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Efficiency of subspace-based estimators for elliptical symmetric distributions

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ABSTRACT

Subspace-based algorithms that exploit the orthogonality between a sample subspace and a parameter-dependent subspace have proved very useful in many applications in signal processing. The purpose of this paper is to complement theoretical results already available on the asymptotic (in the number of measurements) performance of subspace-based estimators derived in the Gaussian context to real elliptical symmetric (RES), circular complex elliptical symmetric (C-CES) and non-circular CES (NC-CES) distributed observations in the same framework. First, the asymptotic distribution of *M*-estimates of the orthogonal projection matrix is derived from those of the *M*-estimates of the covariance matrix. This allows us to characterize the asymptotically minimum variance (AMV) estimator based on estimates of orthogonal projectors associated with different *M*-estimates of the covariance matrix. A closed-form expression is then given for the AMV bound on the parameter of interest characterized by the column subspace of the mixing matrix of general linear mixture models. We also specify the conditions under which the AMV bound based on Tyler's *M*-estimate attains the stochastic Cramér-Rao bound (CRB) for the complex Student *t* and complex generalized Gaussian distributions. Finally, we prove that the AMV bound attains the stochastic CRB in the case of maximum likelihood (ML) *M*-estimate of the covariance matrix for RES, C-CES and NC-CES distributed observations, which is equal to the semiparametric CRB (SCRB) recently introduced.

1. Introduction

Noisy linear mixtures of signals in which the parameter of interest is characterized by the mixing matrix are very common in many applications, including array processing and linear system identification (see e.g., [1–3]). To get rid of the nuisance parameters, subspace-based estimates obtained by exploiting the orthogonality between a sample subspace and a parameter-dependent subspace have been exploited since the seminal paper [4] that introduces the multiple signal classification (MUSIC) algorithm for direction of arrival (DOA) estimation. These methods are always the object of active research in many applications (see e.g., [5,6]), with generally many possible algorithms (see e.g., [7] for special structures of the mixing matrix). In these noisy linear mixtures, two statistical models have been commonly used [8]. If the signals in the mixture are nonrandom, but rather unknown deterministic parameters, the model is called deterministic or conditional and the associated CRB on the parameter of interest is called deterministic CRB. Otherwise, they are random and the model is a stochastic or unconditional model and the associated CRB is called stochastic CRB. Note, however, that, the deterministic CRB is not asymptotically achievable by the maximum likelihood estimator with respect to the number of snapshots, while the stochastic CRB is attainable. Considering the family of subspace-based estimators, it was proved [9] in the context of DOA estimation for circular (C-CG) and generally non-circular complex Gaussian (NC-CG) observations, that there exists among these estimators, an AMV estimator or an asymptotically best consistent estimators (ABC) introduced by Porat and Friedlander [10] and Stoica *et al* [11], respectively, whose covariance attains the stochastic CRB.

We are mainly interested in this paper, to extend the previous results in [9] to both (i) generic noisy linear mixture whose parameters of interest are characterized by the columns space of the mixing matrix, (ii) orthogonal projectors derived from the principal subspace of different *M*-estimates of the covariance, and (iii) RES (see e.g. [12]), C-CES (see e.g. [13]) and NC-CES [14] (introduced in[15] under the name Generalized CES) distributed observations. First, we extend to NC-CES distributions, the asymptotic distribution of the *M*-estimate of the covariance as well as the

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asymptotic distribution of the associated projectors. This allows us to consider the RES, C-CES and NC-CES distributions in the same framework and to give a common closed-form expression of the AMV bound on the parameter of interest based on the projectors. We prove in particular that the AMV bound attains the stochastic CRB in the case of ML M-estimate of the covariance matrix for all RES, C-CES and NC-CES distributions with finite fourth-order moments. We specify the conditions under which the AMV bound associated with the projector derived from Tyler's M-estimate attains the stochastic CRB for the complex Student t and complex generalized Gaussian distributions. Finally, we prove that the SCRB introduced in [16] and the stochastic CRB for the parameters of interest depending on the covariance matrix resulting from a noisy linear mixture model are equal.

This paper is organized as follows. Section 2 specifies the general parametric model of RES, C-CES, and NC-CES distributed noisy mixtures and formalizes any subspace-based algorithm as a mapping linking an M-estimate of the covariance matrix to the estimate of the parameter of interest. The problem formulation and a brief review of AMV estimators are given in Section 3. Section 4 reviews different properties of M-estimates of the covariance matrix for RES and C-CES distributions and extends them to NC-CES distributions. This allows us to deduce the asymptotic distribution of the associated M-estimates of the orthogonal projection matrices and then derive a closed-form expression of the AMV bound based on projector statistics, enabling us to prove that this one attains the CRB in the case of ML M-estimate of the covariance matrix for all RES, C-CES and NC-CES distributions. Section 5 presents simulation results to validate the theoretical results, and finally this paper is concluded in Section 6.

The notations used throughout this paper are the following. Vectors and matrices are denoted by bold-faced lowercase and uppercase letters, respectively. *, ^{T}, and ^{H} respectively represent the conjugate, the transpose and the conjugate transpose operators and the symbol $^+$ stands for ^{T} in the real case and for ^{H} in the complex case. $|.|$, $(.)^\#$ and span(.) are the determinant, Moore-Penrose inverse and range space of a matrix, respectively. $\to_d$ denotes convergence in distribution, $\sim$ means "distributed as" and $=_d$ stands for "shares the same distribution as". $\mathcal{N}_R(\mathbf{0},\mathbf{R})$ and $\mathcal{N}_C(\mathbf{0},\mathbf{R},\mathbf{C})$ denote the zero-mean real (resp., complex) valued Gaussian distributions, where $\mathbf{R}$ and $\mathbf{C}$ are the covariance and complementary covariance matrices, respectively. vec$(\cdot)$ is the vectorization operator that turns a matrix into a vector by stacking the columns of the matrix one below another which is used in conjunction with the Kronecker product $\mathbf{A}\otimes\mathbf{B}$ as the block matrix whose (i,j) block element is $a_{i,j}\mathbf{B}$ and with the vec-permutation matrix $\mathbf{K}_q$ which transforms vec($\mathbf{C}$) to vec($\mathbf{C}^T$) for any $q\times q$ matrix $\mathbf{C}$. The matrix $\mathbf{J}$ is the exchange matrix $\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{pmatrix}$.

2. Data model and subspace-based estimation

2.1. General parametric model

Assume that you have a set of K independent and identically distributed zero-mean N-dimensional RES, C-CES or NC-CES distributed data snapshots $(\mathbf{y}_k)_{k=1,..,K}$, such that the probability density function (p.d.f.) can be written[1] as:

$$p(\mathbf{y}_k)=|\boldsymbol{\Sigma}|^{-1/2}g_r\left(\mathbf{y}_k^T\boldsymbol{\Sigma}^{-1}\mathbf{y}_k\right)\qquad\text{(real case)},\tag{1}$$

$$=|\boldsymbol{\Sigma}|^{-1}g_c\left(\mathbf{y}_k^H\boldsymbol{\Sigma}^{-1}\mathbf{y}_k\right)\qquad\text{(circular complex case)},\tag{2}$$

$$=|\widetilde{\boldsymbol{\Gamma}}|^{-1/2}g_c\left(\frac{1}{2}\tilde{\mathbf{y}}_k^H\widetilde{\boldsymbol{\Gamma}}^{-1}\tilde{\mathbf{y}}_k\right)\qquad\text{(non-circular complex case)},\tag{3}$$

where $\tilde{\mathbf{y}}_k\stackrel{\text{def}}{=}(\mathbf{y}_k^T,\mathbf{y}_k^H)^T$ and $\widetilde{\boldsymbol{\Gamma}}\stackrel{\text{def}}{=}\begin{pmatrix}\boldsymbol{\Sigma} & \boldsymbol{\Omega}\\ \boldsymbol{\Omega}^* & \boldsymbol{\Sigma}^*\end{pmatrix}$ with $\boldsymbol{\Sigma}$ and $\boldsymbol{\Omega}$ are $N\times N$ Hermitian positive definite and complex symmetric matrices, respectively called scatter and pseudo-scatter matrices. The functions $g_r(.)$ and $g_c(.):\mathbb{R}^+\mapsto\mathbb{R}^+$ satisfy $\delta_{N,g_r}\stackrel{\text{def}}{=}\int_0^\infty t^{N/2-1}g_r(t)dt<\infty$ and $\delta_{N,g_c}\stackrel{\text{def}}{=}\int_0^\infty t^{N-1}g_c(t)dt<\infty$. The r.v. $\mathbf{y}_k$ admits the following stochastic representation:

$$\mathbf{y}_k=_d\sqrt{\mathcal{Q}_k}\mathbf{T}\mathbf{u}_k,\qquad\text{(real [12] and circular complex [13] cases)},\tag{4}$$

$$=_d\sqrt{\mathcal{Q}_k}\mathbf{T}\mathbf{v}_k,\qquad\text{(non-circular complex case [14])},\tag{5}$$

where the random variables $\mathcal{Q}_k$ and $\mathbf{u}_k$ [resp. $\mathcal{Q}_k$ and $\mathbf{v}_k$] are independent. $\mathbf{u}_k$ is uniformly distributed on the unit real or complex N-sphere and $\mathbf{v}_k$ is defined by [14] $\mathbf{v}_k=\boldsymbol{\Delta}_1\mathbf{u}_k+\boldsymbol{\Delta}_2\mathbf{u}_k^*$, where $\boldsymbol{\Delta}_1\stackrel{\text{def}}{=}\frac{\boldsymbol{\Delta}_++\boldsymbol{\Delta}_-}{2}$, $\boldsymbol{\Delta}_2\stackrel{\text{def}}{=}\frac{\boldsymbol{\Delta}_+-\boldsymbol{\Delta}_-}{2}$, $\boldsymbol{\Delta}_+\stackrel{\text{def}}{=}\sqrt{\mathbf{I}+\boldsymbol{\Delta}_\kappa}$ and $\boldsymbol{\Delta}_-\stackrel{\text{def}}{=}\sqrt{\mathbf{I}-\boldsymbol{\Delta}_\kappa}$, with $\boldsymbol{\Delta}_\kappa$ is an $N\times N$ diagonal matrix containing the non-circularity coefficients $(\kappa_n)_{n=1,..,N}$ of $\mathbf{y}_k$ [17] satisfying $0\le\kappa_n\le1$, and $\boldsymbol{\Sigma}=\mathbf{T}\mathbf{T}^H$ and $\boldsymbol{\Omega}=\mathbf{T}\boldsymbol{\Delta}_\kappa\mathbf{T}^T$ are factorizations of $\boldsymbol{\Sigma}$ and $\boldsymbol{\Omega}$, respectively, where $\mathbf{T}$ has full rank. We note that Eq. (5) is equivalent to $\tilde{\mathbf{y}}_k=_d\sqrt{\mathcal{Q}_k}\widetilde{\boldsymbol{\Gamma}}^{1/2}\tilde{\mathbf{u}}_k$ with $\tilde{\mathbf{u}}_k\stackrel{\text{def}}{=}(\mathbf{u}_k^T,\mathbf{u}_k^H)^T$ and that in the complex circular case $\mathbf{v}_k$ and Eq. (3) reduce to $\mathbf{u}_k$ and Eq. (2), respectively.

