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Robustness of subspace-based algorithms with respect to the distribution of the noise: Application to DOA estimation

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ABSTRACT

This paper addresses the theoretical analysis of the robustness of subspace-based algorithms with respect to non-Gaussian noise distributions using perturbation expansions. Its purpose is twofold. It aims, first, to derive the asymptotic distribution of the estimated projector matrix obtained from the sample covariance matrix (SCM) for arbitrary distributions of the useful signal and the noise. It proves that this distribution depends only of the second-order statistics of the useful signal, but also on the second and fourth-order statistics of the noise. Second, it derives the asymptotic distribution of the estimated projector matrix obtained from any *M*-estimate of the covariance matrix for both real (RES) and complex elliptical symmetric (CES) distributed observations. Applied to the MUSIC algorithm for direction-of-arrival (DOA) estimation, these theoretical results allow us to theoretically evaluate the performance loss of this algorithm for heavy-tailed noise distributions when it is based on the SCM, which is significant for weak signal-to-noise ratio (SNR) or closely spaced sources. These results also make it possible to prove that this performance loss can be alleviated by replacing the SCM by an *M*-estimate of the covariance for CES distributed observations, which has been observed until now only by numerical experiments.

1. Introduction

Subspace-based algorithms are obtained by exploiting the orthogonality between a sample subspace and a parameter-dependent subspace. They have been proved very useful in many applications, including array processing and linear system identification (see e.g., [1,2]). The purpose of this paper is to complement theoretical results already available on subspace-based estimators. Among them, the asymptotic distribution of the projection matrix was directly derived without any eigendecomposition in [3]. Conditions on robustness to the distribution of the useful signal have been given in [4,5] for Gaussian distributed noise. Equivalence between subspace fitting and subspace matching algorithms has been studied in [6]. It has been proved that the asymptotic minimum variance on parameters estimated by subspace-based algorithms attains the Cramér–Rao bound for Gaussian observations in [7,8]. But all these properties have been derived from the SCM and under the assumption of Gaussian distributed noise, only.

But, for heavy-tailed distributions of the noise, it has been shown (see e.g., [9–12]) by numerical experiments that the performance of MUSIC algorithm for DOA estimation derived from the SCM degrades dramatically. To explain this behavior, the first purpose of this paper is to derive the asymptotic distribution of the estimated projector matrix obtained from the SCM for arbitrary distributions of the useful signal and the noise, to prove that it depends only of the second-order statistics of the useful signal, but also on the second and fourth-order of the noise. The second purpose is to derive the asymptotic distribution of the estimated projector matrix obtained from any *M*-estimate of the covariance matrix, for both RES and CES distributions of the observations.

We take the MUSIC DOA estimation algorithm, which is always the object of active research (see e.g., [13]), as an example of subspace-based algorithms. We first theoretically specify the loss in performance of this algorithm built from the SCM for arbitrary heavy-tailed distributions of strong noise with a particular emphasis to CES distributions. The second step of our analysis, then, is to theoretically assess the robustness of this algorithm built from an *M*-estimate of the covariance for CES distributed observations. These theoretical results are confirmed by some simulations performed with either a complex circular Student *t* distribution or a complex generalized Gaussian distribution of the observations or the noise model.

The paper is organized as follows. Section 2 specifies the general signal model and the problem formulation. The asymptotic

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performance of subspace-based algorithms associated with the SCM and the M-estimate of the covariance matrix are given in Section 3 and 4, respectively, with a particular attention to the asymptotic performance of the MUSIC-based DOA estimation algorithm. Section 5 illustrates the theoretical performance of the MUSIC-based DOA estimation algorithm given in the previous two sections. Finally, the paper is concluded in Section 6.

The notations used throughout the paper are conventional. Matrices and vectors are represented by bold upper case and bold lower case letters, respectively. $\mathbf{I}_N$ is the identity matrix of order N and vectors are by default in column orientation, while T, H and * stand for transpose, conjugate transpose and conjugate, respectively. E(.), Re(.), |.| and # are the expectation, real part operator, determinant and Moore–Penrose inverse, respectively. vec(·) is the vectorization operator that turns a matrix into a vector by stacking the columns of the matrix one below another which is used in conjunction with the Kronecker product $\mathbf{A}\otimes\mathbf{B}$ as the block matrix whose (i, j) block element is $a_{i,j}\mathbf{B}$ and with the vec-permutation matrix $\mathbf{K}$ which transforms vec($\mathbf{C}$) to vec($\mathbf{C}^T$) for any square matrix $\mathbf{C}$.

2. Signal model and problem formulation

2.1. Signal model

Consider the following model[1]

$$\mathbf{y}_t = \mathbf{A}(\boldsymbol{\theta})\mathbf{x}_t + \mathbf{n}_t \in \mathbb{C}^N \text{ or } \mathbb{R}^N, \quad t = 1, \dots T, \tag{1}$$

where $(\mathbf{y}_t)_{t=1,\dots,T}$ are independent identically distributed. $\mathbf{n}_t$ is an additive noise, which is assumed zero-mean circular (in the complex case) with finite fourth-order moments, spatially uncorrelated with $\mathrm{E}(\mathbf{n}_t\mathbf{n}_t^+) = \mathbf{R}_n = \sigma_n^2\mathbf{I}_N$ where the symbol $^+$ stands for ^{T} in the real case and for ^{H} in the complex one. The noise $\mathbf{n}_t$ is independent from the signals $x_{t,k}$ where $\mathbf{x}_t \stackrel{\text{def}}{=} (x_{t,1}, \dots, x_{t,K})^T$. $(x_{t,k})_{k=1,\dots,K,1,\dots,T}$ are either deterministic unknown parameters (in the so-called conditional or deterministic model), or zero-mean circular (in the complex case) random with finite fourth-order moments where $\mathrm{E}(\mathbf{x}_t\mathbf{x}_t^+) = \mathbf{R}_x$ is nonsingular (in the so-called unconditional or stochastic model). $\mathbf{A}(\boldsymbol{\theta})$ is a full column rank $N \times P$ (with $P < N$) matrix parameterized by the real-valued parameter of interest $\boldsymbol{\theta}$, which is characterized by the subspace generated by its columns. With these assumptions, the covariance matrix of $\mathbf{y}_t$ given in the stochastic model is

$$\mathbf{R}_y = \mathbf{A}(\boldsymbol{\theta})\mathbf{R}_x\mathbf{A}^+(\boldsymbol{\theta}) + \mathbf{R}_n. \tag{2}$$

2.2. Problem formulation

Under the Gaussian distribution of $\mathbf{y}_t$, the maximum likelihood estimate of $\mathbf{R}_y$ is the SCM:

$$\mathbf{R}_{y,T} \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}_t\mathbf{y}_t^+ \tag{3}$$

and any subspace-based algorithm can be considered as the following mapping:

$$\mathbf{R}_{y,T} \mapsto \boldsymbol{\Pi}_{y,T} \stackrel{alg}{\mapsto} \boldsymbol{\theta}_T \tag{4}$$

where $\boldsymbol{\Pi}_{y,T}$ denotes the orthogonal projection matrix associated with the so called noise subspace of $\mathbf{R}_{y,T}$ (built from the SVD of $\mathbf{R}_{y,T}$). The functional dependence $\boldsymbol{\theta}_T = alg(\boldsymbol{\Pi}_{y,T})$ constitutes an extension of the mapping $\boldsymbol{\Pi}_y = \mathbf{I}_N - \mathbf{A}(\boldsymbol{\theta})[\mathbf{A}^+(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta})]^{-1}\mathbf{A}^+(\boldsymbol{\theta}) \stackrel{alg}{\longmapsto} \boldsymbol{\theta}$ in the neighborhood of $\boldsymbol{\Pi}_y$. Each extension alg specifies a particular subspace algorithm, whose MUSIC algorithm is an example.

