_P$ for which identity (66) proved in Appendix A allows us to derive the following expressions from simple algebraic manipulations:

$$\widetilde{\chi}_k = \frac{1}{2N}\, {}_2F_1\left(1, N-\frac{P}{2}, N+1, 1-\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}\right), \ k=1,...,P, \tag{39}$$

$$\widetilde{\chi} \stackrel{\text{def}}{=} \widetilde{\chi}_k = \frac{(\widetilde{\lambda}/\widetilde{\lambda}_1)}{2N}\, {}_2F_1\left(1, N-\frac{P}{2}+1, N+1, 1-\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}\right), \ k=P+1,...,2N, \tag{40}$$

$$\widetilde{\gamma}_{k,k} = \frac{3}{4N(N+1)}\, {}_2F_1\left(2, N-\frac{P}{2}, N+2, 1-\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}\right), \ k=1,...,P, \tag{41}$$

$$\widetilde{\gamma} \stackrel{\text{def}}{=} \widetilde{\gamma}_{k,k} = \frac{3(\widetilde{\lambda}/\widetilde{\lambda}_1)^2}{4N(N+1)}\, {}_2F_1\left(2, N-\frac{P}{2}+2, N+2, 1-\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}\right), \ k=P+1,...,2N, \tag{42}$$

$$\widetilde{\gamma}_k \stackrel{\text{def}}{=} \widetilde{\gamma}_{k,\ell} = \frac{(\widetilde{\lambda}/\widetilde{\lambda}_1)}{4N(N+1)}\, {}_2F_1\left(2, N-\frac{P}{2}+1, N+2, 1-\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}\right), \ k=1,...,P, \ \ell=P+1,...,2N. \tag{43}$$

We note that (39), (40) and (43) are consistent with the expressions [20, (3.16), (3.17)] given for N-dimensional RES distributed data.

5.2. Performance of subspace-based algorithms

We consider here that the scatter matrices $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_{\tilde{x}}$ in (38) are structured as follows:

$$\mathbf{\Sigma}_x = \mathbf{A}(\boldsymbol{\theta})\mathbf{R}_s\mathbf{A}^H(\boldsymbol{\theta}) + \lambda\mathbf{I} \ \text{and} \ \mathbf{\Sigma}_{\tilde{x}} = \widetilde{\mathbf{A}}(\boldsymbol{\theta})\mathbf{R}_{\tilde{s}}\widetilde{\mathbf{A}}^H(\boldsymbol{\theta}) + \lambda\mathbf{I}, \tag{44}$$

where the real-valued parameter of interest $\boldsymbol{\theta}$ is characterized by the subspace generated by the columns of the full column rank matrices $\mathbf{A}(\boldsymbol{\theta})$ and $\widetilde{\mathbf{A}}(\boldsymbol{\theta})$, and where $\mathbf{R}_s$ and $\mathbf{R}_{\tilde{s}}$ are $P \times P$ positive definite Hermitian and real-valued symmetric matrices, respectively. This is in particular the case of the general noisy linear mixture model:

$$\mathbf{x}_t = \mathbf{A}(\boldsymbol{\theta})\mathbf{s}_t + \mathbf{n}_t. \tag{45}$$

This parametric model finds wide applications in various areas such as communication systems and array processing as explained in [27, Section II]. We note that $\mathbf{s}_t$ and $\mathbf{n}_t$ cannot be both CES distributed as the family of CES distributions is not closed under summation except for the Gaussian distribution. But fixing both the structures (44) and the CES distribution of $\mathbf{x}_t$, (2) and (3) can be considered as good approximations thanks to the flexibility of the family of the elliptical symmetric distributions.

In these conditions any subspace-based algorithms can be considered as the following mapping:

$$(\mathbf{x}_1, .., \mathbf{x}_t, .., \mathbf{x}_T) \longmapsto \widehat{\mathbf{\Sigma}} \longmapsto \widehat{\mathbf{\Pi}} \stackrel{\text{alg}}{\longmapsto} \widehat{\boldsymbol{\theta}}, \tag{46}$$

where $\widehat{\mathbf{\Sigma}}$ can be the SSCM, SCM, ML and Tyler's M estimate of $\mathbf{\Sigma}_x$ for C-CES distributed data, or of $\mathbf{\Sigma}_{\tilde{x}}$ for NC-CES distributed data, and $\widehat{\mathbf{\Pi}}$ denotes the orthogonal projection matrix associated with the so-called noise subspace of $\widehat{\mathbf{\Sigma}}$. The functional dependence $\widehat{\boldsymbol{\theta}} = \text{alg}(\widehat{\mathbf{\Pi}})$ constitutes an extension of the mapping

$$\mathbf{\Pi}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \mathbf{I} - \mathbf{B}(\boldsymbol{\theta})[\mathbf{B}^H(\boldsymbol{\theta})\mathbf{B}(\boldsymbol{\theta})]^{-1}\mathbf{B}^H(\boldsymbol{\theta}) \stackrel{\text{alg}}{\longmapsto} \boldsymbol{\theta}, \tag{47}$$

in the neighborhood of $\mathbf{\Pi}(\boldsymbol{\theta})$ with $\mathbf{B}(\boldsymbol{\theta})$ is either $\mathbf{A}(\boldsymbol{\theta})$ or $\widetilde{\mathbf{A}}(\boldsymbol{\theta})$. Each extension alg(.) specifies a particular subspace-based algorithm which is assumed asymptotically unbiased and differentiable w.r.t. $(\text{Re}(\mathbf{\Pi}(\boldsymbol{\theta}), \text{Im}(\mathbf{\Pi}(\boldsymbol{\theta}))$, whose MUSIC algorithm is an example. Among these algorithms, the asymptotically minimum variance (AMV) algorithm (introduced in [35] and [36]), which minimizes the covariance matrix of the asymptotic distribution of the estimate $\widehat{\boldsymbol{\theta}}$ plays a role of benchmark. The covariance of the Gaussian asymptotic distribution of the estimate $\widehat{\boldsymbol{\theta}}$ given by these subspace-based algorithms is given by [27]:

$$\mathbf{R}_{\hat{\theta}} = \mathbf{D}_{\text{alg}}\mathbf{R}_{\widehat{\Pi}}\mathbf{D}_{\text{alg}}^H, \tag{48}$$

where $\mathbf{D}_{\text{alg}}$ is the differential matrix[3] of the algorithm and $\mathbf{R}_{\widehat{\Pi}}$ is the covariance of the asymptotic distribution of the different estimates of the projector matrices on the noise subspace built on the SSCM, SCM, ML and Tyler's M estimates (27)-(31). Consequently, from (32)-(35), the covariance matrices $\mathbf{R}_{\hat{\theta}}$ associated with the SCM, ML and Tyler's M estimates for arbitrary elliptical distributed data are equal up to a multiplicative factor ϑ, to the covariance matrix $\mathbf{R}_{\hat{\theta}}$ associated with the SCM and ML estimates for Gaussian distributed data. Besides, the covariance of the asymptotic distribution of the AMV algorithms takes the particular expression [27]:

$$\mathbf{R}_{\hat{\theta}} = (\mathbf{\Pi}'^H(\boldsymbol{\theta})\mathbf{R}_{\widehat{\Pi}}^{\#}\mathbf{\Pi}'(\boldsymbol{\theta}))^{-1}, \tag{49}$$

where $\mathbf{\Pi}'(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \frac{d\text{vec}(\mathbf{\Pi}(\boldsymbol{\theta}))}{d\boldsymbol{\theta}}$. It has been proved in [27] that the AMV estimates $\widehat{\boldsymbol{\theta}}$ derived from the estimates $\widehat{\mathbf{\Pi}}_{\text{ML}}$ and $\widehat{\widetilde{\mathbf{\Pi}}}_{\text{ML}}$ built on the ML estimate of $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_{\tilde{x}}$, respectively, are asymptotically efficient, i.e., their covariance matrices $\mathbf{R}_{\hat{\theta},\text{ML}}$ reach the Cramér-Rao bound (CRB) of the parameter $\boldsymbol{\theta}$. Consequently the following theorem is deduced from (36), (37), (48) and (49):

[3] This differential matrix $\mathbf{D}$ is defined by the relation $\widehat{\boldsymbol{\theta}} = \text{alg}(\widehat{\mathbf{\Pi}}) = \underbrace{\text{alg}(\mathbf{\Pi}(\boldsymbol{\theta}))}_{\theta} + \mathbf{D}\text{vec}(\widehat{\mathbf{\Pi}} - \mathbf{\Pi}(\boldsymbol{\theta})) + o(\widehat{\mathbf{\Pi}} - \mathbf{\Pi}(\boldsymbol{\theta}))$.

Theorem 4. *The covariance of the Gaussian asymptotic distribution of the estimated parameter $\widehat{\boldsymbol{\theta}}$ derived for any subspace-based algorithm built on the SSCM is bounded below by those built on Tyler's M estimate for C-CES and NC-CES data. These two covariance matrices being themselves bounded below by the CRB.*

$$T \times \text{CRB}(\boldsymbol{\theta}) = (\boldsymbol{\Pi}'^H(\boldsymbol{\theta})\mathbf{R}^{\#}_{\widehat{\Pi},\text{ML}}\boldsymbol{\Pi}'(\boldsymbol{\theta}))^{-1} = \mathbf{R}_{\hat{\theta},\text{ML}} \leq \mathbf{R}_{\hat{\theta},\text{Ty}} \leq \mathbf{R}_{\hat{\theta},\text{SSCM}}. \tag{50}$$

Note that similarly to Tyler's M estimate, $\mathbf{R}_{\hat{\theta},\text{SSCM}}$ is distribution-free and that for any subspace-based algorithm there is no general order relation between $\mathbf{R}_{\hat{\theta},\text{SSCM}}$ and $\mathbf{R}_{\hat{\theta},\text{SCM}}$. However, since the SCM is very sensitive to heavy-tailed CES distributions, $\mathbf{R}_{\hat{\theta},\text{SSCM}}$ can be bounded above by $\mathbf{R}_{\hat{\theta},\text{SCM}}$ for such distributions. This point will be illustrated in Subsection 5.4.

Moreover, since the covariances of asymptotic distributions of the projectors given by (27) and (28) are structured in a similar way to those associated with the SCM, ML and Tyler's M estimate (30) and (31), it follows that all the analytical results concerning the asymptotic distributions of subspace-based algorithms resulting from the SCM, ML and Tyler's M estimates immediately extend to subspace-based algorithms resulting from the SSCM, especially in DOA estimation.

Furthermore, to gain insight into the effect of P, N and eigenvalue spectra on the degradation of the SSCM-based estimates compared to the Tyler's M-based estimates, we consider in the next subsection the special case in which $\lambda_1 = \ldots = \lambda_P$.

5.3. Asymptotic efficiency of the SSCM relative to Tyler's M estimate

In this special case which includes the case $P=1$, the proportionality of $\mathbf{U}_{\text{SSCM},(j)}$ and $\mathbf{U}_{(j)}$, and of $\widetilde{\mathbf{U}}_{\text{SSCM},(j)}$ and $\widetilde{\mathbf{U}}_{(j)}$ given by (29) and (32) implies that the covariance matrices of the asymptotic distribution of any subspace-based algorithms derived from the SSCM, SCM, ML and Tyler's M estimates are proportional for both C-CES and NC-CES distributed data.

$$\mathbf{R}_{\hat{\theta},\text{SSCM}} = \frac{\gamma_1}{(\chi_1-\chi)^2}\frac{(\lambda_1-\lambda)^2}{\lambda_1\lambda}\frac{1}{\vartheta}\mathbf{R}_{\hat{\theta}} \text{ and } \mathbf{R}_{\hat{\theta},\text{SSCM}} = \frac{\widetilde{\gamma}_1}{(\widetilde{\chi}_1-\widetilde{\chi})^2}\frac{(\widetilde{\lambda}_1-\widetilde{\lambda})^2}{\widetilde{\lambda}_1\widetilde{\lambda}}\frac{1}{\vartheta}\mathbf{R}_{\hat{\theta}}, \tag{51}$$

where ϑ is given by (33)-(35) for SCM, ML and Tyler's M estimates and $\mathbf{R}_{\hat{\theta}}$ denotes the associated covariance matrix of the estimated parameter. We see from (51) that for very heavy tailed CES data for which ϑ (33) is not upper-bounded for the SCM, the covariance matrix $\mathbf{R}_{\hat{\theta},\text{SSCM}}$ can be upper-bounded by the covariance matrix $\mathbf{R}_{\hat{\theta},\text{SCM}}$. This point is specified in Subsection 5.4 with $P=1$.