It follows from Eqs. (4) and (5) that the quadratic/Hermitian forms

$$\mathbf{y}_k^+\boldsymbol{\Sigma}^{-1}\mathbf{y}_k=_d\mathcal{Q}_k\qquad\text{(real and complex circular cases)},\tag{6}$$

$$\frac{1}{2}\tilde{\mathbf{y}}_k^H\widetilde{\boldsymbol{\Gamma}}^{-1}\tilde{\mathbf{y}}_k=_d\mathcal{Q}_k\qquad\text{(non-circular complex case)},\tag{7}$$

and hence the p.d.f. of the 2nd-order modular variate $\mathcal{Q}_k$ (or the quadratic/Hermitian forms) is given by

$$p(q_k)=\delta_{N,g_r}^{-1}q_k^{N/2-1}g_r(q_k)\qquad\text{(real case)},\tag{8}$$

$$=\delta_{N,g_c}^{-1}q_k^{N-1}g_c(q_k)\qquad\text{(complex case)}.\tag{9}$$

Furthermore, to remove the so-called scale ambiguity, the density generators g_r and g_c are here constrained such that $\delta_{N+1,g_r}/\delta_{N,g_r}=\delta_{N+1,g_c}/\delta_{N,g_c}=N$ or equivalently $\mathrm{E}(\mathcal{Q}_k)=N$ given that 2nd-order moments exist [13, (20)], to ensure that the scatter matrix $\boldsymbol{\Sigma}$ and the extended scatter matrix $\widetilde{\boldsymbol{\Gamma}}$ are equal to the covariance matrix $\mathbf{R}_y\stackrel{\text{def}}{=}\mathrm{E}(\mathbf{y}_k\mathbf{y}_k^H)$ and the extended covariance matrix $\mathbf{R}_{\tilde{y}}\stackrel{\text{def}}{=}\mathrm{E}(\tilde{\mathbf{y}}_k\tilde{\mathbf{y}}_k^H)$, respectively.

We assume that the covariance $\boldsymbol{\Sigma}$ matrix in Eqs. (1) and (2) takes the following structured form:

$$\boldsymbol{\Sigma}=\mathbf{A}(\boldsymbol{\theta})\mathbf{R}_x\mathbf{A}^+(\boldsymbol{\theta})+\sigma_n^2\mathbf{I},\tag{10}$$

where $\mathbf{R}_x$ is a $P\times P$ (with $P<N$) positive definite, real-valued symmetric or Hermitian matrix in the real and circular complex case, respectively. In the non-circular complex case, we assume that the extended covariance matrix $\widetilde{\boldsymbol{\Gamma}}$ in Eq. (3) takes one of the following structured forms:

$$\widetilde{\boldsymbol{\Gamma}}=\widetilde{\mathbf{A}}_r(\boldsymbol{\theta})\mathbf{R}_r\widetilde{\mathbf{A}}_r^H(\boldsymbol{\theta})+\sigma_n^2\mathbf{I},\tag{11}$$

$$\widetilde{\boldsymbol{\Gamma}}=\widetilde{\mathbf{A}}_c(\boldsymbol{\theta})\mathbf{R}_{\tilde{x}}\widetilde{\mathbf{A}}_c^H(\boldsymbol{\theta})+\sigma_n^2\mathbf{I},\tag{12}$$

where $\mathbf{R}_r$ is a $P\times P$ (with $P<2N$) positive definite, real-valued symmetric matrix and $\mathbf{R}_{\tilde{x}}$ is a $2P\times2P$ (with $P<N$) positive definite Hermitian matrix structured as $\begin{pmatrix}\mathbf{R}_x & \mathbf{C}_x\\ \mathbf{C}_x^* & \mathbf{R}_x^*\end{pmatrix}$. $\widetilde{\mathbf{A}}_r(\boldsymbol{\theta})$ and $\widetilde{\mathbf{A}}_c(\boldsymbol{\theta})$ are

[1] These expressions are consistent with the ones given in [12], [13] and [14] for the RES, C-CES and NC-CES, respectively, because the normalizing constant is here included in the functions g_r and g_c.

structured $2N \times P$ and $2N \times 2P$ matrices, respectively, with $\widetilde{\mathbf{A}}_r(\boldsymbol{\theta}) = \begin{pmatrix} \mathbf{A}(\boldsymbol{\theta}) \\ \mathbf{A}^*(\boldsymbol{\theta}) \end{pmatrix}$ and $\widetilde{\mathbf{A}}_c(\boldsymbol{\theta}) = \begin{pmatrix} \mathbf{A}(\boldsymbol{\theta}) & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^*(\boldsymbol{\theta}) \end{pmatrix}$.

We assume that the real-valued parameter of interest $\boldsymbol{\theta} \in \mathbb{R}^L$ is characterized by the subspace generated by the columns of the full column rank matrices $\mathbf{A}(\boldsymbol{\theta})$, $\widetilde{\mathbf{A}}_r(\boldsymbol{\theta})$ and $\widetilde{\mathbf{A}}_c(\boldsymbol{\theta})$. The nuisance parameters are $\boldsymbol{\rho}$ and σ_n^2 where $\boldsymbol{\rho}$ collects the real and imaginary parts of the unknown matrices $\mathbf{R}_x$, $\mathbf{R}_r$ or $\mathbf{R}_{\tilde{x}}$.

This case of a low-rank plus identity covariance matrix is commonly used in signal processing to account for low dimensional signals embedded in white noise. This is in particular the case of the general noisy linear mixture model:

$$\mathbf{y}_k = \mathbf{A}(\boldsymbol{\theta})\mathbf{x}_k + \mathbf{n}_k. \quad (13)$$

This low-rank signal in full-rank noise data model Eq.(13) encompasses many far or near-field, narrow or wide-band DOA models with scalar or vector-sensors for an arbitrary number of parameters per source $x_{k,p}$ (with $\mathbf{x}_k \stackrel{\text{def}}{=} (x_{k,1}, .., x_{k,P})^T$) and many other models as the bandlimited SISO, SIMO [2] and MIMO [3] channel models. For example, parametrization Eq. (11) can be applied for DOA estimation modeling with rectilinear or strictly second-order sources and for SIMO channels estimation modeling with BPSK or MSK symbols [18] where $\boldsymbol{\theta}$ represents both the localization parameters (azimuth, elevation, range) and the phase of the sources, and the real and imaginary parts of channel impulse response coefficients, respectively. Whereas, parametrization Eq. (12) is used for DOA modeling with generally non-circular complex sources.

We note that $\mathbf{x}_k$ and $\mathbf{n}_k$ cannot be both elliptical symmetric distributed as the family of elliptical symmetric distributions is not closed under summation except for the Gaussian distribution. But fixing both the structure of the covariance matrices $\boldsymbol{\Sigma}$ (10) or $\widetilde{\boldsymbol{\Gamma}}$ Eqs. (11) and (12) and the elliptical symmetric distribution of $\mathbf{y}_k$ Eqs. (1), (2) or (3) can be considered as good approximations thanks to the flexibility of the family of the elliptical symmetric distributions. Furthermore, this family of distributions offers robustness to outliers and heavy tailed samples.

2.2. Subspace-based estimation

Since the parameter of interest $\boldsymbol{\theta}$ is characterized by the subspace generated by the columns of the full column rank matrices $\mathbf{A}(\boldsymbol{\theta})$, $\widetilde{\mathbf{A}}_r(\boldsymbol{\theta})$ or $\widetilde{\mathbf{A}}_c(\boldsymbol{\theta})$, a simple way to get rid of the nuisance parameters $\boldsymbol{\rho}$ and σ_k^2, is to consider subspace-based algorithms as the following mapping:

$$(\mathbf{y}_1, .., \mathbf{y}_k, .., \mathbf{y}_K) \longmapsto \mathbf{R}_K \longmapsto \boldsymbol{\Pi}_K \stackrel{\text{alg}}{\longmapsto} \widehat{\boldsymbol{\theta}}_K, \quad (14)$$

where $\mathbf{R}_K$ can be either any estimate $\mathbf{R}_{y,K}$ of $\mathbf{R}_y$ or any estimate $\mathbf{R}_{\tilde{y},K}$ of $\mathbf{R}_{\tilde{y}} \stackrel{\text{def}}{=} \mathrm{E}(\tilde{\mathbf{y}}_k\tilde{\mathbf{y}}_k^H)$, and $\boldsymbol{\Pi}_K$ denotes either the orthogonal projection matrix $\boldsymbol{\Pi}_{y,K}$ associated with the so-called noise subspace of $\mathbf{R}_{y,K}$ or the orthogonal projection matrix $\boldsymbol{\Pi}_{\tilde{y},K}$ associated with the so-called noise subspace of $\mathbf{R}_{\tilde{y},K}$. The functional dependence $\widehat{\boldsymbol{\theta}}_K = \mathrm{alg}(\boldsymbol{\Pi}_K)$ constitutes an extension of the mapping

$$\boldsymbol{\Pi}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \mathbf{I} - \mathbf{B}(\boldsymbol{\theta})[\mathbf{B}^+(\boldsymbol{\theta})\mathbf{B}(\boldsymbol{\theta})]^{-1}\mathbf{B}^+(\boldsymbol{\theta}) \stackrel{\text{alg}}{\longmapsto} \boldsymbol{\theta}, \quad (15)$$

in the neighborhood of $\boldsymbol{\Pi}(\boldsymbol{\theta})$ with $\mathbf{B}(\boldsymbol{\theta})$ can either be $\mathbf{A}(\boldsymbol{\theta})$, $\tilde{\mathbf{A}}_r(\boldsymbol{\theta})$ or $\tilde{\mathbf{A}}_c(\boldsymbol{\theta})$. Each extension alg(.) specifies a particular subspace algorithm, whose conventional MUSIC algorithm [4] based on $\boldsymbol{\Pi}_{y,K}$ and non-circular MUSIC algorithms [19] based on $\boldsymbol{\Pi}_{\tilde{y},K}$ for parametrization Eq. (11) can be seen as examples in DOA estimation.

3. Problem formulation and brief review of AMV estimators

3.1. Problem formulation

The existence of a lower bound for the covariance of the asymptotic distribution of DOA-estimates given by an arbitrary weakly consistent subspace-based algorithm has been proved in [9]. This bound can be used as a benchmark against which to assess the asymptotic statistical accuracy of any subspace-based algorithms. This bound which is itself generally lower bounded by the stochastic CRB derived from the arbitrary likelihood functions related to the observations, and it has been proved in [9] to be equal to the stochastic CRB in the case of circular and non-circular Gaussian observations associated with the parameterizations Eqs. (10) and (12), respectively. The problem, we tackle here, is to extend these results to the subspace-based algorithms built from different M-estimates of scatter matrix in Eq. (10) [resp. in Eqs. (11) and (12)] of RES/C-CES [resp. of NC-CES] distributed observations where the parameter of interest $\boldsymbol{\theta}$ is characterized by the subspace generated either by the columns of the full column rank matrices $\mathbf{A}(\boldsymbol{\theta})$, $\widetilde{\mathbf{A}}_r(\boldsymbol{\theta})$ or $\widetilde{\mathbf{A}}_c(\boldsymbol{\theta})$ for arbitrary parametrizations.