It has been proved in [5], that under the above assumptions on stochastic model (1), the asymptotic distribution of any subspace-based estimate $\boldsymbol{\theta}_T$ derived form the SCM depends only of the second-order statistics of $\mathbf{x}_t$ when the noise $\mathbf{n}_t$ is Gaussian distributed. The problem we are dealing with here is to clarify how the performance is affected when the noise is not Gaussian distributed and how the possible loss of performance can be mitigated by using an M-estimate of $\mathbf{R}_y$, instead of the SCM.

3. Asymptotic performance of subspace-based algorithms associated with the SCM

3.1. Asymptotic distribution of the SCM

Using the central limit theorem applied to the sequence of independent random variables $\mathrm{vec}(\mathbf{y}_t\mathbf{y}_t^+) = \mathbf{y}_t^* \otimes \mathbf{y}_t$ in the complex case [resp., $\mathbf{y}_t\otimes\mathbf{y}_t$ in the real case], which are identically [resp., non-identically] distributed in the stochastic [resp. deterministic] model, we get the following convergence in distribution:

$$\sqrt{T}\Big(\mathrm{vec}(\mathbf{R}_{y,T}) - \mathrm{vec}(\mathbf{R}_y)\Big) \xrightarrow{\mathcal{L}} \mathcal{N}\Big(\mathbf{0}, \mathbf{R}_{r_y}, \mathbf{C}_{r_y}\Big) \text{ (complex case)},$$
$$\mathcal{N}\Big(\mathbf{0}, \mathbf{R}_{r_y}\Big) \text{ (real case)}, \tag{5}$$

where $\mathbf{R}_y$ is given by (2) in the stochastic model and by $\mathbf{A}(\boldsymbol{\theta})\mathbf{R}_{x,\infty}\mathbf{A}^+(\boldsymbol{\theta}) + \mathbf{R}_n$ with $\mathbf{R}_{x,\infty} \stackrel{\text{def}}{=} \lim_{t\to\infty}\frac{1}{T}\sum_{t=1}^{T}\mathbf{x}_t\mathbf{x}_t^+$ (if it exists) in the deterministic model. In the stochastic model, $\mathbf{R}_{r_y}$ and $\mathbf{C}_{r_y}$ denotes the covariance and the complementary covariance or pseudo covariance of $\mathbf{y}_t^* \otimes \mathbf{y}_t$, respectively. They are given by[2]:

$$\mathbf{R}_{r_y} = (\mathbf{A}^* \otimes \mathbf{A})\mathbf{R}_{r_x}(\mathbf{A}^T \otimes \mathbf{A}^+) + (\mathbf{A}^*\mathbf{R}_x^*\mathbf{A}^T) \otimes \mathbf{R}_n$$
$$+\mathbf{R}_n^* \otimes (\mathbf{A}\mathbf{R}_x\mathbf{A}^+) + \mathbf{R}_{r_n} \tag{6}$$

$$\mathbf{C}_{r_y} = \mathbf{R}_{r_y}\mathbf{K}, \tag{7}$$

where $\mathbf{A} \stackrel{\text{def}}{=} \mathbf{A}(\boldsymbol{\theta})$ for short, $\mathbf{R}_{r_x}$ and $\mathbf{R}_{r_n}$ are the covariance matrices of $\mathrm{vec}(\mathbf{x}_t\mathbf{x}_t^+)$ and $\mathrm{vec}(\mathbf{n}_t\mathbf{n}_t^+)$, respectively. Whereas in the deterministic model, $\mathbf{R}_{r_y}$ and $\mathbf{C}_{r_y}$ are given by:

$$\mathbf{R}_{r_y} = (\mathbf{A}^*\mathbf{R}_{x,\infty}^*\mathbf{A}^T) \otimes \mathbf{R}_n + \mathbf{R}_n^* \otimes (\mathbf{A}\mathbf{R}_{x,\infty}\mathbf{A}^+) + \mathbf{R}_{r_n} \quad \text{and} \quad \mathbf{C}_{r_y} = \mathbf{R}_{r_y}\mathbf{K}. \tag{8}$$

Considering the statistics of the noise: $\mathbf{R}_n = \sigma_n^2\mathbf{I}_N$ and

$$\mathbf{R}_{r_n} = (\mathbf{R}_n \otimes \mathbf{R}_n)\mathbf{K}' + \mathbf{Q}_n = \sigma_n^4\mathbf{K}' + \mathbf{Q}_n, \tag{9}$$

where

$$\mathbf{K}' \stackrel{\text{def}}{=} \mathbf{I}_{N^2} \quad [\text{resp., } \mathbf{K}' \stackrel{\text{def}}{=} \mathbf{I}_{N^2} + \mathbf{K}] \quad \text{in the complex [resp., real] case} \tag{10}$$

and $\mathbf{Q}_n$ is the quadrivariance of the noise defined by $[\mathbf{Q}_n]_{i+(j-1)N,k+(l-1)N} = \mathrm{Cum}(n_{t,j}^*, n_{t,i}, n_{t,\ell}, n_{t,k}^*)$ with $\mathbf{n}_t = (n_{t,1}, \dots, n_{t,N})^T$. To go further, we must specify these cumulants. Here we consider the following usual assumptions: (i) the components $(n_{t,i})_{i=1,\dots,N}$ are identically distributed and (ii) the only non-zero expectations $\mathrm{E}(n_{t,j}^*n_{t,i}n_{t,\ell}n_{t,k}^*)$ are obtained for $i = j = k = \ell$ and for two indices equal two by two. For example, in the complex case:

$$\mathrm{E}(n_{t,j}^*n_{t,i}n_{t,\ell}n_{t,k}^*) = \begin{cases} \mathrm{E}|n_{t,i}|^4 = \kappa\sigma_n^4 & \text{for } i = j = k = \ell \\ (\mathrm{E}|n_{t,i}|^2)^2 = \sigma_n^4 & \text{for } i = j \neq k = \ell \text{ and } i = k \neq j = \ell \\ 0 & \text{elsewhere,} \end{cases} \tag{11}$$

[1] This model is an extension of the conditional and unconditional models described in [14] for DOA estimation.

[2] Note that all asymptotic covariance $\mathbf{R}$ and complementary covariance $\mathbf{C}$ of Hermitian structured estimators are related by $\mathbf{C} = \mathbf{R}\mathbf{K}$.

where κ is the kurtosis of the noise. In this case, it is straightforward to prove that:

$$\mathbf{Q}_n = \sigma_n^4(\kappa - \rho)\boldsymbol{\Lambda}_N, \tag{12}$$

where $\rho = 3$ [resp., $\rho = 2$] in the real [resp., complex] case with $\boldsymbol{\Lambda}_N \stackrel{\text{def}}{=} \sum_{i=1}^N (\mathbf{e}_i\mathbf{e}_i^T) \otimes (\mathbf{e}_i\mathbf{e}_i^T)$, where $\mathbf{e}_i$ is the Nth vector with one in the position i and zeros elsewhere. Note that $\kappa - \rho = 0$ for real or circular complex Gaussian distributions and $\kappa - \rho$ is strictly positive or negative for respectively, super-Gaussian or sub-Gaussian distribution of the noise.