From proportionality (51), we can define the asymptotic efficiency of the SSCM relative to Tyler's M estimate by the ratios $r_c \stackrel{\text{def}}{=} \frac{(\chi_1-\chi)^2}{\gamma_1}\frac{\lambda_1\lambda}{(\lambda_1-\lambda)^2}\frac{N+1}{N} \leq 1$ and $r_{nc} \stackrel{\text{def}}{=} \frac{(\widetilde{\chi}_1-\widetilde{\chi})^2}{\widetilde{\gamma}_1}\frac{\widetilde{\lambda}_1\widetilde{\lambda}}{(\widetilde{\lambda}_1-\widetilde{\lambda})^2}\frac{N+1}{N} \leq 1$ for C-CES and NC-CES distributed data, respectively, for which the following theorem is proved in Appendix A.

Theorem 5. *Under the assumption $\lambda_1 = \ldots = \lambda_P$ and $\widetilde{\lambda}_1 = \ldots = \widetilde{\lambda}_P$, the ratios r_c and r_{nc} are given respectively by:*

$$r_c = \frac{[{}_2F_1(1, N-P+1, N+2, 1-\frac{\lambda}{\lambda_1})]^2}{{}_2F_1(2, N-P+1, N+2, 1-\frac{\lambda}{\lambda_1})} \text{ and } r_{nc} = \frac{[{}_2F_1(1, N-\frac{P}{2}+1, N+2, 1-\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1})]^2}{{}_2F_1(2, N-\frac{P}{2}+1, N+2, 1-\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1})}. \tag{52}$$

These ratios are monotonic increasing functions of respectively $\frac{\lambda}{\lambda_1}$ and $\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}$ from the intervals (0.1) to (0,1).

In the neighborhood of $\frac{\lambda}{\lambda_1}=1$, $\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}=1$ and $\frac{\lambda}{\lambda_1}=0$, $\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}=0$, we have respectively:

$$r_c = 1 - \frac{(N-P+1)(P+1)}{(N+2)^2(N+3)}\left(1-\frac{\lambda}{\lambda_1}\right)^2 + o\left(1-\frac{\lambda}{\lambda_1}\right)^2, \tag{53}$$

$$r_{nc} = 1 - \left(\frac{(2N-P+2)(P+2)}{4(N+2)^2(N+3)}\right)\left(1-\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}\right)^2 + o\left(1-\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}\right)^2, \tag{54}$$

and

$$r_c = \begin{cases} o_{N,1}(1) & \text{for } P=1 \\ (1+\frac{1}{N})(1-\frac{1}{P})(1+o_{N,P}(1)) & \text{for } P>1 \end{cases} \tag{55}$$

$$r_{nc} = \begin{cases} \widetilde{o}_{N,P}(1) & \text{for } P=1,2 \\ (1+\frac{1}{N})(1-\frac{2}{P})(1+\widetilde{o}_{N,P}(1)) & \text{for } P>2 \end{cases}, \tag{56}$$

where $\lim_{\lambda/\lambda_1\to 0} o_{N,P}(1) = \lim_{\widetilde{\lambda}/\widetilde{\lambda}_1\to 0}\widetilde{o}_{N,P}(1) = 0$ with

$$o_{N,P}(1) = \begin{cases} -\frac{N+1}{N}\frac{1}{\log\frac{\lambda}{\lambda_1}} + o\left(\frac{1}{\log\frac{\lambda}{\lambda_1}}\right) & \text{for } P=1 \\ -2(N-1)\frac{\lambda}{\lambda_1}\log\frac{\lambda}{\lambda_1} + o\left(\frac{\lambda}{\lambda_1}\log\frac{\lambda}{\lambda_1}\right) & \text{for } P=2 \\ \frac{2(N-P+1)}{(P+1)(P+2)}\left(\frac{\lambda}{\lambda_1}\right) + o\left(\frac{\lambda}{\lambda_1}\right) & \text{for } P>2 \end{cases} \tag{57}$$

$$\widetilde{o}_{N,P}(1) = \begin{cases} \frac{4\Gamma(N+\frac{1}{2})}{\sqrt{\pi}\Gamma(N)}\left(\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}\right)^{1/2} + o\left(\left(\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}\right)^{1/2}\right) & \text{for } P=1 \\ -\frac{N+1}{N}\frac{1}{\log\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}} + o\left(\frac{1}{\log\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}}\right) & \text{for } P=2 \\ \frac{3\sqrt{\pi}\Gamma(N)}{2\Gamma(N-\frac{1}{2})}\left(\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}\right)^{1/2} + o\left(\left(\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}\right)^{1/2}\right) & \text{for } P=3 \\ -2(N-1)\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}\log\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1} + o\left(\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}\log\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}\right) & \text{for } P=4 \\ \frac{4(2N^2+4N-2NP-P+2)}{(N+1)(P-2)(P-4)}\left(\frac{\lambda}{\lambda_1}\right) + o\left(\frac{\lambda}{\lambda_1}\right) & \text{for } P>4 \end{cases}. \quad (58)$$

It follows from (53) and (54) that the performance of the subspace-based algorithms derived from SSCM and Tyler's M estimate are very similar for close eigenvalues, and particularly for large values of N and P. This property is consistent with (23) and (24). It follows, conversely, from (55) and (56), that for well-separated eigenvalues, the performance of the SSCM-based subspace algorithms are largely outperformed by those derived from Tyler's M estimate for $P=1$ and $P=1,2$ for C-CES and NC-CES distributed data because r_c and r_{nc} tend to zero when λ/λ_1 and $\widetilde{\lambda}/\widetilde{\lambda}_1$ tend to zero, respectively. But comparing $\widetilde{o}_{N,2}(1)$ to $\widetilde{o}_{N,1}(1)$, r_{nc} tends to zero less rapidly for $P=2$ than for $P=1$. Note however that for N and P large, the performance of the subspace-based algorithms derived from SSCM and Tyler's M estimate are very similar because r_c and r_{nc} tend to $(1+\frac{1}{N})(1-\frac{1}{P})<1$ and $(1+\frac{1}{N})(1-\frac{2}{P})<1$ when λ/λ_1 and $\widetilde{\lambda}/\widetilde{\lambda}_1$ tend to zero, respectively.

These points are highlighted in Fig. 1 that represents the ratios r_c and r_{nc} as functions respectively of λ/λ_1 and $\widetilde{\lambda}/\widetilde{\lambda}_1$, for $P=1,3,5$ with different values of N. This figure confirms the analysis of the behavior of r_c and r_{nc} from the analytical results (53), (54) and (55), (56) proved in the neighborhood of one and zero, in all the domain (0,1) of the ratio of eigenvalues.

Consequently, despite the asymptotic performance of all subspace-based algorithms built from Tyler's M estimate outperforming those of the SSCM-based algorithms, these performances are close in particular for large values of N and not too small values of P, and, therefore, conclude that SSCM estimate is of great interest from the point of view of its lower computational complexity for large values of N.

5.4. Subspace-based DOA estimation

We illustrate here the relative inefficiency of the SSCM relative to Tyler's M estimate in the worst case of $P=1$. Let us consider a narrowband signal source s_t with power σ_s^2 which impinges on a uniform linear array of N sensors separated by a half-wavelength for which the steering vectors are given by $\mathbf{a}(\theta)=(1,e^{j\theta},\ldots,e^{j(M-1)\theta})^T$ where $\theta=\pi\sin(\omega)$, with ω is the DOAs relative to the normal of array broadside, with a spatially circular white noise $\mathbf{n}_t$ with power σ_n^2. The array output $\mathbf{x}_t=\mathbf{a}(\theta)s_t+\mathbf{n}_t$ is assumed to be either circular or non-circular complex Student's t-distributed with parameter $\nu>0$ associated with a circular or rectilinear ($s_t=e^{i\phi}r_t$ with r_t is real-valued and ϕ is unknown and fixed) source. This distribution has finite 2nd-order moments if $\nu>2$ and finite fourth order moments if $\nu>4$ in which case $\vartheta=\frac{\nu-2}{\nu-4}$ and $\vartheta=\frac{N+\nu/2+1}{N+\nu/2}$ for the SCM and ML estimate, respectively [37]. The complex Student's t-distribution has heavier tails than the Gaussian one. The limiting case $\nu\to\infty$ yields the Gaussian distribution. We also remind the reader that $\vartheta=\frac{N+1}{N}$ for Tyler's M estimate (see (35)).

In this model, the scatter matrices $\mathbf{\Sigma}_x$ and $\mathbf{\Sigma}_{\tilde{x}}$ (44) are given by

$$\mathbf{\Sigma}_x=\sigma_s^2\mathbf{a}(\theta)\mathbf{a}^H(\theta)+\sigma_n^2\mathbf{I} \text{ and } \mathbf{\Sigma}_{\tilde{x}}=\sigma_s^2\widetilde{\mathbf{a}}(\theta)\widetilde{\mathbf{a}}^H(\theta)+\sigma_n^2\mathbf{I} \quad (59)$$

where $\widetilde{\mathbf{a}}(\theta)\stackrel{\text{def}}{=}(e^{i\phi}\mathbf{a}^T(\theta),e^{-i\phi}\mathbf{a}^H(\theta))^T$ and Eqs. (76) and (84) [resp., (39) and (41)] are applied here with $\lambda_1=N\sigma_s^2+\sigma_n^2$ [resp., $\widetilde{\lambda}_1=2N\sigma_s^2+\sigma_n^2$] and $\lambda=\sigma_n^2$. In this example, it is well known that the conventional MUSIC [38] and NC MUSIC [30] algorithms are efficient for Gaussian distributed data. This property has been extended to C-CES and NC-CES distributed data in [27], [39]. Consequently the variance $r_{\hat{\theta}}$ of the asymptotic distribution of the estimated DOA by MUSIC and NC MUSIC algorithms are given respectively by

$$r_{\hat{\theta}}=T\times\text{CRB}(\theta)=\frac{\vartheta}{\alpha}\frac{\sigma_n^2}{\sigma_s^2}\left(1+\frac{\sigma_n^2}{N\sigma_s^2}\right) \text{ and } r_{\hat{\theta}}=T\times\text{CRB}(\theta)=\frac{\vartheta}{\alpha}\frac{\sigma_n^2}{\sigma_s^2}\left(1+\frac{\sigma_n^2}{2N\sigma_s^2}\right), \quad (60)$$

where $\alpha\stackrel{\text{def}}{=}2\mathbf{a}'^H(\theta)\mathbf{\Pi}\mathbf{a}'(\theta)$ and ϑ is given by (34).

Figs. 2 and 3 compare the theoretical asymptotic variance $\frac{r_{\hat{\theta}}}{T}$ and MSEs of conventional and NC MUSIC algorithms based on SCM, SSCM and Tyler's M estimate versus SNR for complex Student's t-distributed data of different values of the parameter ν. Note first that for $N=6$ and $\nu=2$, we get from the above expressions of ϑ that $\vartheta=\frac{N+1}{N}=7/6$ and $\vartheta=\frac{N+\nu/2+1}{N+\nu/2}=8/7$ for Tyler's M and ML estimators, respectively. So the asymptotic variance of Tyler's M estimator and the CRB are too close to be distinguishable in Figs. 2 and 3, which are equal up to a multiplicative factor ϑ to the asymptotic variance of the SCM estimate associated with Gaussian distributed data. These figures also show that the theoretical asymptotic variances given by the MUSIC algorithms based on SSCM and Tyler's M estimates from (48), are very close to each other and to their MSE for a weak SNR and for $\nu>4$ in the worst-case scenario of $P=1$. On the other hand, for $2<\nu\le 4$, for which the fourth-order moments of the data do not exist, and hence the asymptotic distribution of the MUSIC estimates based on the SCM is not available, the associated MSE increases strongly when ν approaches 2, for which the data are no longer of the second-order.

