Asymptotic distribution of circularity coefficients estimate of complex random variables

Jean Pierre Delmas *, Habti Abeida

Institut TELECOM, TELECOM SudParis, Département CITI, CNRS UMR 5157, 91011 Evry Cedex, France

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ABSTRACT

In this correspondence we mainly consider the asymptotic distribution of the estimator of circularity coefficients of scalar and multidimensional complex random variables. A particular attention is paid to rectilinear RV. After deriving new properties of the circularity coefficients, the maximum likelihood estimate of the circularity coefficients in the Gaussian case and asymptotic distribution of this estimate for arbitrary distributions are given. Finally, an illustrative example is presented in order to strengthen the obtained theoretical results.

1. Introduction

Many papers (see, e.g. [1–3]) show that significant performance gains can be achieved by the second-order algorithms exploiting the statistical information contained in the complementary covariance [4] matrix $\mathbf{R}'_z = E(\mathbf{z}\mathbf{z}^T)$ provided it is non-zero (also termed as relation matrix in [5] and pseudo-covariance matrix in [6]) in addition to that contained in the standard covariance matrix $\mathbf{R}_z = E(\mathbf{z}\mathbf{z}^H)$. Signals such that $\mathbf{R}'_z \neq \mathbf{O}$, referred to as improper [4] or second-order non-circular [5,6], occur in many signal processing applications. Particularly many digital modulations of practical interest, such as BPSK, M-ASK, OQPSK MSK and GMSK contain improper processes. But these signals have received a resurgence of interest. To assess detection or estimation performance of

algorithms adapted to improper signals, the circularity spectrum, whose elements are referred to as circularity coefficients in [6] and fruitfully interpreted as canonical correlations between $\mathbf{z}$ and $\mathbf{z}^*$ simultaneously and independently introduced in [7], has been introduced. These parameters have been studied from different points of view. They allow one to specify conditions for identifiability, separability and uniqueness in complex-valued independent component analysis in [6]. They are used to design a generalized likelihood ratio test (GLRT) for impropriety in [7] through a specific measure for the degree of impropriety. Different measures of this degree of impropriety have been proposed in [8], where upper and lower bounds have been derived. Finally, it has been proved in [9] that two random variables (RVs) with identical covariance matrix $\mathbf{R}_z = E(\mathbf{z}\mathbf{z}^H)$ and identical circularity coefficients can behave differently in second-order estimation and detection. Note that the results of this paper have already been partially presented in [10].

The aim of this correspondence is twofold. First, it aims in Section 2 to complement previously available theoretical results by introducing the non-circularity phase for

* Corresponding author. Tel.: +33 1 60 76 46 32; fax: +33 1 60 76 44 33.
  E-mail addresses: jean-pierre.delmas@int-evry.fr,
jean-pierre.delmas@it-sudparis.eu (J.P. Delmas), abeida3@yahoo.fr
(H. Abeida).

scalar complex RV and attempts to extend it to multi-dimensional complex RVs with particular attention paid to rectilinear complex RVs. Second, since the knowledge of the circularity coefficients are generally crucial for using specific second-order algorithms dedicated to improper signals (e.g., in direction of arrival estimation, the specific algorithms [11,12] dedicated to rectilinear sources fails when they are used for quasi-rectilinear sources), the maximum likelihood (ML) estimate of the circularity coefficients in the Gaussian case and asymptotic distribution of this estimate for arbitrary distributions useful in practice for deriving confidence intervals are considered in Section 3. We note that ML estimates have also been considered in [7]. However, the problem addressed was not the ML estimate of the circularity coefficients, but rather the GLRT for impropriety of complex signals where only the ML estimates of $\mathbf{R}_z$ and $\mathbf{R}_z'$ are needed to derive the GLRT.