3.2. Brief review of AMV estimators

For the reader's convenience, we briefly summarize here the necessary background of the AMV estimators. Let $\mathbf{s}_{y,K}$ be a sequence of statistics which is a weakly consistent[2] estimate of $\mathbf{s}(\boldsymbol{\theta})$ for which $\boldsymbol{\theta}$ is identifiable from $\mathbf{s}(\boldsymbol{\theta})$. We suppose that $\mathbf{s}_{y,K}$ (function of $(\mathbf{y}_1, \ldots, \mathbf{y}_K)$) is asymptotically Gaussian distributed with zero mean and a possibly singular covariance matrix $\mathbf{R}_s$, i.e., $\sqrt{K}(\mathbf{s}_{y,K} - \mathbf{s}(\boldsymbol{\theta})) \to_d \mathcal{N}_R(\mathbf{0}, \mathbf{R}_s)$ (real case), $\mathcal{N}_C(\mathbf{0}, \mathbf{R}_s, \mathbf{C}_s)$ (complex case). Let $\widehat{\boldsymbol{\theta}}_K$ be an estimator of the unknown parameter $\boldsymbol{\theta}$ defined by a mapping alg(.): $\mathbf{s}_{y,K} \stackrel{\text{alg}}{\longmapsto} \widehat{\boldsymbol{\theta}}_K$, which is differentiable w.r.t. (Re($\mathbf{s}(\boldsymbol{\theta})$), Im($\mathbf{s}(\boldsymbol{\theta})$)) whose differential matrix[3] is denoted by $\mathbf{D}$, we therefore get by the standard theorem of continuity (see e.g., [20, p. 122]) $\sqrt{K}(\widehat{\boldsymbol{\theta}}_K - \boldsymbol{\theta}) \to_d \mathcal{N}_R(\mathbf{0}, \mathbf{R}_{\boldsymbol{\theta}})$ where $\mathbf{R}_{\boldsymbol{\theta}}$ satisfies the following theorem proved in [21].

Theorem 1. *The covariance matrix $\mathbf{R}_{\boldsymbol{\theta}}$ of the asymptotic distribution of a weakly consistent estimate $\widehat{\boldsymbol{\theta}}_K$ of $\boldsymbol{\theta}$ given by any algorithm considered as a differentiable mapping $\mathbf{s}_{y,K} \mapsto \widehat{\boldsymbol{\theta}}_K = \mathrm{alg}(\mathbf{s}_{y,K})$ is bounded below by the real symmetric matrix $\mathbf{R}_{\boldsymbol{\theta}}^{\mathrm{AMV}(\mathbf{s})} = (\mathcal{S}^+\mathbf{R}_s^{\#}\mathcal{S})^{-1}$ with $\mathcal{S} \stackrel{\text{def}}{=} \frac{d\mathbf{s}(\boldsymbol{\theta})}{d\boldsymbol{\theta}}$:*

$$\mathbf{R}_{\boldsymbol{\theta}} = \mathbf{D}\mathbf{R}_s\mathbf{D}^+ \geq (\mathcal{S}^+\mathbf{R}_s^{\#}\mathcal{S})^{-1}, \quad (16)$$

if the following two conditions hold:

$$\mathrm{span}(\mathcal{S}) \subset \mathrm{span}(\mathbf{R}_s) \quad \text{and} \quad \mathbf{s}_{y,K}^* = \mathbf{P}\mathbf{s}_{y,K}, \quad (17)$$

where $\mathbf{P}$ is a permutation matrix.

Furthermore, under the assumptions of Theorem 1, it has been also proved in [21], that the following nonlinear least square estimate achieves the lower bound (16):

$$\widehat{\boldsymbol{\theta}}_K = \arg\min_{\boldsymbol{\omega} \in \mathbb{R}^L} [\mathbf{s}_{y,K} - \mathbf{s}(\boldsymbol{\omega})]^+ \mathbf{R}_s^{\#} [\mathbf{s}_{y,K} - \mathbf{s}(\boldsymbol{\omega})]. \quad (18)$$

We note that the asymptotic covariance of the nonlinear least square estimate Eq. (18) is preserved if the weighting matrix is replaced by any weakly consistent estimate $\mathbf{W}_T$ of $\mathbf{R}_s^{\#}$ [21].

[2] We remind that a sequence of estimators of a parameter is weakly consistent if it converges in probability to this parameter.

[3] This differential matrix $\mathbf{D}$ is defined by the relation $\widehat{\boldsymbol{\theta}}_K = \mathrm{alg}(\mathbf{s}_{y,K}) = \underbrace{\mathrm{alg}(\mathbf{s}(\boldsymbol{\theta}))}_{\boldsymbol{\theta}} + \mathbf{D}(\mathbf{s}_{y,K} - \mathbf{s}(\boldsymbol{\theta})) + o(\mathbf{s}_{y,K} - \mathbf{s}(\boldsymbol{\theta}))$.

4. Efficiency of projector-based estimators

4.1. M-estimate of covariance matrices

Let us first focus our attention on the estimation of the covariance matrix $\mathbf{R}_y$. The practical applications of array processing generally require the use of a sample covariance matrix (SCM) $\mathbf{R}_{y,K} \stackrel{\text{def}}{=} \frac{1}{K}\sum_{k=1}^{K}\mathbf{y}_k\mathbf{y}_k^+$, which is the ML estimator for real or circular complex Gaussian distributed observations. However, the performance of SCM-based subspace algorithms can be drastically degraded in heavy-tailed scenarios, as shown in [13, sec.VII.C] with MUSIC DOA estimation algorithm. In these scenarios, if the density generator $g_r(.)$ of the RES Eq. (1) distributions [resp., $g_c(.)$ of C-CES distributions Eq. (2)] is known, the ML estimate of $\mathbf{R}_y$ is solution of the implicit equation in $\mathbf{\Sigma}_T$:

$$\mathbf{\Sigma}_K = \frac{1}{K}\sum_{k=1}^{K}\phi(\mathbf{y}_k^+\mathbf{\Sigma}_K^{-1}\mathbf{y}_k)\mathbf{y}_k\mathbf{y}_k^+, \tag{19}$$

where $\phi(t) \stackrel{\text{def}}{=} -\frac{2}{g_r(t)}\frac{dg_r(t)}{dt}$ [resp. $\phi(t) \stackrel{\text{def}}{=} -\frac{1}{g_c(t)}\frac{dg_c(t)}{dt}$] for RES [resp. C-CES] distributions. The solution of Eq. (19) is unique. It can be obtained by an iterative fix point algorithm, given any initial symmetric or positive definite Hermitian matrix $\mathbf{\Sigma}_0$ and that the observations, $\mathbf{y}_k$, fulfill certain mild regularity conditions [24] [13, sec.V.A]. When the density generator $g_r(.)$ of the RES distributions and [resp., $g_c(.)$ of the C-CES distributions] is unknown, M-estimators have been proposed to estimate $\mathbf{R}_y$. They are also solutions of the implicit equation Eq.(19), where $\phi(.)$ in Eq. (19) is replaced by a real-valued non-negative weight function $u(.)$ which is not related to a particular RES or C-CES distribution. Tyler's and Huber's M-estimators are examples of such estimators (see e.g., [13, sec.V.C]). Existence and uniqueness of the solution $\mathbf{\Sigma}_K^u$ of Eq. (19) have been proved in the real case provided that $u(.)$ satisfies a set of general conditions (called Maronna conditions) stated by Maronna in [22]. These conditions have been extended to the complex case in [23] and [13]. Under these conditions, it has been also proved in the real case that the solution of Eq. (19) can be derived by an iterative fix point algorithm [24]. The sequence $\mathbf{\Sigma}_K^u$ of solutions of Eq. (19) converges in probability to $\mathbf{\Sigma}^u$ proportional to $\mathbf{R}_y$ [13, (45)]:

$$\mathbf{\Sigma}^u = \sigma_u\mathbf{R}_y, \tag{20}$$

where σ_u depending on $u(.)$ and the RES [28, sec.3 ex.3] or C-CES [13, (46)] distribution of $\mathbf{y}_k$, is solution of

$$\mathrm{E}[u(\mathcal{Q}_k/\sigma_u)\mathcal{Q}_k/\sigma_u] = N, \tag{21}$$

where $\mathcal{Q}_k$ has the same distribution as the symmetric/Hermitian form Eq. (6), and has p.d.f. Eq. (8) in real case and Eq. (9) in complex case.

Consider now the estimate of the extended covariance matrix $\mathbf{R}_{\tilde{y}}$ associated with the NC-CES distribution Eq. (3). Following similar proof than Eq. (19) from the p.d.f. Eq. (3) [13, sec. V.A], we get the following implicit equation:

$$\widetilde{\mathbf{\Gamma}}_K = \frac{1}{K}\sum_{k=1}^{K}\phi\Big(\frac{1}{2}\tilde{\mathbf{y}}_k^H\widetilde{\mathbf{\Gamma}}_K^{-1}\tilde{\mathbf{y}}_k\Big)\tilde{\mathbf{y}}_k\tilde{\mathbf{y}}_k^H. \tag{22}$$

where $\phi(t) \stackrel{\text{def}}{=} -\frac{1}{g_c(t)}\frac{d_c(t)}{dt}$. For the NC-CG distribution (i.e., $g_c(t) = \exp(-t)$), we have $\phi(t) = 1$, which yields the extended SCM $\widetilde{\mathbf{\Gamma}}_K = \frac{1}{K}\sum_{k=1}^{K}\tilde{\mathbf{y}}_k\tilde{\mathbf{y}}_k^H$ as the unique ML of $\widetilde{\mathbf{\Gamma}}$. Similarly, an extended M-estimator of $\mathbf{R}_{\tilde{y}}$ denoted by $\widetilde{\mathbf{\Gamma}}_T^u$ is defined to be any positive definite Hermitian matrix that solves Eq. (22) with $\phi(t)$ is replaced by $u(t)$ (as defined above) such that

$$\widetilde{\mathbf{\Gamma}}_K^u = \frac{1}{K}\sum_{k=1}^{K}u\Big(\frac{1}{2}\tilde{\mathbf{y}}_k^H\widetilde{\mathbf{\Gamma}}_K^{u^{-1}}\tilde{\mathbf{y}}_k\Big)\tilde{\mathbf{y}}_k\tilde{\mathbf{y}}_k^H. \tag{23}$$

Using the one-to-one mapping $\tilde{\mathbf{y}}_k \mapsto \bar{\mathbf{y}}_k$ defined by $\bar{\mathbf{y}}_k = \mathbf{M}\tilde{\mathbf{y}}_k$ with $\mathbf{M} \stackrel{\text{def}}{=} \frac{1}{2}\begin{pmatrix}\mathbf{I} & \mathbf{I}\\ -i\mathbf{I} & i\mathbf{I}\end{pmatrix}$ and $\bar{\mathbf{y}}_k \stackrel{\text{def}}{=} [\mathrm{Re}(\mathbf{y}_k^T), \mathrm{Im}(\mathbf{y}_k^T)]^T$, it follows that $\tilde{\mathbf{y}}_k^H\widetilde{\mathbf{\Gamma}}_K^{u^{-1}}\tilde{\mathbf{y}}_k = \bar{\mathbf{y}}_k^H\bar{\mathbf{\Gamma}}_K^{u^{-1}}\bar{\mathbf{y}}_k$ with

$$\bar{\mathbf{\Gamma}}_K^u \stackrel{\text{def}}{=} \mathbf{M}\widetilde{\mathbf{\Gamma}}_K^u\mathbf{M}^H \tag{24}$$

and therefore Eq. (23) is tantamount to:

$$\bar{\mathbf{\Gamma}}_K^u = \frac{1}{K}\sum_{k=1}^{K}u\Big(\frac{1}{2}\bar{\mathbf{y}}_k^T\bar{\mathbf{\Gamma}}_K^{u^{-1}}\bar{\mathbf{y}}_k\Big)\bar{\mathbf{y}}_k\bar{\mathbf{y}}_k^T. \tag{25}$$