6. Conclusion

We have presented in this paper an asymptotic performance analysis of the SSCM by giving analytical closed-form expressions of the expectation and covariance of the SSCM by analytically solving one-dimensional integrals for arbitrary eigenvalue spectra of the associated

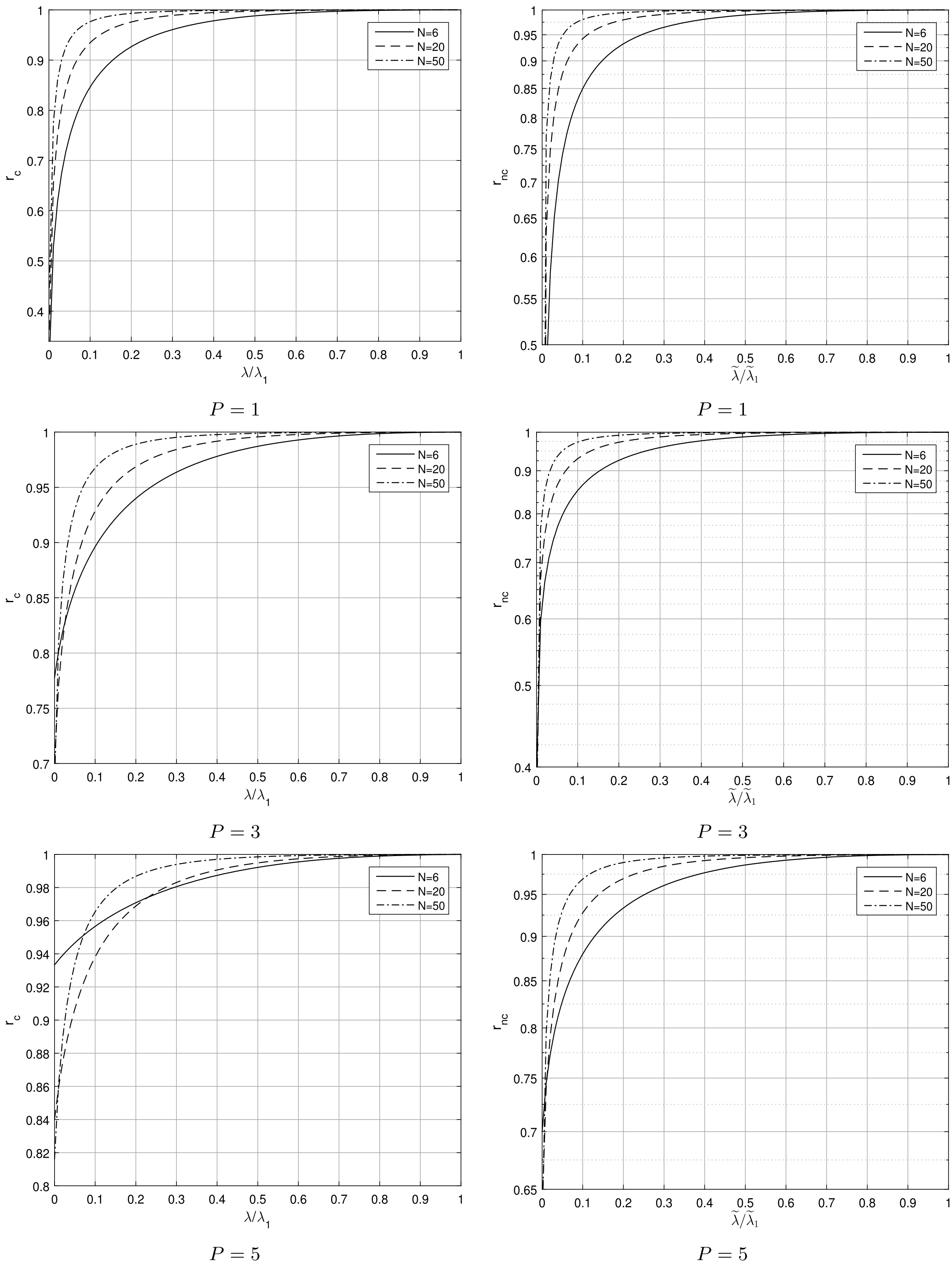
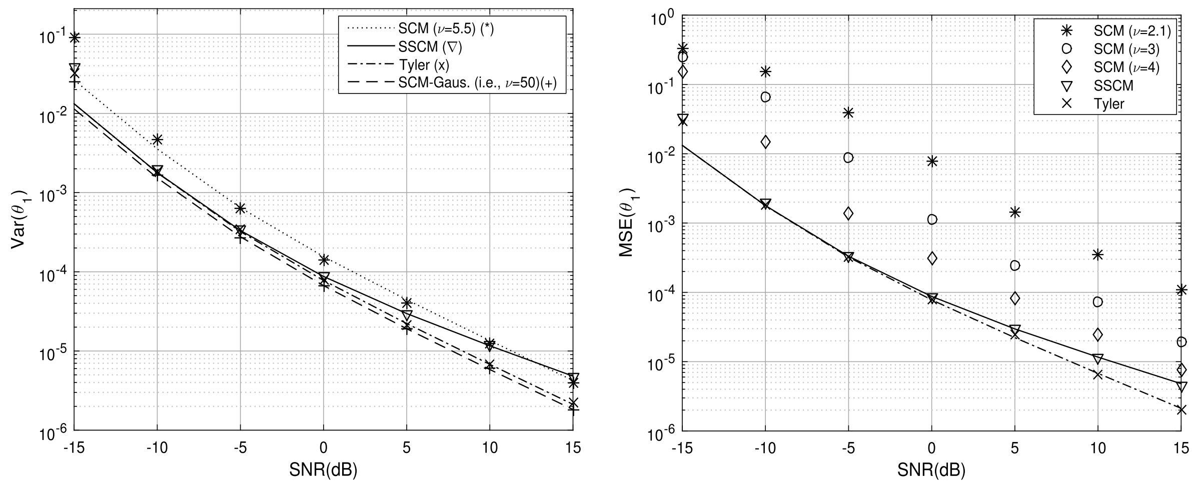


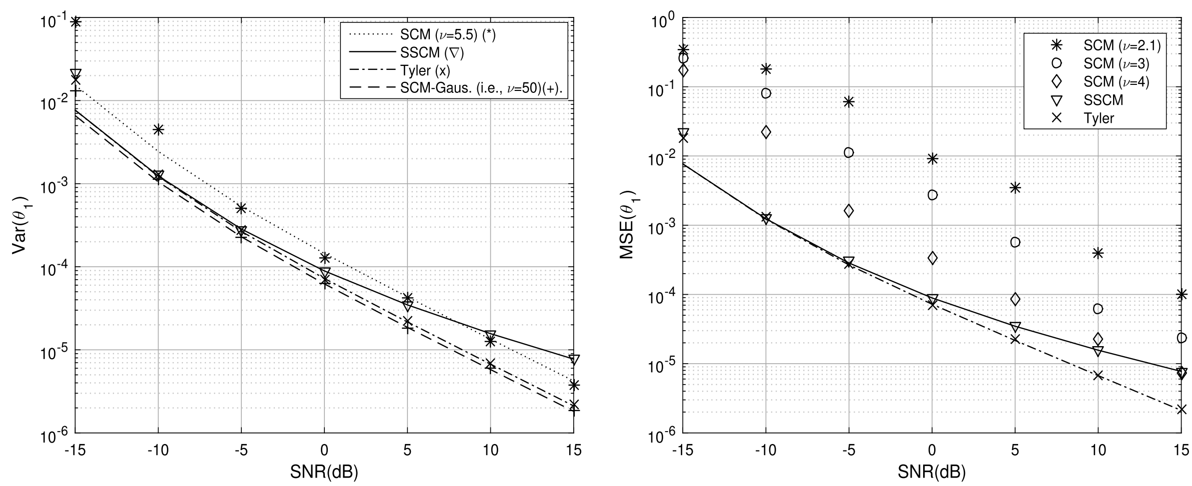
Fig. 1. Ratios r_c and r_{nc} versus respectively λ/λ_1 and $\widetilde{\lambda}/\widetilde{\lambda}_1$ for different values of N and $P = 1, 3, 5$.

SCM for C-CES and NC-CES distributed data. We then conducted an asymptotic performance analysis of the associated projectors and of subspace-based algorithms associated with the SSCM. Finally, this asymptotic performance based on SSCM has been compared to that based on Tyler's M estimate, where an analytical analysis of the efficiency of the SSCM relative to Tyler's M estimate is studied. These results lead us to conclude that the performances of the SSCM and Tyler's M estimate are close for a high-dimensional data and not too small dimension of the principal component space and, therefore, to deduce that the SSCM estimate is of great interest from the point of view of their lower computational complexity for high-dimensional data. Finally, this result opens up the interest of a future analysis of



(a) case $\nu > 4$ (b) case $2 < \nu \leq 4$

Fig. 2. Theoretical asymptotic variance $\frac{r_{\hat{\theta}}}{T}$ and MSEs (with 2000 Monte Carlo runs) of conventional MUSIC algorithm based on SCM, SSCM and Tyler's M estimate versus SNR for circular complex Student's t-distributed data (with $T = 500$) for either $\nu > 4$ or $2 < \nu \leq 4$ and $N = 6$.



(a) case $\nu > 4$ (b) case $2 < \nu \leq 4$

Fig. 3. Theoretical asymptotic variance $\frac{r_{\hat{\theta}}}{T}$ and MSEs (with 2000 Monte Carlo runs) of NC MUSIC algorithm based on SCM, SSCM and Tyler's M estimate versus SNR for NC complex Student's t-distributed data (with $T = 500$) for either $\nu > 4$ or $2 < \nu \leq 4$ with $\phi = \pi/3$ and $N = 6$.

the asymptotic performance of the SSCM in the regime where both the observation dimension N and the number of samples T converge to infinity in such a way that the ratio N/T converges to a positive constant.

CRediT authorship contribution statement

All authors certify that they have participated sufficiently in the work to take public responsibility for the content, including participation in the concept, design, analysis, writing, or revision of the manuscript. Furthermore, each author certifies that this material or similar material has not been and will not be submitted to or published in any other publication.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A

Proof of Theorem 1. Let us first consider the case of C-CES distributed data. Taking the square root $\mathbf{\Sigma}_x^{1/2}=\mathbf{V}\mathbf{\Lambda}^{1/2}$, we get from (8) $\mathrm{E}(\widehat{\mathbf{\Sigma}}_{x,S})=\mathbf{V}\mathbf{\Lambda}^{1/2}\mathrm{E}(\frac{\mathbf{u}_t\mathbf{u}_t^H}{\mathbf{u}_t^H\mathbf{\Lambda}\mathbf{u}_t})\mathbf{\Lambda}^{1/2}\mathbf{V}^H$. Then, using the symmetries of the p.d.f. of the r.v. $\frac{\mathbf{u}_t\mathbf{u}_t^H}{\mathbf{u}_t^H\mathbf{\Lambda}\mathbf{u}_t}$, it is easy to prove that $\mathrm{E}(\frac{\mathbf{u}_t\mathbf{u}_t^H}{\mathbf{u}_t^H\mathbf{\Lambda}\mathbf{u}_t})$ is diagonal. Consequently $\mathrm{E}(\widehat{\mathbf{\Sigma}}_{x,S})=\mathbf{V}\mathbf{\Delta}\mathbf{V}^H$ where $\mathbf{\Delta}=\mathrm{Diag}(\chi_1,..,\chi_N)$ and where $\chi_k=\lambda_k\mathrm{E}\left(\frac{|(\mathbf{u}_t)_k|^2}{\mathbf{u}_t^H\mathbf{\Lambda}\mathbf{u}_t}\right), k=1,...,N$. Because $\mathbf{u}_t$ $U(\mathbb{C}S^N)$ distributed is equivalent to $\bar{\mathbf{u}}_t\stackrel{\text{def}}{=}(\mathrm{Re}^T(\mathbf{u}_t),\mathrm{Im}^T(\mathbf{u}_t))^T$ $U(\mathbb{R}S^{2N})$ distributed and that $|(\mathbf{u}_t)_k|^2=((\bar{\mathbf{u}}_t)_k)^2+((\bar{\mathbf{u}}_t)_{N+k})^2$ and $\mathbf{u}_t^H\mathbf{\Lambda}\mathbf{u}_t=\sum_{n=1}^N\lambda_n((\bar{\mathbf{u}}_t)_n)^2+\sum_{n=N+1}^{2N}\lambda_{n-N}((\bar{\mathbf{u}}_t)_n)^2=\bar{\mathbf{u}}_t^T\mathbf{\Lambda}'\bar{\mathbf{u}}_t$ with $\mathbf{\Lambda}'\stackrel{\text{def}}{=}\mathrm{Diag}(\mathbf{\Lambda},\mathbf{\Lambda})$. Therefore, we get $\frac{\chi_k}{\lambda_k}=2\mathrm{E}\left(\frac{((\bar{\mathbf{u}}_t)_k)^2}{\bar{\mathbf{u}}_t^T\mathbf{\Lambda}'\bar{\mathbf{u}}_t}\right), k=1,...,N$ which is given from [21, proposition 3] for $2N$-dimensional RES distributions by $\int_0^\infty\frac{1}{(1+\lambda_k x)\prod_{n=1}^N(1+\lambda_n x)}dx$. Grouping the multiple eigenvalues $\lambda_k\in s_j$ $j,...,J$, (11) is obtained. ■