2. Rectilinear complex random variable

2.1. Scalar complex random variable

Let $z = x + iy$ denote a zero-mean second-order scalar complex RV with variance $\sigma_z^2 \overset{\text{def}}{=} E(|z|^2)$ and complementary variance $E(z^2)$. The circularity coefficient ρ of z that is real-valued, non-negative and bounded above by 1 (see e.g. [6, lemma 4]) is defined by

$$\rho e^{2i\phi} \overset{\text{def}}{=} \frac{E(z^2)}{E(|z|^2)}, \tag{1}$$

where $\phi \in [0, \pi)$ is referred to as the non-circularity phase of z in [13]. We note that this non-circularity phase has been introduced as a crucial parameter to study the statistical performance of MUSIC-like algorithms for estimating DOA of narrowband complex non-circular sources in [13] and then to characterize the resolution of closely spaced sources in [14].

If $\rho = 0$, z is referred to as proper in [15,4] or circular to the second-order in [5,6] and if $\rho = 1$, z is referred to as rectilinear in [16] (because in this case $z = |z|e^{i\phi}$ and z lies in one line of $\mathscr{C}$) or most improper in [9]. If $\rho_{co} \overset{\text{def}}{=} E(xy)/\sigma_x \sigma_y$ with $\sigma_x \overset{\text{def}}{=} \sqrt{E(x^2)}$ and $\sigma_y \overset{\text{def}}{=} \sqrt{E(y^2)}$, denotes the correlation coefficient between the real x and imaginary y parts of z, the following relations between ρ and ρ_{co} are proved in the Appendix.[1]

Result 1. The circularity coefficient ρ of a scalar complex non-degenerate ($\sigma_x \neq 0$ and $\sigma_y \neq 0$) RV z and the correlation coefficient ρ_{co} between its real x and imaginary y parts are related by the following relations:

- $\rho = 1 \Leftrightarrow \rho_{co} = \pm 1$,
- $\rho = 0 \Rightarrow \rho_{co} = 0$, the converse is false because $\rho_{co} = 0$ does not imply $\sigma_x = \sigma_y$.
- $\rho \geq |\rho_{co}|$ and $\rho = |\rho_{co}|$ when $\sigma_x = \sigma_y$.

To interpret the non-circularity phase ϕ of z, the following result is proved in the Appendix:

Result 2. For a non-circular scalar complex RV z, the orthogonal regression line of the couple (x, y) has a direction given by the non-circularity phase ϕ and the mean square orthogonal distance to this line is given[2] by

$$E(d^2) = \frac{\sigma_z^2}{2}(1 - \rho). \tag{2}$$

Consequently, the larger ρ is, the smaller the mean square distance of (x, y) to the orthogonal regression line will be and this distance is zero if and only if z is rectilinear along this orthogonal regression line whose direction is given by the non-circularity phase ϕ.

2.2. Multidimensional complex random variable

Consider now a full K-dimensional zero-mean second-order complex RV $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ (i.e., with $\mathbf{R}_z$ nonsingular). The canonical correlations between $\mathbf{z}$ and $\mathbf{z}^*$ i.e., the circularity coefficients of $\mathbf{z}$, denoted by $(\rho_k)_{k=1,\dots,K}$ has been defined in [6,7] as the singular values of the coherence matrix[3] $\mathbf{M} = \mathbf{R}_z^{-1/2}\mathbf{R}_z'\mathbf{R}_z^{-T/2}$ associated with $\mathbf{z}$ and $\mathbf{z}^*$, that arranged in decreasing order satisfy $1 = \rho_1 = \cdots = \rho_r > \rho_{r+1} \geq \cdots \geq \rho_K \geq 0$ where r denotes the number of circularity coefficients equal to 1.