The assumptions (i) and (ii) are also satisfied when $\mathbf{n}_t$ is RES or CES distributed. Note that the real (RCG) or complex compound Gaussian (CCG) distribution (also referred to as spherically invariant random vectors (SIRV) in the engineering literature that has been widely used for modeling radar clutter), is a subclass of the RES or CES distributions. Using the stochastic representation theorem of these distributions (see e.g., [15, th.3 and def.3] in the complex case), $\mathbf{n}_t$ is distributed as

$$\sqrt{\mathcal{Q}_t}\ \boldsymbol{\Sigma}^{1/2}\mathbf{u}_t \text{ for RES/CES distributions,}$$
$$\sqrt{\tau_t}\ \boldsymbol{\Sigma}^{1/2}\mathbf{w}_t \text{ for RCG/CCG distributions,} \tag{13}$$

where $\mathcal{Q}_t$ and τ_t are non-negative real random variables, $\mathbf{u}_t$ and $\mathbf{w}_t$ are respectively uniformly distributed on the unit real (or complex) N-sphere and Gaussian distributed $\mathcal{N}(\mathbf{0}, \mathbf{I}_N)$ (or $\mathcal{N}(\mathbf{0}, \mathbf{I}_N, \mathbf{0})$), $\mathcal{Q}_t$ [resp., τ_t] and $\mathbf{u}_t$ [resp., $\mathbf{w}_t$] are independent and $\boldsymbol{\Sigma}$ is the scatter matrix of the distribution of $\mathbf{n}_t$. Because here the second-order moments of $\mathbf{n}_t$ are finite, the density generator of the distribution of $\mathbf{n}_t$ can always be definite such that the scatter matrix is equal to the covariance matrix, i.e.,

$$\boldsymbol{\Sigma} = \mathbf{R}_n = \sigma_n^2\mathbf{I}_N. \tag{14}$$

In this case, it is straightforward to prove that for RES/CES and RCG/CCG distributions:

$$\mathbf{Q}_n = \sigma_n^4(\eta - 1)\left(\mathbf{K}' + \text{vec}(\mathbf{I}_N)\text{vec}^T(\mathbf{I}_N)\right), \tag{15}$$

where

$$\eta = \frac{\text{E}(\mathcal{Q}_t^2)}{N(N+1)} \text{ [resp., } \eta = \text{E}(\tau_t^2)]$$
$$\text{for RES/CES [resp., RCG/CCG] distributions.} \tag{16}$$

3.2. Asymptotic distribution of the associated projector

Then using the standard perturbation result for orthogonal projectors [16] (see also [3]) applied to $\boldsymbol{\Pi}_y$ associated with the noise subspace of $\mathbf{R}_y$:

$$\delta(\boldsymbol{\Pi}_y) = -\boldsymbol{\Pi}_y\delta(\mathbf{R}_y)\mathbf{S}^{\#} - \mathbf{S}^{\#}\delta(\mathbf{R}_y)\boldsymbol{\Pi}_y + o(\delta(\mathbf{R}_y)), \tag{17}$$

where $\mathbf{S} \stackrel{\text{def}}{=} \mathbf{A}\mathbf{R}_x\mathbf{A}^+$ [resp., $\mathbf{A}\mathbf{R}_{x,\infty}\mathbf{A}^+$] in the stochastic [resp. deterministic] model, the asymptotic behaviors of $\boldsymbol{\Pi}_{y,T}$ and $\mathbf{R}_{y,T}$ are directly related. The standard theorem of continuity (see e.g., [17, p. 122]) on regular functions of asymptotically Gaussian statistics applies:

$$\sqrt{T}\left(\text{vec}(\boldsymbol{\Pi}_{y,T}) - \text{vec}(\boldsymbol{\Pi}_y)\right) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\mathbf{0}, \mathbf{R}_{\pi_y}, \mathbf{C}_{\pi_y}\right)$$
$$\text{(complex case)}, \quad \mathcal{N}\left(\mathbf{0}, \mathbf{R}_{\pi_y}\right) \text{ (real case)}, \tag{18}$$

with

$$\mathbf{R}_{\pi_y} = [(\mathbf{S}^{T^\#} \otimes \boldsymbol{\Pi}_y) + (\boldsymbol{\Pi}_y^T \otimes \mathbf{S}^{\#})]\mathbf{R}_{r_y}[(\mathbf{S}^{T^\#} \otimes \boldsymbol{\Pi}_y) + (\boldsymbol{\Pi}_y^T \otimes \mathbf{S}^{\#})]. \tag{19}$$

Then using (6) and (9) with (12) and (15), we get the following result:

Result 1. In the stochastic and deterministic models, the sequence $\sqrt{T}\left(\text{vec}(\boldsymbol{\Pi}_{y,T}) - \text{vec}(\boldsymbol{\Pi}_y)\right)$ where $\boldsymbol{\Pi}_{y,T}$ denotes the noise projector associated with the SCM (3) converges in distribution to the zero-mean Gaussian distribution of covariance given for nonparameterized distributed noise by:

$$\begin{aligned}\mathbf{R}_{\pi_y} &= [(\mathbf{U}^T \otimes \boldsymbol{\Pi}_y) + (\boldsymbol{\Pi}_y^T \otimes \mathbf{U})]\mathbf{K}' \\ &\quad + \sigma_n^4(\kappa - \rho)[(\mathbf{S}^{T^\#} \otimes \boldsymbol{\Pi}_y) \\ &\qquad + (\boldsymbol{\Pi}_y^T \otimes \mathbf{S}^{\#})]\boldsymbol{\Lambda}_N[(\mathbf{S}^{T^\#} \otimes \boldsymbol{\Pi}_y) + (\boldsymbol{\Pi}_y^T \otimes \mathbf{S}^{\#})]\end{aligned} \tag{20}$$

with $\mathbf{U} \stackrel{\text{def}}{=} \sigma_n^2\mathbf{S}^{\#}\mathbf{R}_y\mathbf{S}^{\#}$ and for RES/CES/RCG/CCG distributed noise by:

$$\begin{aligned}\mathbf{R}_{\pi_y} &= [(\mathbf{U}^T \otimes \boldsymbol{\Pi}_y) + (\boldsymbol{\Pi}_y^T \otimes \mathbf{U})]\mathbf{K}' \\ &\quad + (\eta - 1)[(\mathbf{U}'^T \otimes \boldsymbol{\Pi}_y) + (\boldsymbol{\Pi}_y^T \otimes \mathbf{U}')],\end{aligned} \tag{21}$$

with $\mathbf{U}' \stackrel{\text{def}}{=} \sigma_n^4(\mathbf{S}^{\#})^2$, which is inversely proportional to the square of the SNR.