Now, let us consider the NC-CES distributed data $\mathbf{x}_t$ case. Note that by definition of the NC-CES distribution, the r.v.s $\bar{\mathbf{x}}_t$ defined by the one to one mapping $\widetilde{\mathbf{x}}_t=\sqrt{2}\mathbf{M}\bar{\mathbf{x}}_t$ where $\mathbf{M}\stackrel{\text{def}}{=}\frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{I} & i\mathbf{I}\\ \mathbf{I} & -i\mathbf{I}\end{pmatrix}$ is a unitary matrix, are RES distributed with stochastic representation $\bar{\mathbf{x}}_t=_d\sqrt{2}\sqrt{\mathcal{Q}_t}\mathbf{\Sigma}_{\tilde{x}}^{1/2}\bar{\mathbf{u}}_t$ with scatter matrix $\mathbf{\Sigma}_{\tilde{x}}=\bar{\mathbf{V}}(\frac{1}{2}\widetilde{\mathbf{\Lambda}})\bar{\mathbf{V}}^T$ where $\bar{\mathbf{V}}$ is a orthogonal matrix. Taking the square root $\mathbf{\Sigma}_{\tilde{x}}^{1/2}=\bar{\mathbf{V}}(\frac{1}{2}\widetilde{\mathbf{\Lambda}})^{1/2}$ in (9), we get

$$\mathrm{E}(\widehat{\mathbf{\Sigma}}_{\tilde{x},S})=\mathrm{E}\left(\frac{\widetilde{\mathbf{x}}_t\widetilde{\mathbf{x}}_t^H}{\|\widetilde{\mathbf{x}}_t\|^2}\right)=\mathrm{E}\left(\frac{\mathbf{M}\bar{\mathbf{x}}_t\bar{\mathbf{x}}_t^H\mathbf{M}^H}{\|\bar{\mathbf{x}}_t\|^2}\right)=\mathbf{M}\bar{\mathbf{V}}(\frac{1}{2}\widetilde{\mathbf{\Lambda}})^{1/2}\mathrm{E}(\frac{\bar{\mathbf{u}}_t\bar{\mathbf{u}}_t^T}{\bar{\mathbf{u}}_t^T(\frac{1}{2}\widetilde{\mathbf{\Lambda}})\bar{\mathbf{u}}_t})(\frac{1}{2}\widetilde{\mathbf{\Lambda}})^{1/2}\bar{\mathbf{V}}^T\mathbf{M}^H.$$

Then, using the symmetries of the p.d.f. of the r.v. $\frac{\bar{\mathbf{u}}_t\bar{\mathbf{u}}_t^T}{\bar{\mathbf{u}}_t^T(\frac{1}{2}\widetilde{\mathbf{\Lambda}})\bar{\mathbf{u}}_t}$, it is easy to prove that $\mathrm{E}(\frac{\bar{\mathbf{u}}_t\bar{\mathbf{u}}_t^T}{\bar{\mathbf{u}}_t^T(\frac{1}{2}\widetilde{\mathbf{\Lambda}})\bar{\mathbf{u}}_t})$ is diagonal. Therefore, $\mathrm{E}(\widehat{\mathbf{\Sigma}}_{\tilde{x},S})=\widetilde{\mathbf{V}}\widetilde{\mathbf{\Delta}}\widetilde{\mathbf{V}}^H$ where $\widetilde{\mathbf{V}}\stackrel{\text{def}}{=}\mathbf{M}\bar{\mathbf{V}}$ is a unitary structured matrix as $\begin{pmatrix}\mathbf{V}_1\\ \mathbf{V}_1^*\end{pmatrix}$ and $\widetilde{\mathbf{\Delta}}=\mathrm{Diag}(\widetilde{\chi}_1,..,\widetilde{\chi}_{2N})$ where $\widetilde{\chi}_k=\widetilde{\lambda}_k\mathrm{E}\left(\frac{|(\bar{\mathbf{u}}_t)_k|^2}{\bar{\mathbf{u}}_t^T\widetilde{\mathbf{\Lambda}}\bar{\mathbf{u}}_t}\right), k=1,...,2N$, which is given from [21, proposition 3] for $2N$-dimensional RES distributions by $\frac{\widetilde{\lambda}_k}{2}\int_0^\infty\frac{x}{(1+\widetilde{\lambda}_k x)\prod_{n=1}^{2N}(1+\widetilde{\lambda}_n x)^{\frac{1}{2}}}dx$. Finally, grouping the multiple eigenvalues $\widetilde{\lambda}_k\in s_j$ $j,...,J$, (12) is derived. ■

Proof of Theorem 2. For C-CES distributed data, $\mathbf{R}_{\widehat{\Sigma}_S}\stackrel{\text{def}}{=}\mathrm{E}[\mathrm{vec}(\frac{\mathbf{x}_t\mathbf{x}_t^H}{\|\mathbf{x}_t\|^2})\mathrm{vec}^H(\frac{\mathbf{x}_t\mathbf{x}_t^H}{\|\mathbf{x}_t\|^2})]-\mathrm{E}[\mathrm{vec}(\frac{\mathbf{x}_t\mathbf{x}_t^H}{\|\mathbf{x}_t\|^2})]\mathrm{E}[\mathrm{vec}^H(\frac{\mathbf{x}_t\mathbf{x}_t^H}{\|\mathbf{x}_t\|^2})]$ with $\mathrm{vec}(\frac{\mathbf{x}_t\mathbf{x}_t^H}{\|\mathbf{x}_t\|^2})=(\mathbf{V}^*\otimes\mathbf{V})\mathrm{vec}(\mathbf{U}_t)$ where $\mathbf{U}_t\stackrel{\text{def}}{=}\frac{\mathbf{\Lambda}^{1/2}\mathbf{u}_t\mathbf{u}_t^H\mathbf{\Lambda}^{1/2}}{\mathbf{u}_t^H\mathbf{\Lambda}\mathbf{u}_t}$ and $\mathrm{E}(\mathbf{U}_t)=\mathbf{\Delta}$ follows from the proof of Theorem 1. Inspired by the proof in Appendix 6.2 [20], we then get $\mathbf{R}_{\widehat{\Sigma}_S}=(\mathbf{V}^*\otimes\mathbf{V})\mathbf{\Omega}(\mathbf{V}^T\otimes\mathbf{V}^H)$ with

$$\begin{aligned}\mathbf{\Omega}&\stackrel{\text{def}}{=}\mathrm{E}[\mathrm{vec}(\mathbf{U}_t)\mathrm{vec}^H(\mathbf{U}_t)]-\mathrm{E}[\mathrm{vec}(\mathbf{U}_t)]\mathrm{E}[\mathrm{vec}^H(\mathbf{U}_t)]\\ &=\sum_{1\le i,j,k,\ell\le N}\mathrm{E}[(\mathbf{U}_t)_{i,j}(\mathbf{U}_t^*)_{k,\ell}](\mathbf{e}_j\otimes\mathbf{e}_i)(\mathbf{e}_\ell^T\otimes\mathbf{e}_k^T)-\sum_{1\le i,j\le N}\mathrm{E}[(\mathbf{U}_t)_{i,i}]\mathrm{E}[(\mathbf{U}_t^*)_{j,j}](\mathbf{e}_i\otimes\mathbf{e}_i)(\mathbf{e}_j^T\otimes\mathbf{e}_j^T).\end{aligned}$$

It follows from the symmetries of the distribution of the terms of the Hermitian random matrix $\mathbf{U}_t$, the only non-zero terms $\mathrm{E}[(\mathbf{U}_t)_{i,j}(\mathbf{U}_t^*)_{k,\ell}]$ are those corresponding to the indices $i=j=k=\ell$, $i=k$ & $j=\ell$ and $i=j$ & $k=\ell$, for which $\mathrm{E}[(\mathbf{U}_t)_{i,i}(\mathbf{U}_t^*)_{i,i}]=\gamma_{i,i}$ and $\mathrm{E}[(\mathbf{U}_t)_{i,j}(\mathbf{U}_t^*)_{i,j}]=\mathrm{E}[(\mathbf{U}_t)_{i,i}(\mathbf{U}_t^*)_{j,j}]=\gamma_{i,j}$. Consequently

$$\begin{aligned}\mathbf{\Omega}=&\sum_{1\le i,j\le N}\gamma_{i,j}(\mathbf{e}_j\otimes\mathbf{e}_i)(\mathbf{e}_j^T\otimes\mathbf{e}_i^T)+\sum_{1\le i,j\le N}(\gamma_{i,j}-\chi_i\chi_j)(\mathbf{e}_i\otimes\mathbf{e}_i)(\mathbf{e}_j^T\otimes\mathbf{e}_j^T)\\ &-\sum_{i=1}^N\gamma_{i,i}(\mathbf{e}_i\otimes\mathbf{e}_i)(\mathbf{e}_i^T\otimes\mathbf{e}_i^T),\end{aligned}$$

and thus (15) follows.

With the notation used in the proof of Theorem 1, $\gamma_{i,j}$ can be expressed for N-dimensional C-CES distributions as: $\gamma_{i,j}=\lambda_i\lambda_j\mathrm{E}\left(\frac{(((\bar{\mathbf{u}}_t)_i)^2+((\bar{\mathbf{u}}_t)_{N+i})^2)(((\bar{\mathbf{u}}_t)_j)^2+((\bar{\mathbf{u}}_t)_{N+j})^2)}{(\bar{\mathbf{u}}_t^T\mathbf{\Lambda}'\bar{\mathbf{u}}_t)^2})\right), i=1,...,N, j=1,...,N$, which yields that $\frac{\gamma_{i,j}}{\lambda_i\lambda_j}=4\mathrm{E}\left(\frac{((\bar{\mathbf{u}}_t)_i)^4}{(\bar{\mathbf{u}}_t^T\mathbf{\Lambda}'\bar{\mathbf{u}}_t)^2}\right)$ for $i\neq j$ and $\frac{\gamma_{i,i}}{\lambda_i^2}=2\mathrm{E}\left(\frac{((\bar{\mathbf{u}}_t)_i)^4}{(\bar{\mathbf{u}}_t^T\mathbf{\Lambda}'\bar{\mathbf{u}}_t)^2}\right)+2\mathrm{E}\left(\frac{((\bar{\mathbf{u}}_t)_i)^2((\bar{\mathbf{u}}_t)_{N+i})^2}{(\bar{\mathbf{u}}_t^T\mathbf{\Lambda}'\bar{\mathbf{u}}_t)^2}\right)$.