To attempt to extend the notion of non-circularity phase introduced in the scalar case, we first interpret the specific case $r = K$ introduced in [9] and referred to as most improper complex RVs $\mathbf{z}$. Using [9, rel. (12)] and [7, rel. (18), (19)] with $\mathbf{K} = \mathbf{I}$, it is straightforward to prove the following equivalences using the definitions $\tilde{\mathbf{z}} \overset{\text{def}}{=} [\mathbf{z}^T, \mathbf{z}^H]^T$, $\mathbf{w} \overset{\text{def}}{=} [\mathbf{x}^T, \mathbf{y}^T]^T$, $\mathbf{R}_{\tilde{z}} \overset{\text{def}}{=} E(\tilde{\mathbf{z}}\tilde{\mathbf{z}}^H)$ and $\mathbf{R}_w \overset{\text{def}}{=} E(\mathbf{w}\mathbf{w}^T)$:

(i) the circularity spectrum is maximum, i.e., $\rho_1 = \rho_2 = \cdots = \rho_K = 1$,
(ii) rank($\mathbf{R}_{\tilde{z}}$) = K (i.e., $\tilde{\mathbf{z}}$ belongs to a K-dimensional subspace of $\mathscr{C}^{2K}$),
(iii) rank($\mathbf{R}_w$) = K (i.e., $\mathbf{w}$ belongs to a K-dimensional subspace of $\mathscr{R}^{2K}$),
(iv) there exists a square root $\mathbf{R}_z^{1/2}$ of $\mathbf{R}_z$ suchthat $\mathbf{R}_z' = \mathbf{R}_z^{1/2}\mathbf{R}_z^{*/2}$,
(v) there exist square roots $\mathbf{R}_x^{1/2}$ and $\mathbf{R}_y^{1/2}$ of $\mathbf{R}_x$ and $\mathbf{R}_y$, respectively, such that $\mathbf{R}_{x,y} = \mathbf{R}_x^{1/2}\mathbf{R}_y^{1/2}$.

By analogy with the scalar case, we propose to call *rectilinear* such complex multidimensional RVs $\mathbf{z}$ whose circularity spectrum is maximum. Note that if the components $(z_1, \dots, z_K)$ of $\mathbf{z}$ are all rectilinear, there are K linear relations $y_k = \tan(\phi_{z_k})x_k$ $(k = 1, \dots, K)$ between

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[1] Note that there is a typo in the third property of Result 1 of [10].

[2] Note that the expression $((\sigma_x^2 + \sigma_y^2) - \sqrt{(\sigma_x^2 + \sigma_y^2)^2 - 4\sigma_x^2\sigma_y^2(1 - \rho_{co}^2)})/2$ of this distance as a function of the correlation coefficient ρ_{co} given by the minimum eigenvalue of $\mathbf{R}_w$ is much involved.

[3] Note that the coherence matrix $\mathbf{M}$ depends on the specific square root $\mathbf{R}_z^{1/2}$ of $\mathbf{R}_z$, unique only if it is imposed to be positive definite Hermitian, in contrast to the circularity coefficients $(\rho_1, \dots, \rho_K)$ which are always unique [6, th. 2].

the components of $\mathbf{w}$, consequently rank$(\mathbf{R}_w) = K$ and $\mathbf{z}$ is rectilinear.[4] But the converse is not true: if $\mathbf{z}$ is rectilinear, its components $(z_k)_{k=1,\ldots,K}$ need not have maximum circularity coefficients ρ_{z_k}. For example, let $\mathbf{z} = (z_1, z_2)^T$ where z_1 is circular and $z_2 = x_2 + iy_2$ with $x_2 = ax_1$ and $y_2 = y_1$. $\mathbf{z}$ is rectilinear because $\mathbf{w}$ belongs to a two-dimensional subspace of $\mathcal{R}^4$ but the circularity coefficients of z_1 and z_2 are $\rho_{z_1} = 0$ and $\rho_{z_2} = |a^2 - 1|/(a^2 + 1)$.