We note that when $\mathbf{n}_t$ is Gaussian distributed, $\kappa = 3$ [resp., $\kappa = 2$] in the real [resp. complex] case, $\mathcal{Q}_t$ is Gamma(1,N) distributed and $\tau_t = 1$ which implies $\eta = 1$ for RES/CES [resp., RCG/CCG] distributions and we check that $\mathbf{R}_{\pi_y}$ reduces to the first term $[(\mathbf{U}^T \otimes \boldsymbol{\Pi}_y) + (\boldsymbol{\Pi}_y^T \otimes \mathbf{U})]\mathbf{K}'$ derived for the Gaussian distribution of the noise.

3.3. Asymptotic performance of DOA estimated by the MUSIC algorithm

In the DOA application, $\mathbf{A}$ is the steering matrix $[\mathbf{a}_1, \ldots, \mathbf{a}_P]$, where each vector $(\mathbf{a}_k)_{k=1,\ldots,P}$ is parameterized by θ_k which is the DOA of the kth source. We prove in Appendix A the following result:

Result 2. In the stochastic and deterministic models, the sequence $\sqrt{T}\left(\boldsymbol{\theta}_T - \boldsymbol{\theta}\right)$, where $\boldsymbol{\theta}_T$ is the DOA estimate given by MUSIC algorithm, converges in distribution to the zero-mean Gaussian distribution of covariance matrix given for the nonparametric noise model by:

$$[\mathbf{R}_{\text{NG}}^{\text{SCM}}(\boldsymbol{\theta})]_{k,\ell} = [\mathbf{R}_{\text{G}}^{\text{SCM}}(\boldsymbol{\theta})]_{k,\ell} + \frac{\sigma_n^4(\kappa - 2)}{\alpha_{k,k}\alpha_{\ell,\ell}}\mathbf{g}_k^H\boldsymbol{\Lambda}_N\mathbf{g}_\ell, \tag{22}$$

where $\mathbf{g}_k \stackrel{\text{def}}{=} \mathbf{a}_k^T\mathbf{S}^{\#T} \otimes \mathbf{a}_k'\boldsymbol{\Pi}_y + \mathbf{a}_k'^T\boldsymbol{\Pi}_y^T \otimes \mathbf{a}_k^H\mathbf{S}^{\#}$, and $\alpha_{k,\ell} \stackrel{\text{def}}{=} 2\mathbf{a}_k'^H\boldsymbol{\Pi}_y\mathbf{a}_\ell'$ where $\mathbf{a}_k' \stackrel{\text{def}}{=} d\mathbf{a}_k/d\theta_k$ and for CES/CCG parametric noise model, by:

$$[\mathbf{R}_{\text{CES/CCG}}^{\text{SCM}}(\boldsymbol{\theta})]_{k,\ell} = [\mathbf{R}_{\text{G}}^{\text{SCM}}(\boldsymbol{\theta})]_{k,\ell} + \frac{(\eta - 1)}{\alpha_{k,k}\alpha_{\ell,\ell}}\text{Re}(\alpha_{k,\ell}^*\mathbf{a}_k^H\mathbf{U}'\mathbf{a}_\ell), \tag{23}$$

where

$$[\mathbf{R}_{\text{G}}^{\text{SCM}}(\boldsymbol{\theta})]_{k,\ell} = \frac{1}{\alpha_{k,k}\alpha_{\ell,\ell}}\text{Re}(\alpha_{k,\ell}^*\mathbf{a}_k^H\mathbf{U}\mathbf{a}_\ell), \tag{24}$$

is the asymptotic covariance matrix of DOA estimate derived for the first time in [18, th.3.1] for circular Gaussian distributed noise.

For a single source, (22) and (23) become respectively:

$$R_{\text{NG}}^{\text{SCM}}(\theta_1) = \frac{1}{\alpha_{1,1}}\left[\frac{\sigma_n^2}{\sigma_1^2} + \frac{1}{\|\mathbf{a}_1\|^2}\frac{\sigma_n^4}{\sigma_1^4}\right] + \frac{\beta_1}{\alpha_{1,1}^2}\left[(\kappa - 2)\frac{\sigma_n^4}{\sigma_1^4}\right], \tag{25}$$

$$R_{\text{CES/CCG}}^{\text{SCM}}(\theta_1) = \frac{1}{\alpha_{1,1}}\left[\frac{\sigma_n^2}{\sigma_1^2} + \frac{1}{\|\mathbf{a}_1\|^2}\frac{\sigma_n^4}{\sigma_1^4}\right] + \frac{(\eta - 1)}{\alpha_{1,1}}\frac{1}{\|\mathbf{a}_1\|^2}\frac{\sigma_n^4}{\sigma_1^4}, \tag{26}$$

where β_1 is a positive purely geometric factor and $\sigma_1^2 \stackrel{\text{def}}{=} \text{E}|x_t^2|$ [resp., $\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^T x_t^2$] in the stochastic [resp. deterministic] model. Note that the additive term in (22), (23), (25) and (26) is inversely proportional to the square of the SNR. Therefore, non-Gaussian noise effects can occur mainly at low SNR values. Furthermore, note that for sub-Gaussian ($\kappa < 2$) noise distributions, this asymptotic variance is slightly reduced as $\kappa \geq 1$. On the other hand, it can be largely increased for super-Gaussian ($\kappa > 2$) noise distributions because κ is not upper-bounded, and can also be very large for heavy-tailed noise distributions and for low SNRs.

4. Asymptotic performance of subspace-based algorithms associated with the *M*-estimate of covariance

To mitigate the loss of performance of subspace-based algorithms for strong non-Gaussian distributed noise, the SCM can be replaced by the ML estimate of $\mathbf{R}_y$. However, this estimate cannot be obtained for specific non-Gaussian RES/CES distributions of $\mathbf{x}_t$ and $\mathbf{n}_t$ in (1) because the family of RES/CES distributions are not closed under summation of independent RES/CES distributed random variables. To overcome this difficulty, we assume in this section that the observations $\mathbf{y}_t$ in (1) are independent zero-mean RES/CES identically distributed, with stochastic representation (13) and finite fourth-order moments and whose covariance $\mathbf{R}_y$ is always given by (2). Note that this model is approximately satisfied in the strong noise scenario where the RES/CES distributed noise fixes the distribution of $\mathbf{y}_t$. With this new model, we first recast the derivations of the previous section to compare the asymptotic performance of subspace-based algorithms based on the SCM and *M*-estimate.