We deduce, thanks to [21, proposition 3] which gives the expressions of $\gamma_{i,j}$ and $\gamma_{i,i}$ in one-dimensional integral representations for $2N$-dimensional RES distributions, that $\gamma_{i,j}=\lambda_i\lambda_j\int_0^\infty\frac{x}{(1+\lambda_i x)(1+\lambda_j x)\prod_{n=1}^N(1+\lambda_n x)}dx$ and $\gamma_{i,i}=2\lambda_i^2\int_0^\infty\frac{x}{(1+\lambda_i x)\prod_{n=1}^N(1+\lambda_n x)}dx$. Finally, grouping the multiple eigenvalues $\lambda_k\in s_j$ $j,...,J$, (17) and (18) are derived. ■

For NC-CES distributed data, we get from the proof of Theorem 1, $\frac{\widetilde{\mathbf{x}}_t\widetilde{\mathbf{x}}_t^H}{\|\widetilde{\mathbf{x}}_t\|^2}=\widetilde{\mathbf{V}}(\frac{1}{2}\widetilde{\mathbf{\Lambda}})^{1/2}\frac{\bar{\mathbf{u}}_t\bar{\mathbf{u}}_t^T}{\bar{\mathbf{u}}_t^T(\frac{1}{2}\widetilde{\mathbf{\Lambda}})\bar{\mathbf{u}}_t}(\frac{1}{2}\widetilde{\mathbf{\Lambda}})^{1/2}\widetilde{\mathbf{V}}^H$, which gives $\mathrm{vec}(\frac{\widetilde{\mathbf{x}}_t\widetilde{\mathbf{x}}_t^H}{\|\widetilde{\mathbf{x}}_t\|^2})=(\widetilde{\mathbf{V}}^*\otimes\widetilde{\mathbf{V}})\mathrm{vec}(\widetilde{\mathbf{U}}_t)$, where $\widetilde{\mathbf{U}}_t\stackrel{\text{def}}{=}\frac{\widetilde{\mathbf{\Lambda}}^{1/2}\bar{\mathbf{u}}_t\bar{\mathbf{u}}_t^T\widetilde{\mathbf{\Lambda}}^{1/2}}{\bar{\mathbf{u}}_t\widetilde{\mathbf{\Lambda}}\bar{\mathbf{u}}_t}$ and $\mathrm{E}(\widetilde{\mathbf{U}}_t)=\widetilde{\mathbf{\Delta}}$. Following similar steps as in the case of C-CES distributed data, we get $\mathbf{R}_{\widehat{\Sigma}_S}=(\widetilde{\mathbf{V}}^*\otimes\widetilde{\mathbf{V}})\widetilde{\mathbf{\Omega}}(\widetilde{\mathbf{V}}^T\otimes\widetilde{\mathbf{V}}^H)$ with

$$\widetilde{\boldsymbol{\Omega}} \stackrel{\text{def}}{=} \mathrm{E}[\mathrm{vec}(\widetilde{\mathbf{U}}_t)\mathrm{vec}^T(\widetilde{\mathbf{U}}_t)] - \mathrm{E}[\mathrm{vec}(\widetilde{\mathbf{U}}_t)]\mathrm{E}[\mathrm{vec}^T(\widetilde{\mathbf{U}}_t)]$$
$$= \sum_{1\le i,j,k,\ell\le 2N} \mathrm{E}[(\widetilde{\mathbf{U}}_t)_{i,j}(\widetilde{\mathbf{U}}_t)_{k,\ell}](\widetilde{\mathbf{e}}_j\otimes\widetilde{\mathbf{e}}_i)(\widetilde{\mathbf{e}}_\ell^T\otimes\widetilde{\mathbf{e}}_k^T) - \sum_{1\le i,j\le 2N} \mathrm{E}[(\widetilde{\mathbf{U}}_t)_{i,i}]\mathrm{E}[(\widetilde{\mathbf{U}}_t)_{j,j}](\widetilde{\mathbf{e}}_i\otimes\widetilde{\mathbf{e}}_i)(\widetilde{\mathbf{e}}_j^T\otimes\widetilde{\mathbf{e}}_j^T).$$

Since the entries of the real-valued symmetric random matrix $\widetilde{\mathbf{U}}_t$ are symmetrically distributed, it follows that $\mathrm{E}[(\widetilde{\mathbf{U}}_t)_{i,j}(\widetilde{\mathbf{U}}_t)_{k,\ell}] = 0$ unless $i=j=k=\ell$, $i=k$ & $j=\ell$, $i=j$ & $k=\ell$ and $i=\ell$ & $j=k$ for which $\mathrm{E}[(\widetilde{\mathbf{U}}_t)_{i,i}(\widetilde{\mathbf{U}}_t)_{i,i}] = \widetilde{\gamma}_{i,i}$ and $\mathrm{E}[(\widetilde{\mathbf{U}}_t)_{i,j}(\widetilde{\mathbf{U}}_t)_{i,j}] = \mathrm{E}[(\widetilde{\mathbf{U}}_t)_{i,i}(\widetilde{\mathbf{U}}_t)_{j,j}] = \mathrm{E}[(\widetilde{\mathbf{U}}_t)_{i,j}(\widetilde{\mathbf{U}}_t)_{j,i}] = \widetilde{\gamma}_{i,j}$. Consequently

$$\widetilde{\boldsymbol{\Omega}} = \sum_{1\le i,j\le 2N} \widetilde{\gamma}_{i,j}(\widetilde{\mathbf{e}}_j\otimes\widetilde{\mathbf{e}}_i)(\widetilde{\mathbf{e}}_j^T\otimes\widetilde{\mathbf{e}}_i^T) + \sum_{1\le i,j\le 2N} (\widetilde{\gamma}_{i,j} - \widetilde{\chi}_i\widetilde{\chi}_j)(\widetilde{\mathbf{e}}_i\otimes\widetilde{\mathbf{e}}_i)(\widetilde{\mathbf{e}}_j^T\otimes\widetilde{\mathbf{e}}_j^T)$$
$$+ \sum_{1\le i,j\le 2N} \widetilde{\gamma}_{i,j}(\widetilde{\mathbf{e}}_j\otimes\widetilde{\mathbf{e}}_i)(\widetilde{\mathbf{e}}_i^T\otimes\widetilde{\mathbf{e}}_j^T) - 2\sum_{i=1}^{2N} \widetilde{\gamma}_{i,i}(\widetilde{\mathbf{e}}_i\otimes\widetilde{\mathbf{e}}_i)(\widetilde{\mathbf{e}}_i^T\otimes\widetilde{\mathbf{e}}_i^T)$$

and (16) follows.

We deduce once again from [21, proposition 3] that the expressions of $\widetilde{\gamma}_{i,j}$ for $i \neq j$ and $\widetilde{\gamma}_{i,i}$ are given by $\frac{\widetilde{\lambda}_i\widetilde{\lambda}_j}{4}\int_0^\infty \frac{x}{(1+\widetilde{\lambda}_i x)(1+\widetilde{\lambda}_j x)\prod_{n=1}^{2N}(1+\widetilde{\lambda}_n x)^{\frac{1}{2}}}dx$ and $\frac{3\widetilde{\lambda}_i^2}{4}\int_0^\infty \frac{x}{(1+\widetilde{\lambda}_i x)^2\prod_{n=1}^{2N}(1+\widetilde{\lambda}_n x)^{\frac{1}{2}}}dx$, respectively. Finally, grouping the multiple eigenvalues $\widetilde{\lambda}_k \in s_j$ $j, ..., J$, (18) and (19) are derived. ■

Proof of Theorem 3. For C-CES distributed data, the asymptotic distribution of the eigenprojectors $\widehat{\boldsymbol{\Pi}}_{\mathrm{SSCM},(j)}$ is given by (26) with $\boldsymbol{\Sigma}^{\#}_{(j)} = \sum_{k\notin s_j}\frac{1}{(\chi_k - \chi_{(j)})}\mathbf{v}_k\mathbf{v}_k^H$ and $\boldsymbol{\Pi}_{(j)} = \sum_{k\in s_j}\mathbf{v}_k\mathbf{v}_k^H$. It can be easily simplified thanks to the following identities which are straightforward from the orthonormality of the vectors $\mathbf{v}_k$:

$$(\boldsymbol{\Sigma}^{\#*}_{(j)}\otimes\boldsymbol{\Pi}_{(j)})(\mathbf{v}_k^*\otimes\mathbf{v}_k) = (\boldsymbol{\Pi}^*_{(j)}\otimes\boldsymbol{\Sigma}^{\#}_{(j)})(\mathbf{v}_k^*\otimes\mathbf{v}_k) = \mathbf{0},\quad k=1,...,N, \tag{61}$$

$$(\boldsymbol{\Sigma}^{\#*}_{(j)}\otimes\boldsymbol{\Pi}_{(j)})(\mathbf{v}_\ell^*\otimes\mathbf{v}_k) = \begin{cases} \frac{1}{(\chi_\ell-\chi_{(j)})}(\mathbf{v}_\ell^*\otimes\mathbf{v}_k), & k\in s_j \text{ and } \ell\notin s_j \\ \mathbf{0} & \text{elsewhere}\end{cases} \tag{62}$$

$$(\boldsymbol{\Pi}^*_{(j)}\otimes\boldsymbol{\Sigma}^{\#}_{(j)})(\mathbf{v}_\ell^*\otimes\mathbf{v}_k) = \begin{cases} \frac{1}{(\chi_k-\chi_{(j)})}(\mathbf{v}_\ell^*\otimes\mathbf{v}_k), & k\notin s_j \text{ and } \ell\in s_j \\ \mathbf{0} & \text{elsewhere}\end{cases}. \tag{63}$$

Plugging (61), (62) and (63) into (26), where $\mathbf{R}_{\widehat{\Sigma}_S}$ is given by (15), (27) follows after simple algebra manipulations.

Since in the case of NC-CES distributed data, $\mathbf{R}_{\widehat{\widetilde{\Pi}}_{\mathrm{SSCM},(j)}}$ has a form similar to (26), it follows then that (28) can be obtained following similar steps as above using the orthonormality of $\widetilde{\mathbf{v}}_k$, $\widetilde{\mathbf{v}}_k^* = \mathbf{J}\widetilde{\mathbf{v}}_k$ and $(\widetilde{\mathbf{v}}_\ell^*\otimes\widetilde{\mathbf{v}}_k) = \mathbf{K}_{2N}(\widetilde{\mathbf{v}}_k\otimes\widetilde{\mathbf{v}}_\ell^*)$. ■

Closed-form expressions of (11), (17) *and* (18) *without integral*

For ease of reading, the following identities are used in this proof:

$$\prod_{p=1}^{P}\frac{1}{1+\lambda_p x} = \sum_{p=1}^{P}\frac{c_p}{1+\lambda_p x}\ \text{where}\ c_p = \prod_{j=1,j\neq p}^{P}\left(1-\frac{\lambda_j}{\lambda_p}\right)^{-1}, \tag{64}$$

$$\frac{1}{(1+\lambda_n x)(1+\lambda x)^m} = \frac{1}{\left(1-\frac{\lambda}{\lambda_n}\right)^m}\frac{1}{1+\lambda_n x} - \sum_{\ell=0}^{m-1}\frac{\lambda}{\lambda_n\left(1-\frac{\lambda}{\lambda_n}\right)^{\ell+1}}\frac{1}{(1+\lambda x)^{m-\ell}},\ \forall m\in\mathbb{N}^*, \tag{65}$$

$$\int_0^\infty \frac{x^{\ell-1}}{(x+y)^p(x+z)^q}dx = z^{-q}y^{\ell-p}B(\ell,p+q-\ell)\,{}_2F_1\left(\ell,q,p+q,1-\frac{y}{z}\right),\ \text{for}\ 0<\ell<p+q. \tag{66}$$

Identity (64) is the partial fraction expansion of $\prod_{n=1}^{P}\frac{1}{1+\lambda_n x}$, (65) is proved by induction on m and (66) follows from the change of variables $x = \frac{yt}{1-t}$ and the definition of ${}_2F_1(.,.,.,.)$.