To extend to the multidimensional case, the non-circularity phase ϕ defined in the scalar case by (1), we propose a definition based on the K-dimensional orthogonal regression subspace of $(x_1, \ldots, x_K, y_1, \ldots, y_K)$ which is the support of $\mathbf{w}$ for a maximum circularity spectrum. The canonical angles $(\phi_1, \phi_2, \ldots, \phi_{K^2})$ between this subspace and each of the K hyperspaces $(y_k = 0)_{k=1,\ldots,K}$ of $\mathcal{R}^{2K}$ satisfy this aim. However, two questions remain open. First, how to extend the expression (2) of the mean square orthogonal distance to this K-dimensional orthogonal regression subspace given in Result 2? Second, can on prove that the parameter $(\rho, \phi, \mathbf{R}_z)$ with $\phi \stackrel{\text{def}}{=} (\phi_1, \phi_2, \ldots, \phi_{K^2})^T$ makes up a one to one parametrization of $(\mathbf{R}_z, \mathbf{R}'_z)$?

3. Asymptotic distribution of the circularity spectrum estimate

Let us consider the estimation of the circularity spectrum ρ from T independent identically distributed realizations $(\mathbf{z}_t)_{t=1,\ldots,T}$. The scalar and multidimensional cases are distinguished for the ease of the reader although the derivation mainly follows the same lines.

3.1. Scalar complex random variable

In the scalar-valued case, the following result is proved in the Appendix.

Result 3. When z_t is Gaussian distributed, the maximum likelihood estimate $(\widehat{\rho}_T, \widehat{\phi}_T)$ of (ρ, ϕ) is given by $(|\sum_{t=1}^{T} z_t^2|/\sum_{t=1}^{T} |z_t|^2, \frac{1}{2}\text{Arg}(\sum_{t=1}^{T} z_t^2/\sum_{t=1}^{T} |z_t|^2))$. Furthermore, when z_t is arbitrarily distributed with $\rho < 1$, the sequence $\sqrt{T}(\widehat{\rho}_T - \rho)$ converges in distribution to the zero-mean Gaussian distribution of variance

$$c_\rho = 1 - 2\rho^2 + \rho^4 + \rho^2 \kappa + \frac{\kappa}{2} + \frac{\rho^2 \Re(\kappa')}{2} - 2\rho^2 \Re(\kappa'')$$
$$\text{if } c_\rho \neq 0, \qquad (3)$$

where κ, κ' and κ'' are the normalized-like cumulants $\text{Cum}(z, z, z^*, z^*)/(E(|z|^2))^2$, $\text{Cum}(z, z, z, z)/(E(z^2))^2$ and $\text{Cum}(z, z, z, z^*)/E(|z|^2)E(z^2)$, respectively, which are invariant to any rotation of the distribution of z.

Note that the variance c_ρ of the asymptotic distribution of $\widehat{\rho}_T$ is a decreasing function $1 - 2\rho^2 + \rho^4$ of ρ when z_t is Gaussian distributed ($\kappa = \kappa' = \kappa'' = 0$). Furthermore, note that $c_\rho \leq 1 - 2\rho^2 + \rho^4$ is not valid for an arbitrary

distribution of z (in other words, the Gaussian case is not a worst case for the estimation of ρ). To be convinced, consider $z \stackrel{\text{def}}{=} re^{i\alpha}$ where r and α are, respectively, Bernoulli (p) and uniform on $[0, 2\pi)$ independent RVs. In this case, z is zero-mean and circular to the second-order (i.e., $\rho = 0$) and $E(|z|^4) = E(|z|^2) = p$. Consequently

$$\kappa = \frac{E(|z|^4) - 2(E(|z|^2))^2 - |E(z^2)|^2}{(E(|z|^2))^2} = \frac{1}{p} - 2$$

and

$$c_0 = 1 + \frac{\kappa}{2} = \frac{1}{2p} > 1 - 2\rho^2 + \rho^4_{|\rho=0}$$

for $p < \frac{1}{2}$.