4.1. Asymptotic distribution of the projector associated with the SCM

By following the approach of Section 3.1, the sequence $\sqrt{T}\big(\text{vec}(\mathbf{R}_{y,T}) - \text{vec}(\mathbf{R}_y)\big)$ converges in distribution to the zero-mean Gaussian distribution $\mathcal{N}(\mathbf{0}, \mathbf{R}_{r_y}, \mathbf{C}_{r_y})$ in the complex case and $\mathcal{N}\big(\mathbf{0}, \mathbf{R}_{r_y}\big)$ in the real case, of covariance matrix:

$$\mathbf{R}_{r_y} = \eta(\mathbf{R}_y^T \otimes \mathbf{R}_y)\mathbf{K}' + (\eta - 1)\text{vec}(\mathbf{R}_y)\text{vec}^+(\mathbf{R}_y), \tag{27}$$

where η is given by (16). This gives, following the steps of Section 3.2, the sequence $\sqrt{T}\big(\text{vec}(\mathbf{\Pi}_{y,T}) - \text{vec}(\mathbf{\Pi}_y)\big)$ converges also in distribution to the zero-mean Gaussian distribution $\mathcal{N}(\mathbf{0}, \mathbf{R}_{\pi_y}, \mathbf{C}_{\pi_y})$ of covariance matrix:

$$\mathbf{R}_{\pi_y} = \eta[(\mathbf{U}^T \otimes \mathbf{\Pi}_y) + (\mathbf{\Pi}_y^T \otimes \mathbf{U})]\mathbf{K}'. \tag{28}$$

4.2. Asymptotic distribution of the projector associated with the *M*-estimate

We start to consider the ML estimate of the scatter matrix $\mathbf{\Sigma}_y$ of the zero-mean RES/CES distribution of $\mathbf{y}_t$, which is equal here to its covariance $\mathbf{R}_y$. This ML estimate is solution of the implicit equation:

$$\mathbf{\Sigma}_{y,T} = \frac{1}{T}\sum_{t=1}^{T} u\big(\mathbf{y}_t^+ \mathbf{\Sigma}_{y,T}^{-1}\mathbf{y}_t\big)\mathbf{y}_t\mathbf{y}_t^+, \tag{29}$$

where $u(t) \stackrel{\text{def}}{=} -\frac{1}{g(t)}\frac{dg(t)}{dt}$ is a real-valued non-negative weight function fixed by the density generator $g(.)$ of the underlying RES/CES distribution [15, (38)], whose probability density function can be written as:

$$p(\mathbf{y}_t) \propto |\mathbf{\Sigma}_y|^{-1} g\big(\mathbf{y}_t^H \mathbf{\Sigma}_y^{-1}\mathbf{y}_t\big) \quad \text{(complex case)},$$
$$p(\mathbf{y}_t) \propto |\mathbf{\Sigma}_y|^{-1/2} g\big(\mathbf{y}_t^T \mathbf{\Sigma}_y^{-1}\mathbf{y}_t\big) \quad \text{(real case)}. \tag{30}$$

Note that under specific conditions on the density generator given in [19] for RES and extended to CES in [15], the solution of the implicit Eq. (29) is unique and can be obtained by an iterative fix point algorithm, given any initial positive definite Hermitian matrix $\mathbf{\Sigma}_0$.

Furthermore, this estimate belongs to the class of *M*-estimators of scatter matrices introduced by Maronna [19], where $u(.)$ does not need to be related to the density generator of any particular RES/CES distribution. Existence and uniqueness of the solution $\mathbf{\Sigma}_{y,T}^u$ of (29) have been proved in the real case, provided $u(.)$ satisfies a set of general conditions (called Maronna conditions) stated by Maronna in [19]. These conditions have been extended to the complex case in [11] and [15]. Under these conditions, it has been also proved in the real case that the solution of (29) can be derived by an iterative fix point algorithm. The sequence $\mathbf{\Sigma}_{y,T}^u$ of solutions of (29) converges in probability to $\mathbf{\Sigma}_y^u$ proportional to $\mathbf{\Sigma}_y$ ($\mathbf{\Sigma}_y^u = \sigma_u \mathbf{\Sigma}_y$, where σ_u depends on $u(.)$ and the RES distribution of $\mathbf{y}_t$). The extension of these results to the complex case has been done in [15].

These *M*-estimates of scatter matrices have been proposed for robust estimate of $\mathbf{\Sigma}_y$ against outliers and heavy-tailed non-Gaussian distributions. Under these Maronna conditions, it has been proved in the real case [20] and the complex case [10,15], that the sequence $\sqrt{T}(\text{vec}(\mathbf{\Sigma}_{y,T}^u) - \text{vec}(\mathbf{\Sigma}_y^u))$ converges in distribution to the zero mean Gaussian distribution $\mathcal{N}(\mathbf{0}, \mathbf{R}_{\Sigma_y^u}, \mathbf{C}_{\Sigma_y^u})$ in the complex case and $\mathcal{N}(\mathbf{0}, \mathbf{R}_{\Sigma_y^u})$ in the real case, with covariance:

$$\mathbf{R}_{\Sigma_y^u} = \vartheta_1(\mathbf{\Sigma}_y^T \otimes \mathbf{\Sigma}_y)\mathbf{K}' + \vartheta_2 \text{vec}(\mathbf{\Sigma}_y)\text{vec}^+(\mathbf{\Sigma}_y), \tag{31}$$

where the parameters ϑ_1 and ϑ_2 [15, rels. (48), (49)] and [10, rels. (7), (12)] depend on $u(.)$ and the involved RES/CES distribution. They are given with our notations by:

$$\vartheta_1 = \frac{\text{E}[u^2(\mathcal{Q}_t/\sigma_u)\mathcal{Q}_t^2]}{N(N+1)(1+[N(N+1)]^{-1}c_u)^2} \quad \text{and}$$
$$\vartheta_2 = \frac{\text{E}[(u(\mathcal{Q}_t/\sigma_u)\mathcal{Q}_t - N\sigma_u)^2]}{(N+c_u)^2} - \frac{\vartheta_1}{N}, \tag{32}$$

with $c_u \stackrel{\text{def}}{=} \text{E}[u'(\mathcal{Q}_t/\sigma_u)\mathcal{Q}_t^2/\sigma_u^2]$ [15, (47)] where $u'(x) \stackrel{\text{def}}{=} du(x)/dx$ and σ_u solution of $\text{E}[u(\mathcal{Q}_t/\sigma_u)\mathcal{Q}_t/\sigma_u] = N$[15, (46)]. Note that $\sigma_u = 1$ for finite second-order moments of $\mathbf{y}_t$ and ML estimate of $\mathbf{\Sigma}_y$ (i.e., for $u(t) = -\frac{1}{g(t)}\frac{dg(t)}{dt}$).

Finally note that the covariance matrix $\mathbf{R}_{r_y}$ (27) of the asymptotic distribution of the SCM can be derived form (31) for the weight function $u(t) = 1$ for which $c_u = 0$, which gives: $\vartheta_1 = \text{E}(\mathcal{Q}_t^2)/(N(N+1)) = \eta$ and $\vartheta_2 = \text{E}(\mathcal{Q}_t^2)/(N(N+1)) - 1 = \eta - 1$ from (16).

Now, denote by $\mathbf{\Pi}_{y,T}$, the noise projector built from the SVD of $\mathbf{\Sigma}_{y,T}^u$. Noting that $\mathbf{\Sigma}_y^u$, $\mathbf{\Sigma}_y = \mathbf{R}_y$ are proportional and following the approach of Section 3.1, we get the following result:

Result 3. The sequence $\sqrt{T}\big(\text{vec}(\mathbf{\Pi}_{y,T}) - \text{vec}(\mathbf{\Pi}_y)\big)$ converges in distribution to the zero-mean Gaussian distribution $\mathcal{N}(\mathbf{0}, \mathbf{R}_{\pi_y}, \mathbf{C}_{\pi_y})$ in the complex case and $\mathcal{N}\big(\mathbf{0}, \mathbf{R}_{\pi_y}\big)$ in the real case of covariance matrix:

$$\mathbf{R}_{\pi_y} = \vartheta_1[(\mathbf{U}^T \otimes \mathbf{\Pi}_y) + (\mathbf{\Pi}_y^T \otimes \mathbf{U})]\mathbf{K}'. \tag{33}$$

Following the steps Section 3.3, Result 2 also applies here. Thus the following result is obtained.