Now, let's start to prove (75)-(77). Thanks to (64) and (65), (11) can be expressed as follows:

$$\chi_k = \lambda_k\int_0^\infty \frac{1}{(1+\lambda_k x)^2(1+\lambda x)^{N'}\prod_{p=1,p\neq k}^{P}(1+\lambda_p x)}dx,\qquad k=1,...,P,$$
$$= \lambda_k\sum_{p=1,p\neq k}^{P}c'_p\int_0^\infty\frac{1}{(1+\lambda_k x)^2(1+\lambda_p x)(1+\lambda x)^{N'}}dx$$

$$= \sum_{p=1, p\neq k}^{P} \frac{\lambda_k c'_p}{\left(1-\frac{\lambda}{\lambda_p}\right)^{N'}} \int_0^\infty \frac{1}{(1+\lambda_k x)^2(1+\lambda_p x)} dx$$

$$- \sum_{p=1, p\neq k}^{P} \sum_{\ell=0}^{N'-1} \frac{\lambda_k \lambda c'_p}{\lambda_p \left(1-\frac{\lambda}{\lambda_p}\right)^{\ell+1}} \int_0^\infty \frac{1}{(1+\lambda_k x)^2(1+\lambda x)^{N'-\ell}} dx, \tag{67}$$

where $c'_p \stackrel{\text{def}}{=} (1-\frac{\lambda_k}{\lambda_p})c_p$. It follows from identity (66) that

$$\int_0^\infty \frac{1}{(1+\lambda_k x)^2(1+\lambda_p x)} dx = \frac{1}{1-(\frac{\lambda_p}{\lambda_k})} \left(1 + \frac{\log(\frac{\lambda_k}{\lambda_p})}{1-\frac{\lambda_k}{\lambda_p}}\right), \tag{68}$$

$$\int_0^\infty \frac{1}{(1+\lambda_k x)^2(1+\lambda x)^{N'-l}} dx = \frac{1}{\lambda_k} \frac{1}{N'-\ell+1} {}_2F_1\left(1, N'-\ell, N'-\ell+2, 1-\frac{\lambda}{\lambda_k}\right). \tag{69}$$

Inserting (68) and (69) into (67), we obtain (75).

Similarly, χ defined by (11) which can be expressed thanks to (64) as follows:

$$\chi = \lambda \int_0^\infty \frac{1}{(1+\lambda x)^{N'+1} \prod_{p=1}^{P} (1+\lambda_p x)} dx = \lambda \sum_{p=1}^{P} c_p \int_0^\infty \frac{1}{(1+\lambda x)^2(1+\lambda_p x)(1+\lambda x)^{N'-1}} dx$$

can be obtained by similar steps as in the proof of χ_k by replacing N' by $N'-1$ and λ_k by λ in (75). Noting that now ${}_2F_1\left(1, N'-\ell-1, N'-\ell+1, 0\right) = 1$ in (69), (77) is obtained.

Now, let's prove (78). It follows from (17), using (64) and (65), that

$$\gamma_{k,k} = 2\lambda_k^2 \int_0^\infty \frac{x}{(1+\lambda_k x)^3(1+\lambda x)^{N'} \prod_{p=1, p\neq k}^{P} (1+\lambda_p x)} dx, \qquad k = P+1, ..., N,$$

$$= 2\lambda_k^2 \sum_{p=1, p\neq k}^{P} c'_p \int_0^\infty \frac{x}{(1+\lambda_k x)^3(1+\lambda_p x)(1+\lambda x)^{N'}} dx$$

$$= \sum_{p=1, p\neq k}^{P} \frac{2\lambda_k^2 c'_p}{\left(1-\frac{\lambda}{\lambda_p}\right)^{N'}} \int_0^\infty \frac{x}{(1+\lambda_k x)^3(1+\lambda_p x)} dx$$

$$- \sum_{p=1, p\neq k}^{P} \sum_{\ell=0}^{N'-1} \frac{2\lambda_k^2 \lambda c'_p}{\lambda_p \left(1-\frac{\lambda}{\lambda_p}\right)^{\ell+1}} \int_0^\infty \frac{x}{(1+\lambda_k x)^3(1+\lambda x)^{N'-\ell}} dx. \tag{70}$$

It follows again from identity (66) that

$$\int_0^\infty \frac{x}{(1+\lambda_k x)^3(1+\lambda_p x)} dx = \frac{\lambda_k + \lambda_n}{2\lambda_k\left(\lambda_k - \lambda_n\right)^2} + \frac{\lambda_n\left(\log\left(\lambda_n\right) - \log\left(\lambda_k\right)\right)}{\left(\lambda_k - \lambda_n\right)^3}, \tag{71}$$

$$\int_0^\infty \frac{x}{(1+\lambda_k x)^3(1+\lambda x)^{N'-\ell}} dx = \frac{1}{\lambda_k^2} \frac{1}{(N'-\ell+1)(N'-\ell+2)} {}_2F_1\left(2, N'-\ell, N'-\ell+3, 1-\frac{\lambda}{\lambda_k}\right). \tag{72}$$

Substitute (71) and (72) into (70) proves (78).

Similarly, γ defined by (18) which can be expressed thanks to (64) as follows:

$$\gamma = 2\lambda^2 \int_0^\infty \frac{x}{(1+\lambda x)^{N'+2} \prod_{p=1}^{P} (1+\lambda_p x)} dx = 2\lambda^2 \sum_{p=1}^{P} c_p \int_0^\infty \frac{x}{(1+\lambda x)^3(1+\lambda_p x)(1+\lambda x)^{N'-1}} dx$$

can be obtained by similar steps as in the proof of $\gamma_{k,k}$ by replacing N' by $N'-1$, and λ_k by λ in (78). Noting that now ${}_2F_1\left(2, N'-\ell-1, N'-\ell+2, 0\right) = 1$ in (72), (80) is obtained.

Finally, let us prove (81) and (83). It follows from (18) that

$$\gamma_{k,\ell} = \lambda_k\lambda_\ell \int_0^\infty \frac{x}{(1+\lambda_k x)^2(1+\lambda_\ell x)^2(1+\lambda x)^{N'} \prod_{p=1,p\neq k,p\neq\ell}^{P}(1+\lambda_n x)} dx, \ 1\leq k\neq\ell\leq P$$

$$= \lambda_k\lambda_\ell \sum_{p=1,p\neq k,p\neq\ell}^{P} c_p \int_0^\infty \frac{(1-\frac{\lambda_k}{\lambda_p})(1-\frac{\lambda_\ell}{\lambda_p})x}{(1+\lambda_k x)^2(1+\lambda_\ell x)^2(1+\lambda_p x)(1+\lambda x)^{N'}} dx, \tag{73}$$

and

$$\gamma_{k,\ell} = \lambda_k\lambda \int_0^\infty \frac{x}{(1+\lambda_k x)^2(1+\lambda x)^{N'+1} \prod_{p=1,p\neq k}^{P}(1+\lambda_p x)} dx, \ \ k=1,...,P, \ \ell=P+1,...,N$$

$$= \lambda_k\lambda \sum_{p=1,p\neq k}^{P} c_p \int_0^\infty \frac{(1-\frac{\lambda_k}{\lambda_p})x}{(1+\lambda_k x)^2(1+\lambda_p x)(1+\lambda x)^{N'+1}} dx. \tag{74}$$

Hence (81) and (83) are obtained by using the following partial fraction expansions:

$$\frac{(1-\frac{\lambda_k}{\lambda_p})(1-\frac{\lambda_\ell}{\lambda_p})x}{(1+\lambda_k x)^2(1+\lambda_\ell x)^2(1+\lambda_p x)} = -\frac{\lambda_k^2(\lambda_p-\lambda_\ell)(\lambda_k^2-2\lambda_p\lambda_\ell+\lambda_k\lambda_\ell)}{\lambda_p^2(\lambda_k-\lambda_p)(\lambda_k-\lambda_\ell)^3(1+\lambda_k x)} + \frac{\lambda_k^2(\lambda_p-\lambda_\ell)}{\lambda_p^2(\lambda_k-\lambda_\ell)^2(1+\lambda_k x)^2}$$
$$- \frac{\lambda_\ell^2(\lambda_k-\lambda_p)(2\lambda_k\lambda_p-\lambda_k\lambda_\ell-\lambda_\ell^2)}{\lambda_p^2(\lambda_k-\lambda_\ell)^3(\lambda_p-\lambda_\ell)(1+\lambda_\ell x)} + \frac{\lambda_\ell^2(\lambda_p-\lambda_k)}{\lambda_p^2(\lambda_k-\lambda_\ell)^2(1+\lambda_\ell x)^2}$$
$$+ \frac{\lambda_p}{(\lambda_k-\lambda_p)(\lambda_p-\lambda_\ell)(1+\lambda_p x)}$$

$$\frac{(1-\frac{\lambda_k}{\lambda_p})x}{(1+\lambda_k x)^2(1+\lambda_p x)} = -\frac{\lambda_k}{\lambda_p(\lambda_k-\lambda_p)(1+\lambda_k x)} + \frac{1}{\lambda_p(1+\lambda_k x)^2} + \frac{1}{(\lambda_k-\lambda_p)(1+\lambda_p x)},$$

in respectively (73) and (74), and the following integrals identities deduced from (66)

$$\int_0^\infty \frac{1}{(1+\lambda' x)(1+\lambda x)^{N''}} dx = \frac{1}{\lambda'}\frac{1}{N''}{}_2F_1\left(1, N'', N''+1, 1-\frac{\lambda}{\lambda'}\right).$$

$$\int_0^\infty \frac{1}{(1+\lambda' x)^2(1+\lambda x)^{N''}} dx = \frac{1}{\lambda'}\frac{1}{N''+1}{}_2F_1\left(1, N'', N''+2, 1-\frac{\lambda}{\lambda'}\right).$$

This allows you to prove the following expressions:

$$\chi_k = \sum_{p=1,p\neq k}^{P} \frac{c'_p}{(1-\frac{\lambda}{\lambda_p})^{N'}(1-\frac{\lambda_p}{\lambda_k})}\left(1+\frac{\log\left(\frac{\lambda_k}{\lambda_p}\right)}{1-\frac{\lambda_k}{\lambda_p}}\right)$$
$$-\sum_{\ell=0}^{N'-1}\frac{\lambda}{N'-\ell+1}\left(\sum_{p=1,p\neq k}^{P}\frac{c'_p}{\lambda_p(1-\frac{\lambda}{\lambda_p})^{\ell+1}}{}_2F_1\left(1, N'-\ell, N'-\ell+2, 1-\frac{\lambda}{\lambda_k}\right)\right),$$
$$k=1,...,P \text{ and } P>1, \tag{75}$$

$$\chi_1 = \frac{1}{N}\,{}_2F_1\left(1, N-1, N+1, 1-\frac{\lambda}{\lambda_1}\right), \qquad P=1, \tag{76}$$

$$\chi \stackrel{\text{def}}{=} \chi_k = \sum_{p=1}^{P}\frac{c_p}{(1-\frac{\lambda}{\lambda_p})^{N'-1}(1-\frac{\lambda_p}{\lambda})}\left(1+\frac{\log\left(\frac{\lambda}{\lambda_p}\right)}{1-\frac{\lambda}{\lambda_p}}\right) - \sum_{\ell=0}^{N'-2}\frac{\lambda}{N'-\ell}\left(\sum_{p=1}^{P}\frac{c_p}{\lambda_p(1-\frac{\lambda}{\lambda_p})^{\ell+1}}\right),$$
$$k=P+1,..,N, \tag{77}$$

$$\gamma_{k,k} = \sum_{p=1,p\neq k}^{P}\frac{c'_p\lambda_k}{\lambda_p(1-\frac{\lambda}{\lambda_p})^{N'}(1-\frac{\lambda_k}{\lambda_p})^2}\left(1+\frac{\lambda_k}{\lambda_p}-\frac{2\log\left(\frac{\lambda_k}{\lambda_p}\right)}{1-\frac{\lambda_p}{\lambda_k}}\right)$$