In the particular case of rectilinear RVs for which $\rho = 1$, we have $z_t = r_t e^{i\phi}$ with r_t is a real-valued RV and with ϕ fixed. Consequently, the circularity coefficient and the non-circularity phase are perfectly estimated, i.e., $\widehat{\rho}_T = 1$ and $\widehat{\phi}_T = \phi$. Besides in this case, $\kappa = \kappa' = \kappa''$ are real-valued and we check from (3) that $c_\rho = 0$ for arbitrary distribution. Furthermore, note that it is possible that $c_\rho = 0$ with $\rho < 1$ (such a situation will be illustrated in Section 3.3). In this case, the sequence $T(\widehat{\rho}_T - \rho)$ converges in distribution [19, Th.B, p. 124] to a Hermitian form $\mathbf{r}^H \mathbf{\Omega} \mathbf{r}$, where $\mathbf{r}$ a two-dimensional zero-mean complex Gaussian RV, whose distribution is defined by the right-hand side of (7), and where our first-order analysis does not allow one to specify the matrix $\mathbf{\Omega}$.

3.2. Multidimensional complex random variable

In the multidimensional-valued case, the following result is proved in the Appendix.

Result 4. When $\mathbf{z}_t$ is Gaussian distributed, the ML estimate $\widehat{\boldsymbol{\rho}}_T$ of $\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots, \rho_K)^T$ is given by the vector containing the K singular values of the empirical coherence matrix $\mathbf{M}_T = \mathbf{R}_{z,T}^{-1/2} \mathbf{R}'_{z,T} \mathbf{R}_{z,T}^{-T/2}$ where $\mathbf{R}_{z,T} \stackrel{\text{def}}{=} (1/T)\sum_{t=1}^{T} \mathbf{z}_t \mathbf{z}_t^H$ and $\mathbf{R}'_{z,T} \stackrel{\text{def}}{=} (1/T)\sum_{t=1}^{T} \mathbf{z}_t \mathbf{z}_t^T$. Furthermore, when $\mathbf{z}_t$ is arbitrarily distributed and when the circularity spectrum $\boldsymbol{\rho}$ has distinct elements, the sequence $\sqrt{T}(\widehat{\boldsymbol{\rho}}_T - \boldsymbol{\rho})$ converges in distribution to a zero-mean Gaussian distribution that extends Result 3, whose covariance is given by

$$\mathbf{C}_\rho = 2\Re(\mathbf{A}_\rho \mathbf{C}_M \mathbf{A}_\rho^H + \mathbf{A}_\rho \mathbf{C}'_M \mathbf{A}_\rho^T), \qquad (4)$$

where[5] $\mathbf{A}_\rho = \frac{1}{2}(\mathbf{U} \circ \mathbf{U})^H$ with $\mathbf{U}\Delta\mathbf{U}^T$ is the singular value decomposition (SVD) of the coherence matrix $\mathbf{M}$ and $\mathbf{C}_M$ and $\mathbf{C}'_M$ are the two covariance matrices of the asymptotic distribution of the estimated coherence matrix $\mathbf{M}_T$ given by (12).

3.3. Illustrative example

Consider the baseband signal associated with a BPSK modulation. We assume no frequency offset, a sampling at the symbol rate and an inter-symbol interference of P

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[4] Note that the components $(z_k)_{k=1,\ldots,K}$ of $\mathbf{z}$ do not need to be uncorrelated as it is usually assumed in DOA estimation of non-circular sources (see e.g., [13,11,12]).