Noting that $\bar{\mathbf{y}}_k \in \mathbb{R}^{2N}$ is RES distributed with covariance $\mathbf{R}_{\bar{y}} = \mathrm{E}(\bar{\mathbf{y}}_k\bar{\mathbf{y}}_k^T) = \mathbf{M}\widetilde{\mathbf{\Gamma}}\mathbf{M}^H = \mathbf{M}\mathbf{R}_{\tilde{y}}\mathbf{M}^H$, and therefore the M-estimate solution of Eq. (23) inherits all the properties provided above for the RES distributions. In particular $\bar{\mathbf{\Gamma}}_K^u$ converges in probability to $\bar{\mathbf{\Gamma}}^u$ proportional to $\mathbf{R}_{\bar{y}}$ and thus $\widetilde{\mathbf{\Gamma}}_K^u$ also converges in probability to $\widetilde{\mathbf{\Gamma}}^u$ proportional to $\mathbf{R}_{\tilde{y}}$:

$$\widetilde{\mathbf{\Gamma}}^u = \sigma_u\mathbf{R}_{\tilde{y}} = \sigma_u\widetilde{\mathbf{\Gamma}}, \tag{26}$$

where σ_u is also similarly deduced from [28]:

$$\widetilde{\mathbf{\Gamma}}^u = \mathrm{E}\Big(u\Big(\frac{1}{2}\tilde{\mathbf{y}}_k^H\widetilde{\mathbf{\Gamma}}^{u^{-1}}\tilde{\mathbf{y}}_k\Big)\tilde{\mathbf{y}}_k\tilde{\mathbf{y}}_k^H\Big), \tag{27}$$

which successively gives the following equalities: $\mathbf{I} = \mathrm{E}(u(\frac{1}{2}\tilde{\mathbf{y}}_k^H\widetilde{\mathbf{\Gamma}}^{u^{-1}}\tilde{\mathbf{y}}_k)\widetilde{\mathbf{\Gamma}}^{u^{-1}}\tilde{\mathbf{y}}_k\tilde{\mathbf{y}}_k^H)$, $2N = \mathrm{E}(u(\frac{1}{2}\tilde{\mathbf{y}}_k^H\widetilde{\mathbf{\Gamma}}^{u^{-1}}\tilde{\mathbf{y}}_k)\mathrm{Tr}(\widetilde{\mathbf{\Gamma}}^{u^{-1}}\tilde{\mathbf{y}}_k\tilde{\mathbf{y}}_k^H))$, $N = \mathrm{E}(u(\frac{1}{2}\tilde{\mathbf{y}}_k^H\widetilde{\mathbf{\Gamma}}^{u^{-1}}\tilde{\mathbf{y}}_k)\frac{1}{2}\tilde{\mathbf{y}}_k^H\widetilde{\mathbf{\Gamma}}^{u^{-1}}\tilde{\mathbf{y}}_k)$, $N = \mathrm{E}(u(\frac{1}{2}\tilde{\mathbf{y}}_k^H\widetilde{\mathbf{\Gamma}}^{-1}\tilde{\mathbf{y}}_k/\sigma_u)\frac{1}{2}\tilde{\mathbf{y}}_k^H\widetilde{\mathbf{\Gamma}}^{-1}\tilde{\mathbf{y}}_k/\sigma_u)$, and from Eq. (7),

$$\mathrm{E}[u(\mathcal{Q}_k/\sigma_u)\mathcal{Q}_k/\sigma_u] = N, \tag{28}$$

where $\mathcal{Q}_k$ has p.d.f. Eq. (9).

Finally, note that the normalized SCM estimate studied in [25]: $\mathbf{R}_{y,K} \stackrel{\text{def}}{=} \frac{1}{K}\sum_{k=1}^{K}\mathbf{S}(\mathbf{y}_k)\mathbf{S}^H(\mathbf{y}_k)$ with $\mathbf{S}(\mathbf{y}_k) \stackrel{\text{def}}{=} \mathbf{y}_k/\|\mathbf{y}_k\|$ if $\mathbf{y}_k \neq \mathbf{0}$ and $\mathbf{S}(\mathbf{0}) \stackrel{\text{def}}{=} \mathbf{0}$, which is not an M-estimate of $\mathbf{R}_y$, is not the object of our study.

4.2. Asymptotic distribution of the projector estimator

To apply Theorem 1 to the statistic $\mathbf{s}_{y,K} = \mathrm{vec}(\mathbf{\Pi}_{y,K})$ in the real and circular complex cases and to $\mathbf{s}_{y,K} = \mathrm{vec}(\mathbf{\Pi}_{\tilde{y},K})$ in the non-circular complex case, we need to derive their asymptotic distributions and to check the conditions Eq. (17). Note that the asymptotic distribution of $\mathrm{vec}(\mathbf{\Pi}_{y,K})$ has been given for the real and circular complex case in [26] and in the circular complex case in [27]. This asymptotic distribution has been derived from the asymptotic distribution of any M-estimate of $\mathbf{R}_y$ derived for the real case in [28, sec.3 ex.3], and for the circular complex case in [29, rel. (7) and (12)]. These asymptotic distributions have the following form, when the arbitrary weight function $u(.)$ satisfies the Maronna's conditions [22]:

$$\sqrt{K}(\mathrm{vec}(\mathbf{\Pi}_{y,K}) - \mathrm{vec}(\mathbf{\Pi}_y(\boldsymbol{\theta}))\ \mathcal{N}_{\mathbb{R}}(\mathbf{0}, \mathbf{R}_{\pi_y}) \text{ in the real case} \tag{29}$$

$$\mathcal{N}_C(\mathbf{0}, \mathbf{R}_{\pi_y}, \mathbf{C}_{\pi_y}) \text{ in the circular complex case} \tag{30}$$

with

$$\mathbf{R}_{\pi_y} = \frac{\vartheta_1}{\sigma_u^2}\mathbf{L}[(\mathbf{U}^T \otimes \mathbf{\Pi}_y(\boldsymbol{\theta})) + (\mathbf{\Pi}_y^T(\boldsymbol{\theta}) \otimes \mathbf{U})] \quad \text{and} \quad \mathbf{C}_{\pi_y} = \mathbf{R}_{\pi_y}\mathbf{K}_{N^2} \tag{31}$$

where
$\mathbf{U} \stackrel{\text{def}}{=} \sigma_k^2\mathbf{S}^\#\mathbf{R}_y\mathbf{S}^\#$ with $\mathbf{S} \stackrel{\text{def}}{=} \mathbf{A}(\boldsymbol{\theta})\mathbf{R}_x\mathbf{A}^+(\boldsymbol{\theta})$ and

$$\vartheta_1 \stackrel{\text{def}}{=} \frac{\mathrm{E}[u^2(\mathcal{Q}_k/\sigma_u)\mathcal{Q}_k^2]}{N(N+2)(1+2[N(N+2)]^{-1}c_u)^2} \quad \text{and}$$

$$\mathbf{L} \stackrel{\text{def}}{=} \mathbf{I} + \mathbf{K}_{N^2} \text{ in the real case} \tag{32}$$

$$\vartheta_1 \stackrel{\text{def}}{=} \frac{\mathrm{E}[u^2(\mathcal{Q}_k/\sigma_u)\mathcal{Q}_k^2]}{N(N+1)(1+[N(N+1)]^{-1}c_u)^2} \quad \text{and}$$

$$\mathbf{L} \stackrel{\text{def}}{=} \mathbf{I} \text{ in the circular complex case,} \tag{33}$$

where σ_u is the solution of (21) and $c_u \stackrel{\text{def}}{=} \mathrm{E}[u'(\mathcal{Q}_k/\sigma_u)\mathcal{Q}_k^2/\sigma_u^2]$ [28, sec.3 ex.3], [13, (47)], where $u'(x) \stackrel{\text{def}}{=} du(x)/dx$ and the p.d.f. of $\mathcal{Q}_k$ is given by Eqs. (8) or (9).

As pointed in [30] and [13], Tyler's M-estimator, i.e., solution of Eq. (19) with weight $u(t) = \frac{N}{t}$, does not satisfy Maronna conditions [22]. However, it has been proved for RES distributions in [30] that, after normalizing the solution of Eq. (19) such that $\mathrm{Tr}(\mathbf{R}_y^{-1}\boldsymbol{\Sigma}_K^u) = N$, the sequence $\boldsymbol{\Sigma}_K^u$ converges in probability to $\mathbf{R}_y$ and is asymptotically Gaussian distributed. These properties have been extended to C-CES distributions in [31]. Following the perturbation analysis of projection matrix of [26], the associated projector $\boldsymbol{\Pi}_{y,K}$ is also asymptotically Gaussian distributed and Eqs. (31), (32) and (33) follow with $\sigma_u = 1$ and $\vartheta_1 = \vartheta_{1,\mathrm{Tyler}}$ independently of the RES and C-CES distributions with:

$$\vartheta_{1,\mathrm{Tyler}} = \frac{N+2}{N}, \text{ in the real case} \tag{34}$$

$$= \frac{N+1}{N}, \text{ in the complex case.} \tag{35}$$

The asymptotic distribution of $\mathrm{vec}(\boldsymbol{\Pi}_{\tilde{y},K})$ can be proven similarly to Eq. (33), using the asymptotic distribution of any M-estimate $\mathbf{R}_{\tilde{y},K}$ of $\mathbf{R}_{\tilde{y}}$. This follows from the asymptotic distribution of any M-estimate $\bar{\boldsymbol{\Gamma}}_K^u$ of $\mathbf{R}_{\bar{y}}$ which is given from [28, sec.3 ex.3] by:

$$\sqrt{K}(\mathrm{vec}(\bar{\boldsymbol{\Gamma}}_K^u) - \mathrm{vec}(\sigma_u \mathbf{R}_{\bar{y}})) \xrightarrow{\mathcal{L}} \mathcal{N}_R(\mathbf{0}, \mathbf{R}_{\bar{\Gamma}^u}), \tag{36}$$

with $\mathbf{R}_{\bar{\Gamma}^u} = \vartheta_1(\mathbf{I} + \mathbf{K}_{(2N)^2})(\mathbf{R}_{\bar{y}} \otimes \mathbf{R}_{\bar{y}}) + \vartheta_2 \mathrm{vec}(\mathbf{R}_{\bar{y}})\mathrm{vec}^T(\mathbf{R}_{\bar{y}})$, where both ϑ_1 and ϑ_2 are also specified in[28, sec.3 ex.3] and [29, rel. (7)] by replacing N by $2N$. It follows from Eq. (24), that the sequence $\mathrm{vec}(\tilde{\boldsymbol{\Gamma}}_K^u)$ is also asymptotically Gaussian distributed with asymptotic covariance $\mathbf{R}_{\tilde{\Gamma}^u}$ and complementary covariance $\mathbf{C}_{\tilde{\Gamma}^u}$ given by:

$$\begin{aligned}
\mathbf{R}_{\tilde{\Gamma}^u} &= (\mathbf{M}^{*-1} \otimes \mathbf{M}^{-1})\mathbf{R}_{\bar{\Gamma}^u}(\mathbf{M}^{-T} \otimes \mathbf{M}^{-H}) \\
&= \vartheta_1[(\mathbf{M}^{*-1}\mathbf{R}_{\bar{y}}\mathbf{M}^{-T}) \otimes (\mathbf{M}^{-1}\mathbf{R}_{\bar{y}}\mathbf{M}^{-H}) \\
&\quad + \mathbf{K}_{(2N)^2}\{(\mathbf{M}^{-1}\mathbf{R}_{\bar{y}}\mathbf{M}^{-T}) \otimes (\mathbf{M}^{*-1}\mathbf{R}_{\bar{y}}\mathbf{M}^{-H})\}] \\
&\quad + \vartheta_2 \mathrm{vec}(\mathbf{M}^{-1}\mathbf{R}_{\bar{y}}\mathbf{M}^{-H})\mathrm{vec}^H(\mathbf{M}^{-1}\mathbf{R}_{\bar{y}}\mathbf{M}^{-H}) \\
&= \vartheta_1[(\mathbf{R}_{\tilde{y}}^* \otimes \mathbf{R}_{\tilde{y}}) + \mathbf{K}_{(2N)^2}(\mathbf{C}_{\tilde{y}} \otimes \mathbf{C}_{\tilde{y}}^*)] \\
&\quad + \vartheta_2 \mathrm{vec}(\mathbf{R}_{\tilde{y}})\mathrm{vec}^H(\mathbf{R}_{\tilde{y}}),
\end{aligned} \tag{37}$$

and $\mathbf{C}_{\tilde{\Gamma}^u} = \mathbf{R}_{\tilde{\Gamma}^u}\mathbf{K}_{(2N)^2}$ where $\mathbf{C}_{\tilde{y}} \stackrel{\text{def}}{=} \mathrm{E}(\tilde{\mathbf{y}}_k\tilde{\mathbf{y}}_k^T) = \mathbf{R}_{\tilde{y}}\mathbf{J} = \mathbf{M}\mathbf{R}_{\bar{y}}\mathbf{M}^T$. Then using the standard perturbation result associated with the mapping $\mathbf{R}_{\tilde{y},K} = \mathbf{R}_{\tilde{y}} + \delta\mathbf{R}_{\tilde{y}} \mapsto \boldsymbol{\Pi}_{\tilde{y},K} = \boldsymbol{\Pi}_{\tilde{y}} + \delta\boldsymbol{\Pi}_{\tilde{y}}$ for orthogonal projectors [32] (see also the operator approach in [33]) applied to $\boldsymbol{\Pi}_{\tilde{y}}$ associated with the noise subspace of $\mathbf{R}_{\tilde{y}}$:

$$\delta(\boldsymbol{\Pi}_{\tilde{y}}) = -\boldsymbol{\Pi}_{\tilde{y}}(\boldsymbol{\theta})\delta(\mathbf{R}_{\tilde{y}})\tilde{\mathbf{S}}^{\#} - \tilde{\mathbf{S}}^{\#}\delta(\mathbf{R}_{\tilde{y}})\boldsymbol{\Pi}_{\tilde{y}}(\boldsymbol{\theta}) + o\big(\delta(\mathbf{R}_{\tilde{y}})\big), \tag{38}$$

where $\tilde{\mathbf{S}} \stackrel{\text{def}}{=} \tilde{\mathbf{A}}_r(\boldsymbol{\theta})\mathbf{R}_r\tilde{\mathbf{A}}_r^H(\boldsymbol{\theta})$ or $\tilde{\mathbf{S}} \stackrel{\text{def}}{=} \tilde{\mathbf{A}}_c(\boldsymbol{\theta})\mathbf{R}_{\tilde{x}}\tilde{\mathbf{A}}_c^H(\boldsymbol{\theta})$, the asymptotic behaviors of $\boldsymbol{\Pi}_{\tilde{y},K}$ and $\tilde{\boldsymbol{\Gamma}}_K^u$ are directly related. The standard theorem of continuity (see e.g., [20, p. 122]) on regular functions of asymptotically Gaussian statistics applies: $\sqrt{K}\big(\mathrm{vec}(\boldsymbol{\Pi}_{\tilde{y},K}) - \mathrm{vec}(\boldsymbol{\Pi}_{\tilde{y}}(\boldsymbol{\theta}))\big) \xrightarrow{\mathcal{L}} \mathcal{N}_C\big(\mathbf{0}, \mathbf{R}_{\pi_{\tilde{y}}}, \mathbf{C}_{\pi_{\tilde{y}}}\big)$ with

$$\begin{aligned}
\mathbf{R}_{\pi_{\tilde{y}}} &= \frac{\vartheta_1}{\sigma_u^2}[(\tilde{\mathbf{S}}^{T\#} \otimes \boldsymbol{\Pi}_{\tilde{y}}(\boldsymbol{\theta})) + (\boldsymbol{\Pi}_{\tilde{y}}^T(\boldsymbol{\theta}) \otimes \tilde{\mathbf{S}}^{\#})]\mathbf{R}_{\tilde{\Gamma}^u} \\
&\quad \times[(\tilde{\mathbf{S}}^{T\#} \otimes \boldsymbol{\Pi}_{\tilde{y}}(\boldsymbol{\theta})) + (\boldsymbol{\Pi}_{\tilde{y}}^T(\boldsymbol{\theta}) \otimes \tilde{\mathbf{S}}^{\#})],
\end{aligned} \tag{39}$$

and $\mathbf{C}_{\pi_{\tilde{y}}} = \mathbf{R}_{\pi_{\tilde{y}}}\mathbf{K}_{(2K)^2}$. Then plugging Eq. (37) into Eq. (39) and using $\boldsymbol{\Pi}_{\tilde{y}}(\boldsymbol{\theta})\tilde{\mathbf{S}}^{\#} = \mathbf{0}$, $\mathbf{C}_{\tilde{y}} = \mathbf{R}_{\tilde{y}}\mathbf{J}$ and $\tilde{\mathbf{S}}^{\#}\mathbf{R}_{\tilde{y}}\boldsymbol{\Pi}_{\tilde{y}}(\boldsymbol{\theta}) = \mathbf{0}$, we get the following result after simple algebraic manipulations:

Result 1. The sequence $\sqrt{K}\big(\mathrm{vec}(\boldsymbol{\Pi}_{\tilde{y},K}) - \mathrm{vec}(\boldsymbol{\Pi}_{\tilde{y}}(\boldsymbol{\theta}))\big)$ converges in distribution to the zero-mean Gaussian distribution $\mathcal{N}_C(\mathbf{0}, \mathbf{R}_{\pi_{\tilde{y}}}, \mathbf{C}_{\pi_{\tilde{y}}})$ where:

$$\begin{aligned}
\mathbf{R}_{\pi_{\tilde{y}}} &= \frac{\vartheta_1}{\sigma_u^2}(\mathbf{I} + \mathbf{K}_{(2N)^2}(\mathbf{J} \otimes \mathbf{J}))[(\tilde{\mathbf{U}}^T \otimes \boldsymbol{\Pi}_{\tilde{y}}(\boldsymbol{\theta})) \\
&\quad + (\boldsymbol{\Pi}_{\tilde{y}}^T(\boldsymbol{\theta}) \otimes \tilde{\mathbf{U}}) \text{ and } \mathbf{C}_{\pi_y} = \mathbf{R}_{\pi_y}\mathbf{K}_{(2N)^2},
\end{aligned} \tag{40}$$

where ϑ_1 is associated with the $2N$-dimensional RES distributions given in [28, sec.3 ex.3], and can be simplified as:

$$\vartheta_1 = \frac{\mathrm{E}[u^2(\mathcal{Q}_k/\sigma_u)\mathcal{Q}_K^2]}{N(N+1)(1+[N(N+1)]^{-1}c_u)^2}. \tag{41}$$

with $c_u \stackrel{\text{def}}{=} \mathrm{E}[u'(\mathcal{Q}_k/\sigma_u)\mathcal{Q}_k^2/\sigma_u^2]$ and σ_u is solution of Eq. (21) where the p.d.f. of $\mathcal{Q}_k =_d \frac{1}{2}\tilde{\mathbf{y}}_k^H\tilde{\boldsymbol{\Gamma}}^{-1}\tilde{\mathbf{y}}_k = \frac{1}{2}\tilde{\mathbf{y}}_k^H\tilde{\boldsymbol{\Gamma}}^{-1}\tilde{\mathbf{y}}_k$ is given by Eq. (9) and $\tilde{\mathbf{U}} \stackrel{\text{def}}{=} \sigma_k^2\tilde{\mathbf{S}}^{\#}\mathbf{R}_{\tilde{y}}\tilde{\mathbf{S}}^{\#}$.

Note that Result 1 also applies to Tyler's M-estimator from the asymptotic distribution of $\mathbf{R}_{\bar{\Gamma}^u}$ Eq. (36) where ϑ_1 can be obtained from the value associated with the real case Eq. (34) by replacing N by $2N$ with $\sigma_u = 1$ and $\vartheta_{1,\mathrm{Tyler}} = \frac{2N+2}{2N} = \frac{N+1}{N}$ which is independent of the NC-CES distributions.

4.3. Subspace AMV bound

Note that from Eqs. (31) and (40) $\mathbf{R}_{\pi_y} = \frac{\vartheta_1}{\sigma_u^2}\mathbf{R}_{\pi_y}^{\mathrm{C-CG}}$ and $\mathbf{R}_{\pi_{\tilde{y}}} = \frac{\vartheta_1}{\sigma_u^2}\mathbf{R}_{\pi_{\tilde{y}}}^{\mathrm{NC-CG}}$, where $\mathbf{R}_{\pi_y}^{\mathrm{C-CG}}$ and $\mathbf{R}_{\pi_{\tilde{y}}}^{\mathrm{NC-CG}}$ are in the specific DOA modeling Eqs. (10) and (12), the covariances of the asymptotic distributions of the projectors given by [9, rel.(3.4)] and [9, rel.(3.6)], respectively, and which are associated with the SCM estimate for the C-CG and NC-CG distributions, respectively [9, Lemma 1]. Because, under the C-CG and NC-CG distributed observations, the proofs of $\mathrm{span}(\mathcal{S}) \subset \mathrm{span}(\mathbf{R}_{\pi_y}^{\mathrm{CG}})$ and $\mathrm{span}(\mathcal{S}) \subset \mathrm{span}(\mathbf{R}_{\pi_{\tilde{y}}}^{\mathrm{CG}})$ given in [9, Appendix A] are valid for an arbitrary parametrization of $\mathbf{A}(\boldsymbol{\theta})$, $\mathbf{A}_r(\boldsymbol{\theta})$ and $\mathbf{A}_c(\boldsymbol{\theta})$, then the first condition of (17) also holds, i.e., $\mathrm{span}(\mathcal{S}) \subset \mathrm{span}(\mathbf{R}_{\pi_y})$ and $\mathrm{span}(\mathcal{S}) \subset \mathrm{span}(\mathbf{R}_{\pi_{\tilde{y}}})$. The second condition of (17) is trivially valid by the structure of both statistics $\boldsymbol{\Pi}_{y,K}$ and $\boldsymbol{\Pi}_{\tilde{y},K}$. Consequently, Theorem 1 applies to the statistics $\mathrm{vec}(\boldsymbol{\Pi}_{y,K})$ and $\mathrm{vec}(\boldsymbol{\Pi}_{\tilde{y},K})$, and the following result is proved in the Appendix.