$$-\sum_{\ell=0}^{N'-1}\frac{2\lambda}{(N'-\ell+1)(N'-\ell+2)}\left(\sum_{p=1,p\neq k}^{P}\frac{c'_p}{\lambda_p(1-\frac{\lambda}{\lambda_p})^{\ell+1}}{}_2F_1\left(2,N'-\ell,N'-\ell+3,1-\frac{\lambda}{\lambda_k}\right)\right),$$
$$k=1,...,P \text{ and } P>1, \tag{78}$$

$$\gamma_{1,1}=\frac{2}{N(N-1)}\,{}_2F_1\left(2,N-2,N+1,1-\frac{\lambda}{\lambda_1}\right),\qquad P=1, \tag{79}$$

$$\gamma\stackrel{\text{def}}{=}\gamma_{k,k}=\sum_{p=1}^{P}\frac{c_p\lambda}{\lambda_p(1-\frac{\lambda}{\lambda_p})^{N'-1}(1-\frac{\lambda}{\lambda_p})^2}\left(1+\frac{\lambda}{\lambda_p}-\frac{2\log\left(\frac{\lambda}{\lambda_p}\right)}{1-\frac{\lambda_p}{\lambda}}\right)$$
$$-\sum_{\ell=0}^{N'-2}\frac{2\lambda}{(N'-\ell)(N'-\ell+1)}\left(\sum_{p=1}^{P}\frac{c_p}{\lambda_p(1-\frac{\lambda}{\lambda_p})^{\ell+1}}\right),\quad k=P+1,...,N, \tag{80}$$

$$\gamma_{k,\ell}=\sum_{p=1,p\neq k,p\neq\ell}^{P}c_pt_{k,\ell,p},\qquad 1\leq k\neq\ell\leq P \text{ and } P>2, \tag{81}$$

$$\gamma_{1,2}=\lambda_1\lambda_2\left[\frac{(\lambda_1+\lambda_2)\,{}_2F_1\left(1,N',N'+1,1-\frac{\lambda}{\lambda_1}\right)}{N'\,(\lambda_1-\lambda_2)^3}-\frac{{}_2F_1\left(1,N',N'+2,1-\frac{\lambda}{\lambda_1}\right)}{(N'+1)\,(\lambda_1-\lambda_2)^2}\right.$$
$$\left.-\frac{(\lambda_1+\lambda_2)\,{}_2F_1\left(1,N',N'+1,1-\frac{\lambda}{\lambda_2}\right)}{N'\,(\lambda_1-\lambda_2)^3}-\frac{{}_2F_1\left(1,N',N'+2,1-\frac{\lambda}{\lambda_2}\right)}{N'+1}\right],$$
$$P=2, \tag{82}$$

$$\gamma_k\stackrel{\text{def}}{=}\gamma_{k,\ell}=\sum_{p=1,p\neq k}^{P}c_pt_{k,p},\qquad k=1,..,P,\ \ \ell=P+1,...N \text{ and } P>1, \tag{83}$$

$$\gamma_1=\frac{(\lambda/\lambda_1)}{N(N+1)}{}_2F_1\left(2,N,N+2,1-\frac{\lambda}{\lambda_1}\right),\qquad P=1, \tag{84}$$

with

$$t_{k,\ell,p}\stackrel{\text{def}}{=}\lambda_k\lambda_\ell\left[\frac{1}{N'\left(\lambda_k-\lambda_p\right)\left(\lambda_p-\lambda_\ell\right)}{}_2F_1\left(1,N',N'+1,1-\frac{\lambda}{\lambda_p}\right)\right.$$
$$+\frac{\lambda_k\left(\lambda_p-\lambda_\ell\right)}{(N'+1)\lambda_p^2\left(\lambda_k-\lambda_\ell\right)^2}{}_2F_1\left(1,N',N'+2,1-\frac{\lambda}{\lambda_k}\right)$$
$$+\frac{\lambda_\ell\left(\lambda_p-\lambda_k\right)}{(N'+1)\lambda_p^2\left(\lambda_\ell-\lambda_k\right)^2}{}_2F_1\left(1,N',N'+2,1-\frac{\lambda}{\lambda_\ell}\right)$$
$$-\frac{\left(\lambda_p-\lambda_\ell\right)\left(\lambda_k^4-2\lambda_k^2\lambda_p\lambda_\ell+\lambda_k^3\lambda_\ell\right)}{N'\lambda_p^3\left(\lambda_k-\lambda_p\right)\left(\lambda_k-\lambda_\ell\right)^3}{}_2F_1\left(1,N',N'+1,1-\frac{\lambda}{\lambda_k}\right)$$
$$\left.-\frac{\lambda_\ell^2\left(\lambda_k-\lambda_p\right)\left(2\lambda_k\lambda_p-\lambda_k\lambda_\ell-\lambda_\ell^2\right)}{N'\lambda_p^3\left(\lambda_k-\lambda_\ell\right)^3\left(\lambda_p-\lambda_\ell\right)}{}_2F_1\left(1,N',N'+1,1-\frac{\lambda}{\lambda_\ell}\right)\right]$$

and

$$t_{k,p}\stackrel{\text{def}}{=}\lambda_k\lambda\left[\frac{1}{(N'+2)\lambda_p\lambda_k}{}_2F_1\left(1,N'+1,N'+3,1-\frac{\lambda}{\lambda_k}\right)\right.$$
$$+\frac{1}{(N'+1)(\lambda_k-\lambda_p)\lambda_p}{}_2F_1\left(1,N'+1,N'+2,1-\frac{\lambda}{\lambda_p}\right)$$
$$\left.-\frac{1}{(N'+1)(\lambda_k-\lambda_p)\lambda_p}{}_2F_1\left(1,N'+1,N'+2,1-\frac{\lambda}{\lambda_k}\right)\right],$$

where $N'\stackrel{\text{def}}{=}N-P$, $c_p\stackrel{\text{def}}{=}\prod_{j=1,j\neq p}^{P}\left(1-\frac{\lambda_j}{\lambda_p}\right)^{-1}$, $c'_p\stackrel{\text{def}}{=}(1-\frac{\lambda_k}{\lambda_p})c_p$ and the Gauss hypergeometric functions ${}_2F_1\,(1,\ell,\ell+1,s)$, ${}_2F_1\,(1,\ell,\ell+2,s)$ and ${}_2F_1\,(2,\ell,\ell+3,s)$ have the following explicit expressions obtained using partial fraction expansions:

$${}_2F_1(1,\ell,\ell+1,s)\stackrel{\text{def}}{=}\frac{1}{B(\ell,1)}\int_0^1\frac{t^{\ell-1}}{1-st}dt$$

$$= \frac{1}{B(\ell,1)}\left[-\frac{\log(1-s)}{s^\ell} - \left(\sum_{k=1}^{\ell-1}\frac{1}{ks^{\ell-k}}\right)\mathbb{1}_{\ell>1}\right], \tag{85}$$

$${}_2F_1(1,\ell,\ell+2,s) \stackrel{\text{def}}{=} \frac{1}{B(\ell,2)}\int_0^1 \frac{t^{\ell-1}(1-t)}{1-st}dt$$

$$= \frac{1}{B(\ell,2)}\left[\frac{1}{\ell s} + \frac{(1-s)\log(1-s)}{s^{\ell+1}} + \left(\sum_{k=1}^{\ell-1}\frac{1-s}{ks^{\ell-k+1}}\right)\mathbb{1}_{\ell>1}\right], \tag{86}$$

$${}_2F_1(2,\ell,\ell+3,s) \stackrel{\text{def}}{=} \frac{1}{B(\ell,3)}\int_0^1 \frac{t^{\ell-1}(1-t)^2}{(1-st)^2}dt$$

$$= \frac{1}{B(\ell,3)}\left[\frac{1}{\ell s^2} + \frac{1-s}{s^{\ell+1}} + \frac{(1-s)(s+1-s\ell+\ell)\log(1-s)}{s^{\ell+2}}\right.$$
$$\left. + \left(\frac{2(1-s)}{(\ell-1)s^3}\right)\mathbb{1}_{\ell>1} - \left(\sum_{k=1}^{\ell-3}\frac{(1-s)[(\ell-k)(s-1)-2s]}{(k+1)s^{\ell-k+1}}\right)\mathbb{1}_{\ell>2}\right], \tag{87}$$

where $B(\ell,1)=\frac{1}{\ell}$, $B(\ell,2)=\frac{1}{\ell(\ell+1)}$ and $B(\ell,3)=\frac{2}{\ell(\ell+1)(\ell+2)}$. ■

Proof of Theorem 5. Note first that the closed-form expressions (52) of r_c and r_{nc} are proved by using the expressions of χ_1, χ, γ_1 and $\widetilde{\chi}_1$, $\widetilde{\chi}$, $\widetilde{\gamma}_1$ given by (75)-(84) and (39)-(43) expressed in terms of Gauss hypergeometric functions ${}_2F_1(a,b,c,s)$, and the symmetric property ${}_2F_1(a,b,c,s) = {}_2F_1(b,a,c,s)$ and the following identity [40, (15.2.20)], i.e.

$$c(1-s)\,{}_2F_1(a,b,c,s) - c\,{}_2F_1(a-1,b,c,s) + (c-b)s\,{}_2F_1(a,b,c+1,s) = 0, \tag{88}$$

by taking $(a,b,c,s) = (N-P+1,1,N+1,1-\frac{\lambda}{\lambda_1})$ for r_c and $(a,b,c,s) = (N-\frac{P}{2}+1,1,N+1,1-\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1})$ for r_{nc}. ■

To prove that the function $r_c(\frac{\lambda}{\lambda_1})$ is monotonically increasing, consider the derivative $\frac{dr_c}{d\rho}$ with $\rho \stackrel{\text{def}}{=} \frac{\lambda}{\lambda_1}$. Using the identity [40, (15.2.1)]:

$$\frac{d}{ds}\,{}_2F_1(a,b,c,s) = \frac{ab}{c}\,{}_2F_1(a+1,b+1,c+1,s), \tag{89}$$

we straightforwardly get from the expression (52) of r_c:

$$\frac{dr_c}{d\rho} = \frac{2(N-P+1)}{N+2}\frac{{}_2F_1(1,a_1,b_1,z_\rho)}{[{}_2F_1(2,a_1,b_1,z_\rho)]^2}$$
$$\times\,[{}_2F_1(1,a_1,b_1,z_\rho)\,{}_2F_1(3,a_1+1,b_1+1,z_\rho) - {}_2F_1(2,a_1+1,b_1+1,z_\rho)\,{}_2F_1(2,a_1,b_1,z_\rho)], \tag{90}$$

with $a_1 = N-P+1$, $b_1 = N+2$ and $z_\rho = 1-\rho$ and ${}_2F_1(\sigma,a_1,b_1,z_\rho) = \frac{1}{B(a_1,b_1-a_1)}\int_0^1 \frac{x^{a_1-1}(1-x)^{b_1-a_1-1}}{(1-xz_\rho)^\sigma}dx$, while ${}_2F_1(1,a_1,b_1,z_\rho)\,{}_2F_1(3,a_1+1,b_1+1,z_\rho) - {}_2F_1(2,a_1+1,b_1+1,z_\rho)\,{}_2F_1(2,a_1,b_1,z_\rho) \geq 0$ thanks to the following inequality [41, chap. IX, rel(1.1)]

$$\int_0^1 p(x)dx\int_0^1 p(x)f(x)g(x)dx \geq \int_0^1 p(x)f(x)dx\int_0^1 p(x)g(x)dx,$$

with $p(x) = \frac{x^{a_1}(1-x)^{b_1-a_1-1}}{(1-xz_\rho)^3}$, $f(x) = 1-xz_\rho$ and $g(x) = \frac{1-xz_\rho}{x}$ where the function $p(x)$ is positive, while the functions $f(x)$ and $g(x)$ are monotone decreasing for fixed $0<z_\rho<1$ and $0<x<1$. Consequently $\frac{dr_c}{d\rho} \geq 0$. The proof for $\rho \stackrel{\text{def}}{=} \frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}$ follows the same steps. ■

For the first special case of λ/λ_1 and $\widetilde{\lambda}/\widetilde{\lambda}_1$ close to one, (53) and (54) are proved starting from its exact expressions given in (52) by using the third-order expansion

$${}_2F_1(a,b,c,s) = 1 + \frac{ab}{c}s + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{s^2}{2} + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{s^3}{6} + o(s^3), \tag{91}$$

derived from (89).