Result 4. The asymptotic covariance of the DOA estimated by the MUSIC algorithm associated with the SCM and the covariance *M*-estimate (29) are respectively given for CES distributed observations by:

$$[\mathbf{R}_{\text{CES/CCG}}^{\text{SCM}}(\boldsymbol{\theta})]_{k,\ell} = \eta[\mathbf{R}_{\text{G}}^{\text{SCM}}(\boldsymbol{\theta})]_{k,\ell} \quad \text{and}$$
$$[\mathbf{R}_{\text{CES/CCG}}^{\text{M.Est}}(\boldsymbol{\theta})]_{k,\ell} = \vartheta_1[\mathbf{R}_{\text{G}}^{\text{SCM}}(\boldsymbol{\theta})]_{k,\ell}. \tag{34}$$

Therefore, the gain from using an *M*-estimate of the covariance rather than the SCM is uniquely characterized by the factor ϑ_1/η, which depends on the specified non-Gaussian noise distributions.

5. Numerical illustrations

This section illustrates the theoretical performance of the MUSIC-based DOA estimation algorithm given by Results 2 and 4. We consider throughout this section two uncorrelated sources of equal power ($\sigma_1^2 = \sigma_2^2$) are impinging on an ULA with $N = 6$ sensors for which $\mathbf{a}_k = (1, e^{j\theta_k}, \dots, e^{j(N-1)\theta_k})^T$ where $\theta_k = \sin(\pi\alpha_k)$ with α_k are the DOAs relative to the normal of array broadside. The SNR is defined as SNR $= \sigma_1^2/\sigma_n^2$. The following table proved in the Appendix, gives the values of the parameter η and ϑ_1 that are involved in the different asymptotic covariance matrices.

In the first experiment the sources are QPSK distributed and the noise $\mathbf{n}_t$ is either circular complex Student t-distributed (i.e., $\mathbf{n}_t \sim \mathbb{C}t_{N,\nu}(\mathbf{0}, \sigma_n^2\mathbf{I})$) with parameter $\nu > 4$ to have finite fourth-order moment or complex circular Gaussian distributed (i.e., $\mathbf{n}_t \sim \mathcal{N}(\mathbf{0}, \sigma_n^2\mathbf{I}, \mathbf{0})$ obtained also for $\nu \to \infty$). Fig. 1 compares the asymptotic variances of DOA estimates obtained with the MUSIC algorithm based on the SCM, given by (23) for the two previously described noise models. These asymptotic variances are also compared to the corresponding MSEs (where $T = 500$ estimated by 1000 Monte Carlo runs). It can be seen from this figure that the CES distributed noise model cause deeper loss of performance of the MUSIC algorithm based on SCM for weak SNR. Note that the CRB cannot be plotted in this figure because $\mathbf{y}_t$ is distributed here as a mixture of four circular complex Student t-distributions, whose CRB is not available.

Fig. 2 shows the theoretical ratio $r \stackrel{\text{def}}{=} R_{\text{G}}^{\text{SCM}}(\theta_1)/R_{\text{CES/CCG}}^{\text{SCM}}(\theta_1)$ of the asymptotic variances for Gaussian distributed and Student t-distributed noise with respect to the DOA separation $\Delta\theta = |\theta_2 - \theta_1|$ for different values of the parameter ν at SNR $= 10dB$. We see that the performance deteriorates strongly for small DOA separation and for small parameter ν ($\nu \to 4$), i.e., for heavy-tailed noise distributions. Obviously, this ratio tends to 1 for $\nu \to \infty$ (Gaussian case).

In the second experiment, the observations $\mathbf{y}_t$ are CES distributed of covariance $\mathbf{R}_y = \mathbf{\Sigma}_y = \sum_{k=1}^2 \sigma_k^2 \mathbf{a}_k \mathbf{a}_k^H + \sigma_n^2\mathbf{I}$. Figs. 3 and 4 exhibit the ratio $r \stackrel{\text{def}}{=} R_{\text{CES/CCG}}^{\text{M.Est}}(\theta_1)/R_{\text{CES/CCG}}^{\text{SCM}}(\theta_1) = \vartheta_1/\eta$ deduced from (34), versus N for different values of the parameters ν and β for the circular complex Student t-distribution $\mathbb{C}t_{N,\nu}(\mathbf{0}, \mathbf{\Sigma}_y)$ and the circular complex generalized Gaussian distribution $\mathbb{C}GG_{N,\beta}(\mathbf{0}, \mathbf{\Sigma}_y)$, respectively.

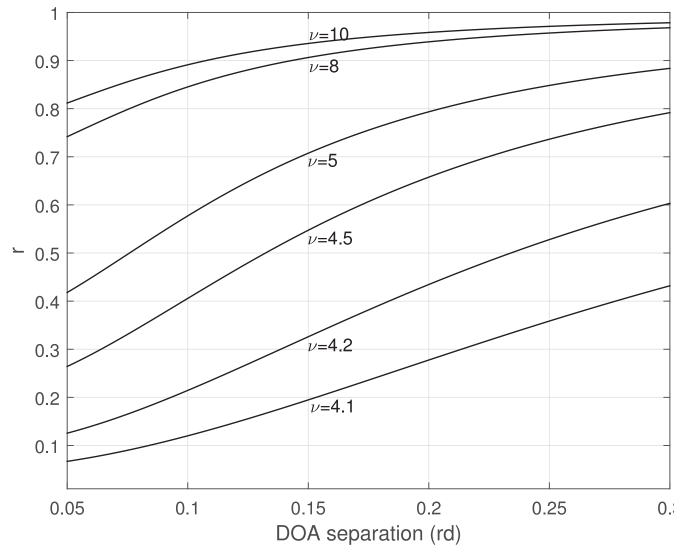

Fig. 2. Theoretical ratio $r \stackrel{\text{def}}{=} R_{\text{G}}^{\text{SCM}}(\theta_1)/R_{\text{CES/CCG}}^{\text{SCM}}(\theta_1)$ as a function of $\Delta\theta$ for SNR = 10dB.

In Fig. 3, we can see performance improvements escalating when the ML estimate of $\mathbf{R}_y$ (M-estimate) is used instead of the SCM with decreasing spikiness parameter ν. For $\nu \to \infty$, the noise tends to be Gaussian distributed (where the SCM is the ML estimate of the covariance matrix) and the performance tends to be equivalent. In Fig. 4, we see that this improvement of performance increases both when $\beta > 1$ (super-Gaussian case) and $\beta < 1$ (sub-Gaussian case), and obviously disappears if the noise is Gaussian distributed (i.e., $\beta = 1$). Moreover, we see that the number N of sensors has little impact on improving the performance for circular complex Student t-distributed observations compared to those distributed with a circular complex generalized Gaussian distribution.