[5] $\mathbf{A} \circ \mathbf{B}$ denotes the Kharti–Rao matrix product (which is a column-wise Kronecker product, see e.g. [17]).
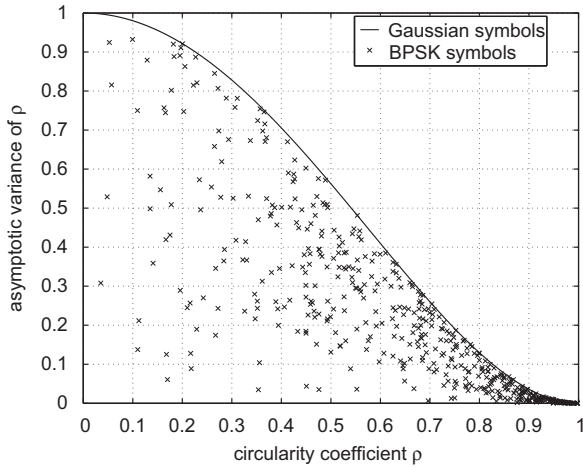
Fig. 1. Asymptotic theoretical variance c_ρ given by (3) as a function of the circularity coefficient ρ for BPSK and Gaussian symbols s_t for $P = 2$ and 500 realizations of a Rayleigh channel whose three coefficients have same variance.
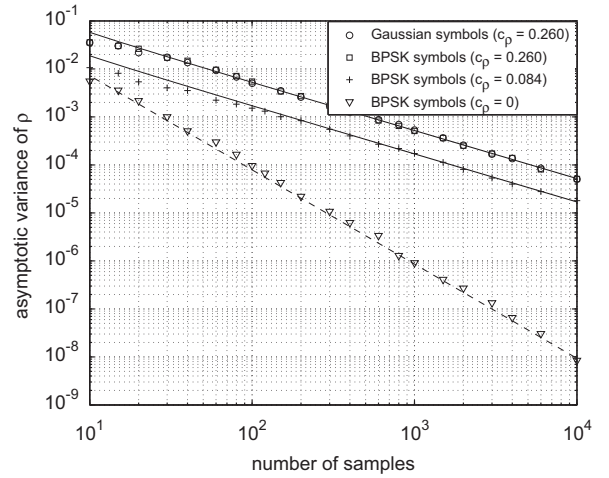

Fig. 2. Asymptotic theoretical variance and empirical (averaging 1000 runs) MSE of the estimate $\widehat{\rho}_T$, compared to the asymptotic theoretical variance c_ρ as a function of the number T of observations for Gaussian and BPSK symbols associated with different channels (h_0, h_1) for $\rho = 0.7$.

symbol, i.e.,

$$z_t = \sum_{i=0}^{P} h_i s_{t-i} \quad \text{with } s_t \in \{-1, +1\} \text{ equiprobable and}$$

$(s_t)_{t=1,2,\dots}$ independent.

Naturally the sequence s_t is rectilinear, but z_t is no longer rectilinear, except for $h_i = a_i h_0$, $i = 1, \dots, P$ with a_i real valued. We note that according to the values of $(h_i)_{i=0,\dots,P}$, the circularity coefficient ρ of z_t can take arbitrary values in $[0, 1]$. For example, z_t becomes circular for $\sum_{i=0}^{P} h_i^2 = 0$.

Regarding the values of the asymptotic variance c_ρ given by (3), we note that contrary to the Gaussian distribution, ρ does not fix c_ρ. In particular $c_\rho = 0$ and $c_\rho = 1 - 2\rho^2 + \rho^4$ can be obtained for $(h_i)_{i=0,\dots,P}$ as roots of polynomials in $(h_i, h_i^*)_{i=0,\dots,P}$. For example, for $P = 1$, it is straightforward to prove that

$$(h_0, h_1) = \left(\sqrt{\frac{1+\rho}{2}}, i\sqrt{\frac{1-\rho}{2}} \right) \quad \text{and}$$

$$(h_0, h_1) = \left(\frac{1}{4}\sqrt{\rho + i\sqrt{1-\rho^2}}, \frac{1}{4}\sqrt{\rho - i\sqrt{1-\rho^2}} \right)$$

gives $c_\rho = 0$ and $c_\rho = 1 - 2\rho^2 + \rho^4$, respectively, for arbitrary $\rho \in [0, 1]$. Furthermore, we note that extensive numerical simulations show that c_ρ is bounded above by its value associated with a Gaussian distribution, i.e.,

$$c_\rho \leq 1 - 2\rho^2 + \rho^4. \tag{5}$$

This property seems valid for arbitrary channel impulse response $(h_i)_{i=0,\dots,P}$, but we have not succeeded to prove it.