For the second special case of λ/λ_1 and $\widetilde{\lambda}/\widetilde{\lambda}_1$ close to zero, we note first that ${}_2F_1(a,b,c,1)$ is not defined for $a+b \geq c$, and thus ${}_2F_1(2,N-P+1,N+2,1-\frac{\lambda}{\lambda_1})$ [resp. ${}_2F_1(2,N-\frac{P}{2}+1,N+2,1-\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1})$] in the expression of r_c [resp. r_{nc}] is not defined for $P=1$ [resp. $P=1,2$]. Taking the limit as $\frac{\lambda}{\lambda_1}$ and $\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}$ tend to 0 and using identity

$${}_2F_1(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-a)\Gamma(c-b)} \tag{92}$$

provided $a+b<c$ [40, (15.1.20)] proves the dominant terms $(1+\frac{1}{N})(1-\frac{1}{P})$ (55) for $P>1$ and $(1+\frac{1}{N})(1-\frac{2}{P})$ (56) for $P>2$.

Using the identity [40, (15.3.3)], i.e.

$$_2F_1(a,b,c,s) = (1-s)^{c-a-b}_2 F_1(c-a,c-b,c,s),$$

with $(a,b,c,s) = (N, \frac{3}{2}, N+2, 1-\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1})$ in (52), we get

$$r_{nc} = \left(\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1}\right)^{1/2} \frac{[_2F_1(1, N+\frac{1}{2}, N+2, 1-\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1})]^2}{_2F_1(N, \frac{3}{2}, N+2, 1-\frac{\widetilde{\lambda}}{\widetilde{\lambda}_1})},$$

where $\frac{[_2F_1(1,N+\frac{1}{2},N+2,1)]^2}{_2F_1(N,\frac{3}{2},N+2,1)} = \frac{4\Gamma(N+\frac{1}{2})}{\sqrt{\pi}\Gamma(N)}$ thanks to (92). Thus the dominant term of (58) for $P=1$ is proved. Furthermore, the expressions of $o_{N,P}(1)$ (57) and $\widetilde{o}_{N,P}(1)$ (58) for $P>1$ are obtained using symbolic mathematical software. ■

References

[1] H. Krim, M. Viberg, Two decades of array signal processing research: the parametric approach, IEEE ASSP Mag. 13 (4) (1996) 67–94.
[2] R.A. Maronna, Robust *M*-estimators of multivariate location and scatter, Ann. Stat. 4 (1976) 51–67.
[3] D.E. Tyler, Robustness and efficiency properties of scatter matrices, Biometrika 70 (2) (1983) 411–420.
[4] D.E. Tyler, A distribution-free *M*-estimator of multivariate scatter, Ann. Stat. 15 (1) (Nov. 1987) 234–251.
[5] E. Ollila, V. Koivunen, Robust antenna array processing using *M*-estimators of pseudo covariance, in: 14th International Symposium on Personal Indoor and Mobile Radio Communication, 2003.
[6] M. Mahot, F. Pascal, P. Forster, J.-P. Ovarlez, Asymptotic properties of robust complex covariance matrix estimates, IEEE Trans. Signal Process. 61 (13) (July 2013) 3348–3356.
[7] B. Mériaux, C. Ren, M.N.E. Korso, A. Breloy, P. Forster, Asymptotic performance of complex *M*-estimators for multivariate location and scatter estimation, IEEE Signal Process. Lett. 26 (2) (Feb. 2019) 367–371.
[8] P.J. Huber, E.M. Ronchetti, Robust Statistics, 2nd ed., Wiley, New York, NY, USA, 2009.
[9] E. Conte, M. Lops, G. Ricci, Adaptive radar detection in compound-Gaussian clutter, in: Proceedings of Eusipco, Edinburgh, Scotland, Sep. 1994, pp. 526–529.
[10] N. Locantore, J.S. Marron, D.G. Simpson, N. Tripoli, J.T. Zhang, K.L. Cohen, Robust principle component analysis for functional data, Test 8 (1999) 1–73.
[11] S. Visuri, V. Koivunen, H. Oja, Sign and rank covariance matrices, J. Stat. Plan. Inference 91 (2000) 557–575.
[12] E. Ollila, H. Oja, C. Croux, The affine equivariant sign covariance matrix: asymptotic behavior and efficiencies, J. Multivar. Anal. 87 (2003) 328–355.
[13] S. Visuri, H. Oja, V. Koivunen, Subspace-based direction of arrival estimation using nonparametric statistics, IEEE Trans. Signal Process. 49 (9) (Sept. 2001) 2060–2073.
[14] F. Gini, M.V. Greco, L. Verrazzani, Detection problem in mixed clutter environment as a Gaussian problem by adaptive pre-processing, Electron. Lett. 31 (14) (Jul. 1995) 1189–1190.
[15] F. Bandiera, D. Orlando, G. Ricci, CFAR detection of extended and multiple point-like targets without assignment of secondary data, IEEE Signal Process. Lett. 13 (4) (April 2006) 240–243.
[16] A. Younsi, A.M. Zoubir, A. Ouldali, Robust adaptive matched subspace CFAR detector for Gaussian signals, in: Proceedings of ICASSP, Toulouse, France, May 2006.
[17] G. Ginolhac, P. Forster, J.-P. Ovarlez, F. Pascal, Spatio-temporal adaptive detector in non-homogeneous and low-rank clutter, in: Proceedings of ICASSP, Taipei, Taiwan, R.O.C, April 2009.
[18] E. Ollila, D.E. Tyler, V. Koivunen, H.V. Poor, Complex elliptically symmetric distributions: survey, new results and applications, IEEE Trans. Signal Process. 60 (11) (Nov. 2012) 5597–5625.
[19] A.F. Magyar, The efficiencies of the spatial median and spatial sign covariance matrix for elliptically symmetric distributions, D.Sc. dissertation, State University of New Jersey, New Brunswick, May 2012.
[20] A.F. Magyar, D.E. Tyler, The asymptotic inadmissibility of the spatial sign covariance matrix for the elliptically symmetric distributions, Biometrika 101 (3) (2014) 673–688.
[21] A. Durre, D.E. Tyler, D. Vogel, On the eigenvalues of the spatial sign covariance matrix in more than two dimensions, Stat. Probab. Lett. 111 (2016) 80–85.
[22] S. Bausson, F. Pascal, P. Forster, J.-P. Ovarlez, P. Larzabal, First- and second-order moments of the normalized sample covariance matrix of spherically invariant random vectors, IEEE Signal Process. Lett. 14 (6) (June 2007) 425–428.
[23] S. Bausson, P. Forster, MUSIC and model-order selection for spherically invariant random vectors, in: Proceedings of Asilomar Conference on Signals, Systems and Computers, Pacific Grove, CA, USA, Nov. 2006.
[24] G. Ginolhac, P. Forster, Performance analysis of a robust low-rank STAP filter in low-rank Gaussian clutter, in: Proceedings of ICASSP, Dallas, Texas, March 2010.
[25] E. Raninen, E. Ollila, Bias adjusted sign covariance matrix, IEEE Signal Process. Lett. (Dec. 2021).
[26] E. Ollila, V. Koivunen, Generalized complex elliptical distributions, in: Proc. SAM Workshop, July 2004, pp. 460–464.
[27] H. Abeida, J.-P. Delmas, Efficiency of subspace-based estimators for elliptical symmetric distributions, Signal Process. 174 (Oct. 2019).
[28] D.E. Tyler, Statistical analysis for the angular central Gaussian distribution on the sphere, Biometrika 74 (1987) 579–589.
[29] E. Ollila, D.E. Tyler, Distribution free detection under complex elliptically symmetric clutter distributions, in: Proc. SAM Workshop, June 2012.
[30] H. Abeida, J.-P. Delmas, MUSIC-like estimation of direction of arrival for noncircular sources, IEEE Trans. Signal Process. 54 (7) (July 2006) 2678–2690.
[31] P.J. Schreier, L.L. Scharf, Statistical Signal Processing of Complex-Valued Data: the Theory of Improper and Noncircular Signals, Cambridge University Press, 2010.
[32] T. Kato, Perturbation Theory for Linear Operators, Springer, Berlin, 1995.
[33] H. Krim, P. Forster, G. Proakis, Operator approach to performance analysis of root-MUSIC and root-min-norm, IEEE Trans. Signal Process. 40 (7) (Jul. 1992) 1687–1696.
[34] R.J. Serfling, Approximation Theorems of Mathematical Statistics, John Wiley and Sons, 1980.
[35] B. Porat, B. Friedlander, Asymptotic accuracy of ARMA parameter estimation methods based on sample covariances, in: Proc. 7th IFAC/IFORS Symposium on Identification and System Parameter Estimation, York, 1985.
[36] P. Stoica, B. Friedlander, T. Söderström, An approximate maximum approach to ARMA spectral estimation, in: Proc. Decision and Control, Fort Lauderdale, 1985.
[37] H. Abeida, J.-P. Delmas, Robustness of subspace-based algorithms with respect to the distribution of the noise: application to DOA estimation, Signal Process. 164 (June 2019) 313–319.
[38] P. Stoica, A. Nehorai, MUSIC, maximum likelihood, and Cramer-Rao bound, IEEE Trans. Acoust. Speech Signal Process. 37 (5) (May 1989) 720–741.
[39] H. Abeida, J.-P. Delmas, Slepian-bangs formula and Cramér Rao bound for circular and non-circular complex elliptical symmetric distributions, IEEE Signal Process. Lett. 26 (10) (Oct. 2019) 1561–1565.
[40] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Doover Publications Inc., New York, 1992, Reprint of the 1972 edition.
[41] D.S. Mitrinovic, J.E. Pecaric, A.M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, 1993.

Habti Abeida received a B.Sc. degree in engineering and applied mathematics from Hassan II University, Casablanca, Morocco, in 2000, and Paris Descartes University, Paris, France, in 2001, the M.S. degree in statistics and signal processing from Pierre and Marie Curie University, Paris, in 2002, and the Ph.D. degree in signal processing and applied mathematics from Pierre and Marie Curie University, in collaboration with Telecom SudParis, France, in 2006. He was a Lecturer with Paris-Dauphine and the University of Paris 8 Vincennes-Saint-Denis, in 2006 and 2007, respectively. He was a

Postdoctoral Research Associate with the Superior School of Electricity, Supelec, Paris, in 2008, the GIPAS-lab, Grenoble, France, in 2009, the Department of Electrical Engineering, KFUPM, Saudi Arabia, in 2010, and the Spectral Analysis Laboratory, University of Florida, USA, in 2011. He is currently working as an Associate Professor with the Department of Electrical Engineering, Taif University, Saudi Arabia. His research interests include the areas of statistical signal processing with application to the antenna array, radar application, and communication systems.

Jean-Pierre Delmas received the engineering degree from Ecole Centrale de Lyon, France in 1973, the Certificat d'Etudes Supérieures from Ecole Nationale Supérieure des Télécommunications, Paris, France in 1982 and the Habilitation à diriger des recherches degree from the University of Paris XI, Orsay, France in 2001. Since 1980, he has been with Telecom SudParis where he is currently a Professor with the CITI department. He was the deputy director (2005-2010) and the director (2011-2014) of UMR 5157 (CNRS laboratory). His teaching and research interest lie in statistical methods for signal processing with emphasis on asymptotic performances analysis and array processing applied to multi-sensor systems in the context of communications. He is author or co-author of more than 140 publications (journal, conference and chapter of book, book). He was an Associate Editor for the IEEE Transactions on Signal Processing (2002-2006) and (2010-2014), for Signal Processing (Elsevier) (2009-2020), and currently for IEEE Signal Processing Letters. From 2011 to 2016, he was a member of the IEEE Sensor Array and Multichannel Technical Committee.