Fig. 5 compares the CRB derived in [21] and [22], the asymptotic variances of DOA estimates obtained with the MUSIC algorithm based on SCM and M-estimate of the covariance given by (34) for $\mathbb{C}GGN, \beta(0, \Sigma)$ observations, and the corresponding MSEs (where $T = 500$ estimated by 2000 Monte Carlo runs). It can be

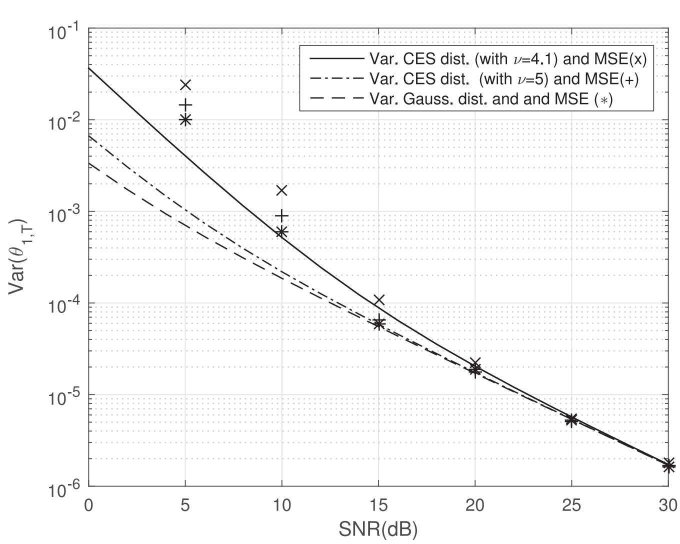

Fig. 1. Asymptotic variances Var($\theta_{1,T}$) given by (23) and (24) and its associated MSEs versus SNR with $\Delta\theta = 0.25(rd)$.

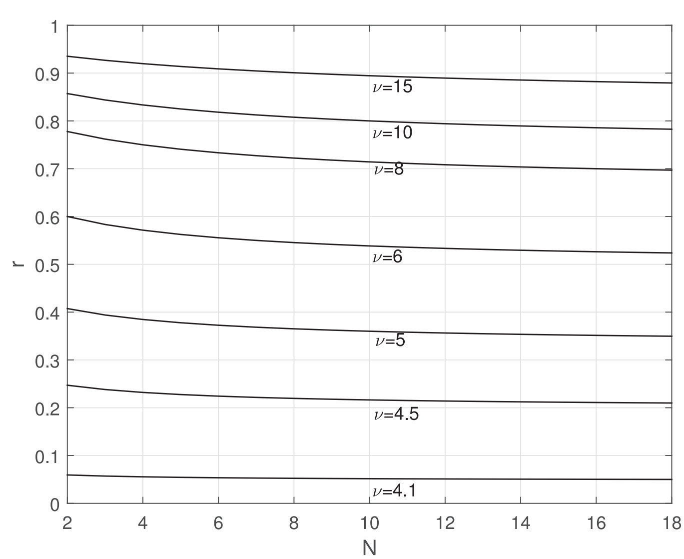

Fig. 3. $r = R_{\text{CES/CCG}}^{\text{M.Est}}(\theta_1)/R_{\text{CES/CCG}}^{\text{SCM}}(\theta_1)$ versus N for different values of ν for the $\mathbb{C}t_{N,\nu}(\mathbf{0}, \mathbf{\Sigma}_y)$ distribution.

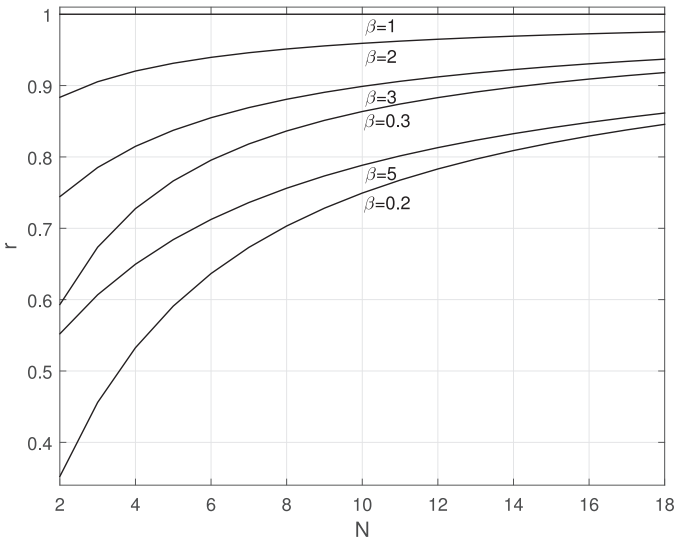

Fig. 4. $r = R^{\text{M.Est}}_{\text{CES/CCG}}(\theta_1)/R^{\text{SCM}}_{\text{CES/CCG}}(\theta_1)$ versus N for different values of β ($\beta > 1$ for super-Gaussian case and $\beta < 1$ for sub-Gaussian case) for the $\mathbb{C}GG_{N,\beta}(\mathbf{0}, \boldsymbol{\Sigma}_y)$ distribution.

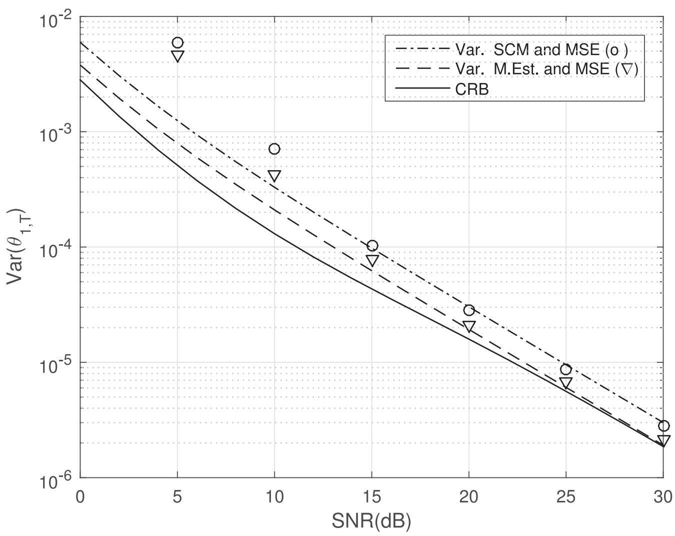

Fig. 5. Asymptotic variances given by (34), CRB and associated MSE versus SNR with $\Delta\theta = 0.25(rd)$ for the $\mathbb{C}GGN, \beta(0, \Sigma y)$ distribution.

seen from this figure the validity of the derived theoretical asymptotic variances in (34) which are in good agreement with the corresponding MSEs. We also observe, in comparison with the CRB, that the MUSIC based on M-estimate of the scatter matrix is asymptotically efficient at high SNR values.

6. Conclusion

This paper has provided theoretical tools for analyzing the performance loss of SCM-based subspace algorithms with respect to non-Gaussian heavy-tailed noise distributions, which has been analyzed until now only by numerical experiments. We have proved that this loss of performance can be mitigated by replacing the SCM with an M-estimate of the covariance for RES/CES distributions. Simulation results of the MUSIC DOA estimation algorithm have been presented to illustrate the theoretical analysis when the observations are distributed according to either circular complex Student t-distribution or circular complex generalized Gaussian distribution. But these results can be applied to many other subspace-based algorithms. Finally, note that our analysis assumes that both conditions (i) distributions have finite fourth-order moments and (ii) the M-estimate of the covariance matrix belongs to the RES/CES family of distributions are satisfied, but otherwise, they will certainly open up new avenues for performance analysis.