Result 2. The covariance matrix $\mathbf{R}_{\boldsymbol{\theta}}$ of the asymptotic Gaussian distribution of any weakly consistent estimate $\widehat{\boldsymbol{\theta}}_K$ of $\boldsymbol{\theta}$ given by any algorithm considered as a differentiable mapping $\boldsymbol{\Pi}_{y,K} \mapsto \widehat{\boldsymbol{\theta}}_K = \mathrm{alg}(\boldsymbol{\Pi}_{y,K})$ [resp., $\boldsymbol{\Pi}_{\tilde{y},K} \mapsto \widehat{\boldsymbol{\theta}}_K = \mathrm{alg}(\boldsymbol{\Pi}_{\tilde{y},K})$] for RES and C-CES [resp., NC-CES] distributed observations is bounded below by $\mathbf{R}_{\boldsymbol{\theta}}^{\mathrm{AMV}(\boldsymbol{\Pi})}$:

$$\mathbf{R}_{\boldsymbol{\theta}} \geq \mathbf{R}_{\boldsymbol{\theta}}^{\mathrm{AMV}(\boldsymbol{\Pi})} = \vartheta_1\beta\frac{\sigma_n^2}{2}\left[\mathrm{Re}\left(\frac{d\mathbf{a}_\theta^+}{d\boldsymbol{\theta}}(\mathbf{H}^T \otimes \boldsymbol{\Pi}(\boldsymbol{\theta}))\frac{d\mathbf{a}_\theta}{d\boldsymbol{\theta}}\right)\right]^{-1}, \tag{42}$$

where:

$\mathbf{a}_\theta \stackrel{\text{def}}{=} \mathrm{vec}(\mathbf{A}(\boldsymbol{\theta}))$, $\mathbf{H} \stackrel{\text{def}}{=} \mathbf{R}_x^+\mathbf{A}^+(\boldsymbol{\theta})\boldsymbol{\Sigma}^{-1}\mathbf{A}(\boldsymbol{\theta})\mathbf{R}_x$ and $\boldsymbol{\Pi}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \boldsymbol{\Pi}_y(\boldsymbol{\theta})$ in the real and complex circular case,

$\mathbf{a}_\theta \stackrel{\text{def}}{=} \mathrm{vec}(\tilde{\mathbf{A}}_r(\boldsymbol{\theta}))$, $\mathbf{H} \stackrel{\text{def}}{=} \mathbf{R}_r\tilde{\mathbf{A}}_r^H(\boldsymbol{\theta})\tilde{\boldsymbol{\Gamma}}^{-1}\tilde{\mathbf{A}}_r(\boldsymbol{\theta})\mathbf{R}_r$ and $\boldsymbol{\Pi}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \boldsymbol{\Pi}_{\tilde{y}}(\boldsymbol{\theta})$ in the complex non-circular case[4] associated with the structured extended covariance Eq. (11) and

[4] Note that in this case $\frac{d\mathbf{a}_\theta^H}{d\boldsymbol{\theta}}(\mathbf{H}^T \otimes \boldsymbol{\Pi}(\boldsymbol{\theta}))\frac{d\mathbf{a}_\theta}{d\boldsymbol{\theta}}$ is real-valued.

$$\mathbf{a}_\theta \stackrel{\text{def}}{=} \text{vec}(\mathbf{A}(\boldsymbol{\theta})), \quad \mathbf{H} \stackrel{\text{def}}{=} \left(\mathbf{R}_x\mathbf{A}^H(\boldsymbol{\theta}), \mathbf{C}_x\mathbf{A}^T(\boldsymbol{\theta})\right)\widetilde{\boldsymbol{\Gamma}}^{-1}\begin{pmatrix}\mathbf{A}(\boldsymbol{\theta})\mathbf{R}_x\\ \mathbf{A}^*(\boldsymbol{\theta})\mathbf{C}_x^*\end{pmatrix} \quad \text{and}$$

$\boldsymbol{\Pi}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \boldsymbol{\Pi}_y(\boldsymbol{\theta})$ in the non-circular complex case associated with Eq. (12) and $\beta = 2$ [resp., 1] in the real [resp., complex] case.

It is important to note that the cost functional in Eq. (18) depends either on $\mathbf{R}_\pi^\#$ or $\mathbf{R}_{\tilde{\pi}}^\#$ depending on the distribution of observations, which can be replaced by weakly consistent estimates $\mathbf{W}_k$ obtained from consistent estimates of $\boldsymbol{\Pi}_y$, $\mathbf{R}_y$, $\mathbf{S}$, σ_n^2 (or $\boldsymbol{\Pi}_{\tilde{y}}$, $\mathbf{R}_{\tilde{y}}$, $\widetilde{\mathbf{S}}$, σ_n^2).

4.4. Efficiency

Let's consider here that $\mathbf{R}_y$ and $\mathbf{R}_{\tilde{y}}$ are estimated using the ML *M*-estimate as in Eqs. (19) and (22) from RES/C-CES and NC-CES distributed observations, respectively. We prove in the Appendix that $\sigma_u = 1$ in Eqs. (32), (33), and (41), and that these expressions of ϑ_1 reduce to:

$$\vartheta_{1,\text{ML}} = \frac{\text{E}[\phi^2(\mathcal{Q}_k)\mathcal{Q}_k^2]}{N(N+2)(1+2[N(N+2)]^{-1}\text{E}[\phi'(\mathcal{Q}_k)\mathcal{Q}_k^2])^2} \quad \text{with}$$

$$\phi(t) = -\frac{2}{g_r(t)}\frac{dg_r(t)}{dt} \quad \text{in the real case} \tag{43}$$

$$= \frac{\text{E}[\phi^2(\mathcal{Q}_k)\mathcal{Q}_k^2]}{N(N+1)(1+[N(N+1)]^{-1}\text{E}[\phi'(\mathcal{Q}_k)\mathcal{Q}_k^2])^2} \quad \text{with}$$

$$\phi(t) = -\frac{1}{g_c(t)}\frac{dg_c(t)}{dt} \quad \text{in the complex case,} \tag{44}$$

where the p.d.f. of $\mathcal{Q}_k$ is respectively given by Eqs. (8) and (9).

Otherwise, for C-CES and NC-CES distributed observations, the concentrated stochastic CRBs on the parameter of interest $\boldsymbol{\theta}$ characterizing the associated projection matrices have been given in [14]. Following similar steps as in [14], and using the Fisher information matrix derived in [34], the stochastic CRBs for real and complex cases take the following general form (with the same notations as in Result 2):

$$\text{CRB}(\boldsymbol{\theta}) = \frac{\beta}{\xi_2}\frac{\sigma_n^2}{2}\left[\text{Re}\left(\frac{d\mathbf{a}_\theta^H}{d\boldsymbol{\theta}}(\mathbf{H}^T \otimes \boldsymbol{\Pi}_y(\boldsymbol{\theta}))\frac{d\mathbf{a}_\theta}{d\boldsymbol{\theta}}\right)\right]^{-1}, \tag{45}$$

with

$$\xi_2 = \frac{\text{E}[\phi^2(\mathcal{Q}_k)\mathcal{Q}_k^2]}{N(N+2)}, \quad \text{in the real case} \tag{46}$$

$$= \frac{\text{E}[\phi^2(\mathcal{Q}_k)\mathcal{Q}_k^2]}{N(N+1)}, \quad \text{in the complex case.} \tag{47}$$

Note that for DOA modeling with scalar sensor array whose output are C-CG distributed, we have $g(t) = e^{-t}$, $u(t) = 1$ and $\beta = \xi_2 = 1$, and therefore, (45) reduces to the well-known relation for a single parameter per source case:

$$\text{CRB}(\boldsymbol{\theta}) = \frac{\sigma_n^2}{2}\{\text{Re}\left((\mathbf{D}^H(\boldsymbol{\theta})\boldsymbol{\Pi}_y(\boldsymbol{\theta})\mathbf{D}(\boldsymbol{\theta})) \odot \mathbf{H}^T\right)\}^{-1}, \tag{48}$$

where $\mathbf{A}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} [\mathbf{a}(\theta_1), .., \mathbf{a}(\theta_P)]$, $\mathbf{D}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \left[\frac{d\mathbf{a}(\theta_1)}{d\theta_1}, .., \frac{d\mathbf{a}(\theta_P)}{d\theta_P}\right]$ and $\mathbf{a}(\theta_p)_{p=1,.P}$ are the steering vectors.

Comparing Eq. (42) to Eq. (45), the following result is proved in the Appendix

Result 3. For RES, C-CES and NC-CES distributed observations, we have $\vartheta_{1,\text{ML}}\xi_2 = 1$ and thus the AMV bounds Eq. (42) based on the projector statistics associated with the ML estimate of the covariances are equal to the stochastic CRB (45).

$$\mathbf{R}_{\boldsymbol{\theta},\text{ML}}^{\text{AMV}(\boldsymbol{\Pi})} = \text{CRB}(\boldsymbol{\theta}). \tag{49}$$

Therefore, the AMV estimators Eq. (18) based on projectors associated with ML *M*-estimate of the covariance are asymptotically efficient w.r.t. the number *K* of measurements. Furthermore, the equality $\vartheta_1\xi_2 = 1$ and the relations Eqs. (31) and (40) imply that all specific subspace-based algorithms built on the ML estimate of $\mathbf{R}_y$ [resp. $\mathbf{R}_{\tilde{y}}$], that are asymptotically efficient for RG or C-CG [resp. NC-CG] distribution, are also asymptotically efficient for RES or C-CES [resp. NC-CES] distributions. This is particularly the case in the DOA modeling for the conventional MUSIC algorithm applied to a single source [35] and to uncorrelated sources when the signal-to-noise ratio of all sources tend to infinity [36].

The following result is proved in the Appendix:

Result 4. The RES, C-CES and NC-CES ML *M*-estimator dependent asymptotic variance parameter $\vartheta_{1,\text{ML}} = 1/\xi_2$ in (43) and (44) are upper bounded by the one associated with the Tyler's *M*-estimator Eqs. (34) and (35) as

$$\vartheta_{1,\text{ML}} < \vartheta_{1,\text{Tyler}} = \frac{N+2}{N}, \quad \text{in the real case} \tag{50}$$

$$\vartheta_{1,\text{ML}} < \vartheta_{1,\text{Tyler}} = \frac{N+1}{N}, \quad \text{in the complex case} \tag{51}$$

and consequently

$$\text{CRB}(\boldsymbol{\theta}) = \mathbf{R}_{\boldsymbol{\theta},\text{ML}}^{\text{AMV}(\boldsymbol{\Pi})} < \mathbf{R}_{\boldsymbol{\theta},\text{Tyler}}^{\text{AMV}(\boldsymbol{\Pi})}. \tag{52}$$

For example, for the complex generalized Gaussian distribution with exponent $\beta > 0$, it was evaluated that $\vartheta_{1,\text{ML}} = \frac{N+1}{N+\beta}$ [26]. It is clear that $\vartheta_{1,\text{ML}} < \vartheta_{1,\text{Tyler}}$ and $\vartheta_{1,\text{ML}}/\vartheta_{1,\text{Tyler}} \approx 1$ for $N \gg 1$ or small values of β which are associated with heavy-tailed distributions.

Result 4 proves that the AMV subspace estimators based on Tyler's *M*-estimator of the covariance matrix are not efficient. To obtain a truly robust efficient subspace-based estimator, one has to find *M*-estimators with an appropriate $u(t)$ such that ϑ_1 be close or equal to $\vartheta_{1,\text{ML}}$.

In general the stochastic CRB associated with a finite-dimensional parameter of a distribution whose p.d.f. is characterized by a functional form, is lower bounded by the semiparametric CRB (denoted by SCRB($\boldsymbol{\theta}$)) introduced by [37] when this functional form is unknown. This SCRB has been studied for RES and C-CES distributions in [38] and [16], respectively. In particular, a closed-form expression of the semiparametric CRB has been derived in [16] for the DOA parameter of C-CES distributed observations. It is given by

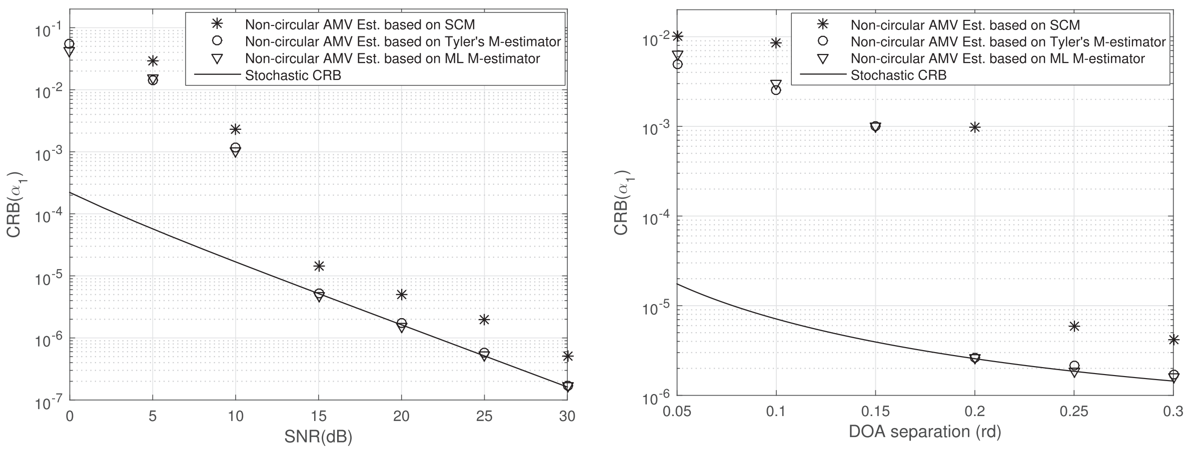
$$\text{SCRB}(\boldsymbol{\theta}) = \frac{1}{\xi_2}\frac{\sigma_n^2}{2}\{\text{Re}\left((\mathbf{D}^H(\boldsymbol{\theta})\boldsymbol{\Pi}_y(\boldsymbol{\theta})\mathbf{D}(\boldsymbol{\theta})) \odot \mathbf{H}^T\right)\}^{-1}, \tag{53}$$

which happens to be equal to the stochastic CRB. This property seems to have been overlooked in [16]. By slightly modifying and extending the proof given in the support document of [16] to general RES, C-CES and NC-CES distributed noisy linear mixture models Eq. (13), we have proved the following result:

Result 5. The stochastic CRB on the parameter of interest $\boldsymbol{\theta}$ that characterizes the column space of $\mathbf{A}(\boldsymbol{\theta})$, $\widetilde{\mathbf{A}}_r(\boldsymbol{\theta})$ or $\widetilde{\mathbf{A}}_c(\boldsymbol{\theta})$ is not reduced when the density generator $g(.)$ of the RES, C-CES or NC-CES distribution is known, viz:

$$\text{CRB}(\boldsymbol{\theta}) = \text{SCRB}(\boldsymbol{\theta}). \tag{54}$$

We note that this property is very specific to the parameter of interest characterized by the column space of the mixing matrix. This property is explained by the fact that this column space does no depend on the density generator $g(t)$. It is important, however, to note that if the AMV estimator Eq. (18) is efficient w.r.t. the stochastic CRB, it is no longer efficient w.r.t. the semiparametric CRB because the AMV estimator is built from the ML *M*-estimate of the covariance matrix based on the knowledge of the density generator $g(t)$.



(a) $\Delta\alpha = 0.2rd$, $\Delta\phi \stackrel{\text{def}}{=} |\phi_2 - \phi_1| = 0.3rd$, $\rho = 0.5$ (b) SNR$= 20dB$, $\Delta\phi = 0.1rd$, $\rho = 0.5$

Fig. 1. Non-circular stochastic CRB (45) and MSEs obtained with AMV subspace-based estimator (18) build from $\mathbf{\Pi}_{\tilde{y},K}$ versus SNR (and versus DOA separation $\Delta\alpha = |\alpha_2 - \alpha_1|$) for non-circular complex Student t-distributed observations with $\nu = 4.1$.

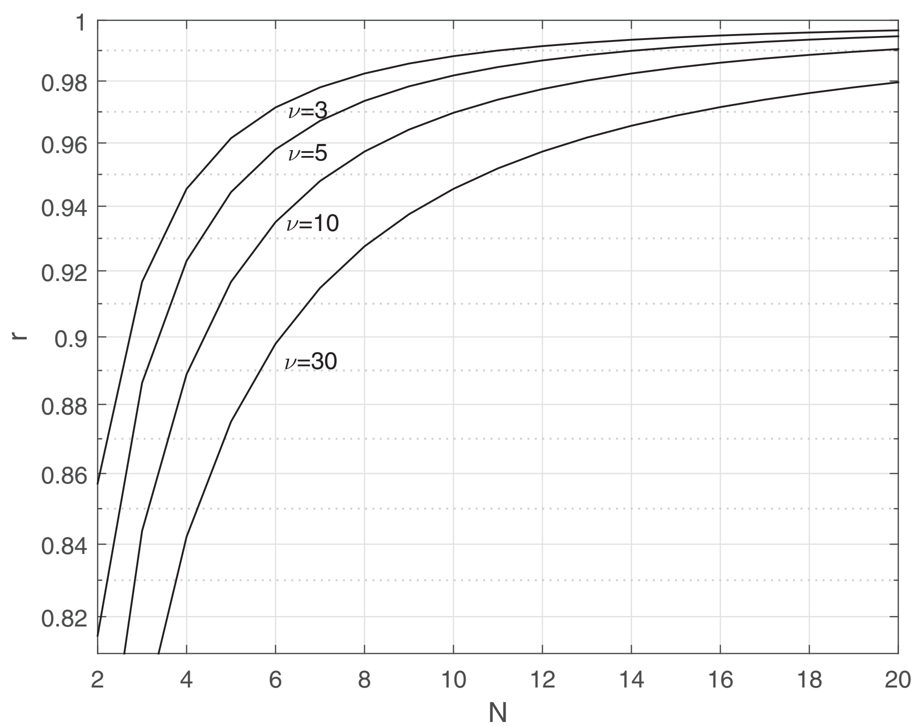


Fig. 2. Ratio $r \stackrel{\text{def}}{=} R^{\text{AMV}(\mathbf{\Pi})}_{\alpha_1,\text{ML}} / R^{\text{AMV}(\mathbf{\Pi})}_{\alpha_1,\text{Tyler}}$ versus N for different values of ν.

5. Numerical illustrations

This section illustrates the theoretical asymptotic results provided in Section 4, focusing on the DOA estimation model for rectilinear correlated signal sources [14], for which the observation data follows a NC-CES distribution with a structured extended covariance matrix $\widetilde{\mathbf{\Gamma}}$ given by Eq. (11). We consider throughout this section that $P = 2$ narrowband equal-power source signals with power σ^2 impinge on an ULA of $N = 6$ sensors for which the steering vectors are $\mathbf{a}(\alpha_k) = (1, e^{i\pi \sin\alpha_k}, \ldots, e^{i(N-1)\pi \sin\alpha_k})^T$, $k = 1, 2$, where α_k are the DOAs relative to the normal of array broadside. The matrices $\mathbf{A}(\boldsymbol{\theta})$ and $\mathbf{R}_r$ in Eq. (11) are given by $\mathbf{A}(\boldsymbol{\theta}) = [\mathbf{a}(\alpha_1)e^{i\phi_1}, \mathbf{a}(\alpha_2)e^{i\phi_2}]$ where the phases ϕ_k associated with different propagation delays are assumed fixed, but unknown during the array observation with $\boldsymbol{\theta} \stackrel{\text{def}}{=} (\alpha_1, \alpha_2, \phi_1, \phi_2)^T$, and $\mathbf{R}_r = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$ with ρ is the correlation factor. The signal-to-noise ratio (SNR) is defined as $10\log_{10}(\sigma^2/\sigma_n^2)$ dB.

We use the AMV estimator in Eq. (18) to estimate the DOA α_1 from the covariance estimate and to calculate the empirical mean squared error (MSE) $\mathrm{E}(\hat{\alpha}_1 - \alpha_1)^2$ from 1000 Monte Carlo runs, by assuming that $\mathbf{y}_k$, $k = 1, \ldots, K = 500$, follows a non-circular complex Student's t distribution with a number of degrees of freedom ν $(0 < \nu < \infty)$. The corresponding stochastic representation is given by (5) for $\mathcal{Q}_k \sim N\mathcal{F}_{2N,\nu}$, where $\mathcal{F}_{n,q}$ denotes the $\mathcal{F}$-distribution with n and q degrees of freedom [13, sec. IV.A], which has finite second and fourth-order moments, respectively, for $\nu > 2$ and $\nu > 4$.

Table 1
Parameter ϑ_1 used in the illustrations.

	Student's ML *M*-estimator	Tyler's *M*-estimator	SCM
ϑ_1	$\frac{N+\nu/2+1}{N+\nu/2}$	$\frac{N+1}{N}$	1

In the illustrations below, we consider three covariance estimates: the complex Student's ML *M*-estimator and the complex Tyler's *M*-estimator (which does not depends on the distribution of $\mathcal{Q}_k$) for which the associated weight functions $\phi(t)$ and $u(t)$ in Eqs. (22) and (23) are respectively defined in [13] by $\phi(t) = \frac{2N+\nu}{\nu+2t}$ and $u(t) = \frac{N}{t}$. The SCM estimator corresponding to the ML in the Gaussian case is obtained with $u(t) = 1$.

The following table gives the values of the parameter ϑ_1 involved in different asymptotic covariance matrices for the complex Student's ML-estimator [26], the complex Tyler's *M*-estimator [31] and the SCM estimator.

Fig. 1(a) and (b) illustrate the validation of theoretical Results 2 and 3. These figures display the non-circular stochastic CRB Eq. (45) and the MSEs of the subspace-based estimator Eq. (18) with consistent estimate of $\mathbf{R}^{\#}_{\tilde{\pi}}$ built from $\mathbf{\Pi}_{\tilde{y},T}$ derived from for Student's ML *M*-estimator, Tyler's *M*-estimator and SCM. It can be seen from these figures that the empirical MSEs associated with the Student's and Tyler's *M*-estimator reach the stochastic CRB as SNR or DOA separation increases. We also observe that the AMV subspace-based estimator based on Tyler's *M*-estimator has close performance as the one based on Student's ML *M*-estimator. On the other hand, as expected, the AMV subspace-based estimator based on SCM has poor performance in the presence of heavy-tailed complex non-circular observations.