Declaration of Competing Interest

None.

Appendix A

Proof of Result 2. Using again the standard theorem of continuity, the DOAs estimated by the MUSIC algorithm based on $\boldsymbol{\Pi}_{y,T}$ are asymptotically Gaussian distributed with covariance $\mathbf{R}(\theta) = \mathbf{D}^g(\theta)\mathbf{R}_{\pi_y}[\mathbf{D}^g(\theta)]^H$ where the Jacobian matrix $\mathbf{D}^g(\boldsymbol{\theta})$ of the mapping g which is the MUSIC algorithm that associates $\boldsymbol{\theta}_T$ to $\boldsymbol{\Pi}_{y,T}$ is given (see e.g., [25]): by

$$\mathbf{D}^g(\theta) = \begin{pmatrix} \mathbf{d}_1^T \\ \vdots \\ \mathbf{d}_P^T \end{pmatrix} \text{ with } \mathbf{d}_k^T = \frac{-1}{\alpha_{k,k}}\left(\mathbf{a}_k'^{\,T} \otimes \mathbf{a}_k^H + \mathbf{a}_k^T \otimes \mathbf{a}_k'^{\,H}\right). \tag{35}$$

Plugging (35) and (20)-(21) in $\mathbf{R}(\theta) = \mathbf{D}^g(\theta)\mathbf{R}_{\pi_y}[\mathbf{D}^g(\theta)]^H$, (22) and (23) follow. □

Proof of Table 1.

(a) Circular complex generalized Gaussian distributions:
The density generator of this distribution is $g(t) = e^{-t^\beta/b}$ which gives the ML weight function $u(t) = (\beta/b)t^{\beta-1}$. Thus, σ_u that is solution of $\mathrm{E}[u(\mathcal{Q}_t/\sigma_u)\mathcal{Q}_t/\sigma_u] = N$[15, (46)] is given by $\sigma_u = (\beta/(bN))\mathrm{E}(\mathcal{Q}_t^\beta) = 1$ from $\mathrm{E}(\mathcal{Q}_t^\beta) = Nb/\beta$ [23, a16], $c_u \stackrel{\text{def}}{=} \mathrm{E}[u'(\mathcal{Q}_t)\mathcal{Q}_t^2] = (\beta(\beta-1)/b)\mathrm{E}(\mathcal{Q}_t^\beta) = N(\beta-1)$, and $\mathrm{E}[u^2(\mathcal{Q}_t)\mathcal{Q}_t^2] = (\beta^2/b^2)e(\mathcal{Q}_t^{2\beta}) = N(\beta+N)$ from $\mathrm{E}(\mathcal{Q}_t^{2\beta}) = (Nb^2/\beta)(1+N/\beta)$ [23, a16]. As a result $\vartheta_1 = (N+1)/(N+\beta)$ from (32).
Because $\mathrm{E}(\mathbf{u}_t\mathbf{u}_t^H) = \mathbf{I}_N/N$ [15, lemma.1], (13) implies that $\mathrm{E}(\mathcal{Q}_t) = N$. Otherwise, $\mathrm{E}(\mathcal{Q}_t) = b^{1/\beta}\frac{\Gamma[(N+1)/\beta]}{\Gamma[N/\beta]}$ and $\mathrm{E}(\mathcal{Q}_t^2) = b^{2/\beta}\frac{\Gamma[(N+2)/\beta]}{\Gamma[N/\beta]}$ from [24, (16)], where $\Gamma(.)$ is the gamma function. Consequently $\eta = \frac{N\mathrm{E}(\mathcal{Q}_t^2)}{(N+1)[\mathrm{E}(\mathcal{Q}_t)]^2} = \frac{N}{N+1}\frac{\Gamma(N/\beta)\Gamma((N+2)/\beta)}{\Gamma((N+1)/\beta)^2}$.

(b) Circular complex Student t-distribution:
The density generator of this distribution is $g(t) = \left(1+\frac{2t}{\nu}\right)^{-(2N+\nu)/2}$ which gives the ML weight function $u(t) = \frac{2N+\nu}{\nu+2t}$. σ_u solution of $\mathrm{E}[u(\mathcal{Q}_t/\sigma_u)\mathcal{Q}_t/\sigma_u] = N$[15, (46)] is equal to unity because $\mathrm{E}[u(\mathcal{Q}_t)\mathcal{Q}_t] = N$ [21, (11)]. Using $\mathrm{E}[\mathcal{Q}_t\psi'(\mathcal{Q}_t)] = \frac{\nu}{2}\frac{N}{N+1+\nu/2}$ and $\mathrm{E}[\psi^2(\mathcal{Q}_t)] = \frac{N(N+1)(N+\nu/2)}{N+1+\nu/2}$ with $\psi(x) \stackrel{\text{def}}{=} xu(x)$ from [26], we get $c_u \stackrel{\text{def}}{=} \mathrm{E}[u'(\mathcal{Q}_t)\mathcal{Q}_t^2] = \frac{\nu}{2}\frac{N}{N+1+\nu/2} - N$ (using also $\mathrm{E}[u(\mathcal{Q}_t)\mathcal{Q}_t] = N$) and $\mathrm{E}[u^2(\mathcal{Q}_t)\mathcal{Q}_t^2] = \frac{N(N+1)(N+\nu/2)}{N+1+\nu/2}$, which give $\vartheta_1 = \frac{N+1+\nu/2}{N+\nu/2}$ from (32).
From [15, (8) and Sec. IV.], $\mathrm{E}|y_{i,t}^4|/[\mathrm{E}|y_{i,t}^2|]^2 - 2 = \frac{4}{\nu-4}$ with $y_{i,t} \stackrel{\text{def}}{=} \mathbf{e}_i^T\mathbf{y}_t$. Then $\mathrm{E}|y_{i,t}^2| = \mathrm{E}(\tau_t)(\mathbf{e}_i^T\boldsymbol{\Sigma}_y\mathbf{e}_i)$ and $\mathrm{E}|y_{i,t}^4| = 2\mathrm{E}(\tau_t^2)(\mathbf{e}_i^T\boldsymbol{\Sigma}_y\mathbf{e}_i)^2$ from (13). Consequently $\frac{4}{\nu-4} + 2 = 2\frac{\mathrm{E}(\tau_t^2)}{[\mathrm{E}(\tau_t)]^2} = 2\mathrm{E}(\tau_t^2)$. This implies $\eta = \frac{\nu-2}{\nu-4}$. □

Table 1
Parameters η and ϑ_1 used in the illustrations.

CES Distributions	η	ϑ_1
$\mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}, \mathbf{0})$	1	1
$\mathbb{C}t_{N,\nu}(\mathbf{0}, \boldsymbol{\Sigma})$	$\frac{\nu-2}{\nu-4}$	$\frac{N+1+\nu/2}{N+\nu/2}$
$\mathbb{C}GG_{N,\beta}(\mathbf{0}, \boldsymbol{\Sigma})$	$\frac{N}{N+1}\frac{\Gamma(N/\beta)\Gamma((N+2)/\beta)}{\Gamma((N+1)/\beta)^2}$	$\frac{N+1}{N+\beta}$

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