Fig. 2 illustrates Result 4, by plotting the ratio $r \stackrel{\text{def}}{=} R^{\text{AMV}(\mathbf{\Pi})}_{\alpha_1,\text{ML}} / R^{\text{AMV}(\mathbf{\Pi})}_{\alpha_1,\text{Tyler}} = \vartheta_{1,\text{ML}}/\vartheta_{1,\text{Tyler}} = \frac{N(N+\nu/2+1)}{(N+1)(N+\nu/2)} < 1$ versus N for different values of ν, where $\vartheta_{1,\text{ML}}$ and $\vartheta_{1,\text{Tyler}}$ are the values of ϑ_1 associated, respectively, with Student's ML *M*-estimator and Tyler's *M*-estimator, given in Table. 1. It can be seen from this figure that, for a small value of ν (heavy-tailed distribution), the AMV bound associated with the Student's ML *M*-estimator becomes closer to the one associated with Tyler's *M*-estimator as N increases. In other words, the AMV estimate built from Tyler's *M*-estimator becomes efficient in the sense that it asymptotically achieves the stochastic CRB when $N \gg 1$ and ν not too large.

6. Conclusion

This paper has derived the asymptotic (in the number of measurements) distribution of estimates of the orthogonal projector associated with different *M*-estimates of the covariance matrix in the context of RES, C-CES, and NC-CES distributed observations whose covariance is low rank structured, in the same framework. Then it has presented the AMV subspace-based estimator of the parameter of interest characterized by the column subspace of the mixing matrix for general linear mixtures models, associated with the *M*-estimates of the covariance matrix. It has given a common closed-form expression of the AMV bound which can be used as a benchmark against which the subspace-based algorithms are tested. This has allowed us to prove that this AMV bound attains the stochastic CRB in the case of ML *M*-estimate of the covariance matrix for RES, C-CES, and NC-CES distributed observations, and to specify the conditions for which the AMV bound based on Tyler's *M*-estimate attains this stochastic CRB for complex Student *t* and complex generalized Gaussian distributions. Finally, it has proved that this stochastic CRB is equal to the semiparametric CRB recently introduced. However, the AMV estimator is not efficient w.r.t. the semiparametric CRB, which raises the question of finding an efficient estimator w.r.t. this CRB, which is a challenge.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in the paper.

Appendix A

Proof. Proof of Result 2: In [9] it has been proved, for the DOA modeling with both C-CG and NC-CG (with parametrization (12)) observations, that $\mathbf{R}^{\text{AMV}(\mathbf{\Pi})}_{\boldsymbol{\theta}} = \frac{\sigma_k^2}{2}\left[\text{Re}\left(\frac{d\mathbf{a}_\theta^H}{d\theta}(\mathbf{H}^T \otimes \mathbf{\Pi}_y(\boldsymbol{\theta}))\frac{d\mathbf{a}_\theta}{d\theta}\right)\right]^{-1}$ with the notations of Result 2 and that, respectively, $\mathbf{R}^{\text{C-CG}}_{\pi_y} = (\mathbf{U}^T \otimes \mathbf{\Pi}_y(\boldsymbol{\theta})) + (\mathbf{\Pi}_y^T(\boldsymbol{\theta}) \otimes \mathbf{U})$ and $\mathbf{R}^{\text{NC-CG}}_{\pi_{\tilde{y}}} = (\widetilde{\mathbf{U}}^T \otimes \mathbf{\Pi}_{\tilde{y}}(\boldsymbol{\theta})) + (\mathbf{\Pi}_{\tilde{y}}^T(\boldsymbol{\theta}) \otimes \widetilde{\mathbf{U}})$. This expression of $\mathbf{R}^{\text{AMV}(\mathbf{\Pi})}_{\boldsymbol{\theta}}$ is straightforwardly extended to the parametrization Eq. (11). As the proof does not depend on the parametrization of $\mathbf{A}(\boldsymbol{\theta})$, Eq. (42) is valid with $\beta = 1$ for the C-CES and NC-CES distributions for both Eqs. (11) and (12) parametrizations from Eq. (31).

For the RES distributions, the derivation of $\mathbf{R}^{\#}_{\pi_y}$ from Eq. (31) is not direct. Because $\mathbf{L}^2 = 2\mathbf{L}$, we get

$$\mathbf{R}_{\pi_y} = \frac{1}{2}\mathbf{L}[(\mathbf{U}^T \otimes \mathbf{\Pi}_y(\boldsymbol{\theta})) + (\mathbf{\Pi}_y^T(\boldsymbol{\theta}) \otimes \mathbf{U})]\mathbf{L}^T = \frac{1}{2}\mathbf{L}\mathbf{R}^{\text{C-CG}}_{\pi_y}\mathbf{L}^T.$$

Following the derivation of the AMV bound given in [10], this bound is the result of the minimization:

$$\mathbf{R}^{\text{AMV}(\mathbf{\Pi})}_{\boldsymbol{\theta}} = \min_{\mathbf{D}\mathcal{S}=\mathbf{I}} \mathbf{D}\mathbf{R}_{\pi_y}\mathbf{D}^T = \frac{1}{2}\min_{\mathbf{D}\mathcal{S}=\mathbf{I}} (\mathbf{D}\mathbf{L})\mathbf{R}^{\text{C-CG}}_{\pi_y}(\mathbf{D}\mathbf{L})^T. \quad (55)$$

Checking that $\mathbf{L}\mathcal{S} = 2\mathcal{S}$ with $\mathcal{S} \stackrel{\text{def}}{=} \frac{d\text{vec}(\mathbf{\Pi}_y)}{d\boldsymbol{\theta}}$, the constraints $\mathbf{D}\mathcal{S} = \mathbf{I}$ and $\mathbf{D}\mathbf{L}\mathcal{S} = 2\mathbf{I}$ are equivalent, and therefore Eq. (55) is tantamount to

$$\mathbf{R}^{\text{AMV}(\mathbf{\Pi})}_{\boldsymbol{\theta}} = 2\min_{(\mathbf{D}\mathbf{L}/2)\mathcal{S}=\mathbf{I}} \left(\frac{\mathbf{D}\mathbf{L}}{2}\right)\mathbf{R}^{\text{C-CG}}_{\pi_y}\left(\frac{\mathbf{D}\mathbf{L}}{2}\right)^T.$$

As a result, the steps of the derivation for the C-CES distributions apply and we get Eq. (42) with $\beta = 2$ for RES distributions. □

Proof. Proof of rel. Eqs. (43) and (44) When the ML estimate of either $\mathbf{R}_y$ or $\mathbf{R}_{\tilde{y}}$ is considered for RES and C-CES distributions or for NC-CES distributions, the solution of either Eq. (21) or Eq. (28) is $\sigma_u = 1$ because $\text{E}[u(\mathcal{Q}_k)\mathcal{Q}_k] = \text{E}[\phi(\mathcal{Q}_k)\mathcal{Q}_k] = N$ from $\phi(t) = -\frac{2}{g_r(t)}\frac{dg_r(t)}{dt}$ [resp., $\phi(t) = -\frac{1}{g_c(t)}\frac{dg_c(t)}{dt}$] with Eq. (8) [resp., Eq. (9)] for RES [resp., C-CES or NC-CES] distributions and consequently Eqs. (43) and (44) are proved. □

Proof. Proof of Result 3 Comparing Eq. (42) to Eq. (45), Eq. (49) is equivalent to $\vartheta_1\xi_2 = 1$. To prove this relation, consider first the complex case for which it is tantamount to:

$$\text{E}[\phi^2(\mathcal{Q}_k)\mathcal{Q}_k^2] = N(N+1) + \text{E}[\phi'(\mathcal{Q}_k)\mathcal{Q}_k^2]. \quad (56)$$

Using the p.d.f. Eq. (9) of the r.v. $\mathcal{Q}_k$, we straightforwardly get:

$$\text{E}[\phi^2(\mathcal{Q}_k)\mathcal{Q}_k^2] - \text{E}[\phi'(\mathcal{Q}_k)\mathcal{Q}_k^2] = \int_0^\infty \delta^{-1}_{N,g_c} q^{N+1}\frac{d^2g_c(q)}{dq^2}dq,$$

where

$$\int_0^\infty \delta^{-1}_{N,g_c} q^{N+1}\frac{d^2g_c(q)}{dq^2}dq = [\delta^{-1}_{N,g_c} q^{N+1}\frac{dg_c(q)}{dq}]_0^\infty - (N+1)\int_0^\infty \delta^{-1}_{N,g_c} q^N\frac{dg_c(q)}{dq}dq.$$

The second term can be simplified as follows

$$\int_0^\infty \delta^{-1}_{N,g_c} q^N\frac{dg_c(q)}{dq}dq = [\delta^{-1}_{N,g_c} q^N g_c(q)]_0^\infty - N\int_0^\infty \delta^{-1}_{N,g_c} q^{N-1}g_c(q)dq = -N,$$

because $\lim_{q\to\infty} q^{N+1}\frac{dg_c(q)}{dq} = \lim_{q\to\infty} q^N g_c(q) = 0$ using the fact that the fourth-order moment of $\mathcal{Q}_k$ is assumed finite and

$\int_0^\infty \delta_{N,g_c}^{-1} q^{N-1} g_c(q) dq = 1$. Hence, $\int_0^\infty \delta_{N,g_c}^{-1} q^{N+1} \frac{d^2 g_c(q)}{dq^2} dq = N(N+1)$, thus concluding the proof. The real case is similarly proved by replacing $N(N+1)$ by $N(N+2)$ in Eq. (56) and using the p.d.f. Eq. (8). □

Proof. Proof of Result 4 Using in the complex case $\vartheta_{1,\mathrm{ML}} = \frac{N(N+1)}{\mathrm{E}[\phi^2(\mathcal{Q}_k)\mathcal{Q}_k^2]}$ from Result 3 and Eq. (46) and $(\mathrm{E}[\phi(\mathcal{Q}_k)\mathcal{Q}_k])^2 = N^2$, the Cauchy-Schwarz inequality $(\mathrm{E}[XY])^2 \leq \mathrm{E}[X^2]\,\mathrm{E}[Y^2]$ with $X = \phi(\mathcal{Q}_k)\mathcal{Q}_k$ and $Y = 1$ gives $\vartheta_{1,\mathrm{ML}} \leq \vartheta_{1,\mathrm{Tyler}}$ with equality if and only if the r.v. $\phi(\mathcal{Q}_k)\mathcal{Q}_k$ is constant. Since $\phi(t) = -\frac{1}{g_c(t)}\frac{dg_c(t)}{dt}$, this property is equivalent to $g_c(t) = t^a$ where a is constant. Since there is no constant a such that $\delta_{N,g_c} \stackrel{\mathrm{def}}{=} \int_0^\infty t^{N-1} t^a dt < \infty$, the equality $\vartheta_{1,\mathrm{ML}} = \vartheta_{1,\mathrm{Tyler}}$ is not possible. The real case is similarly proved. □

CRediT authorship contribution statement

Habti Abeida: Conceptualization, Data curation, Formal analysis, Writing - original draft, Writing - review & editing. **Jean-Pierre Delmas:** Conceptualization, Data curation, Formal analysis, Writing - original draft, Writing - review & editing.

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