Through 500 realizations of a Rayleigh channel, Fig. 1 validates (5) and consequently shows that the asymptotic accuracy of the estimate $\widehat{\rho}_T$ is always better for BPSK than for Gaussian symbols and arbitrary value of $\rho \in [0, 1]$.

Fig. 2 represents the asymptotic theoretical variance and empirical MSE of the estimate $\widehat{\rho}_T$, as a function of T independent observations z_t after decimating the original sequence z_t by a factor of $P + 1$, where P is the memory of the simulated channel impulse response,

for BPSK and Gaussian symbols for $P = 1$ and $\rho = 0.7$. More precisely for Gaussian symbols, $(h_0, h_1) = (0.921, 0.387i)$ with $c_\rho = 0.260$ and for BPSK symbols $(h_0, h_1) = (0.710, 0.492 + 0.502i)$, $(h_0, h_1) = (0.887, 0.225 + 0.402i)$ and $(h_0, h_1) = (0.921, 0.387i)$ are associated with $c_\rho = 0.260, 0.084$ and 0, respectively. We see that the domain of validity of our asymptotic analysis for $c_p \neq 0$ is very large $(T > 100)$. Furthermore, we see that for $c_p = 0$, the empirical MSE varies in $1/T^2$ in contrast to $c_p \neq 0$, for which the asymptotic theoretical variance is c_ρ / T.

Appendix

Proof of Result 1. From the following expression of the circularity coefficient:

$$\rho = \sqrt{\frac{\left(\frac{\sigma_x}{\sigma_y} - \frac{\sigma_y}{\sigma_x}\right)^2}{\left(\frac{\sigma_x}{\sigma_y} + \frac{\sigma_y}{\sigma_x}\right)^2} + 4\rho_{co}^2 \frac{1}{\left(\frac{\sigma_x}{\sigma_y} + \frac{\sigma_y}{\sigma_x}\right)^2}},$$

the first two bullets are straightfowardly proved.

For the third one,

$$\rho^2 = 1 - \frac{4(1 - \rho_{co}^2)}{\left(\frac{\sigma_x}{\sigma_y} + \frac{\sigma_y}{\sigma_x}\right)^2} \geq \rho_{co}^2$$

because

$$\frac{\sigma_x}{\sigma_y} + \frac{\sigma_y}{\sigma_x} \geq 2. \qquad \square$$

Proof of Result 2. The orthogonal regression line (see e.g. [18]) of the couple (x, y) is given by the line orthogonal to the eigenvector $\mathbf{u}$ associated with the minimum eigenvalue λ of the covariance matrix $\mathbf{R}_w$ of $\mathbf{w} \stackrel{\text{def}}{=} [x, y]^T$ and the mean square orthogonal distance $E(d^2)$ to this line is given by λ. To solve easily this problem, it is convenient to work with the augmented vector $\tilde{\mathbf{z}} \stackrel{\text{def}}{=} [z, z^*]^T$ whose covariance

matrix $\mathbf{R}_{\tilde{z}}$ is related to $\mathbf{R}_w$ by $\mathbf{R}_w = \frac{1}{2}\mathbf{T}^H\mathbf{R}_{\tilde{z}}\mathbf{T}$ using $\tilde{\mathbf{z}} = \sqrt{2}\mathbf{T}\mathbf{w}$, where $\mathbf{T}$ is the unitary matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$. Because the minimum eigenvalue and the associated unit eigenvector of $\mathbf{R}_{\tilde{z}} = \sigma_z^2\begin{pmatrix} 1 & \rho e^{2i\phi} \\ \rho e^{-2i\phi} & 1 \end{pmatrix}$ are $\lambda = \sigma_z^2(1-\rho)$ and $\mathbf{u} = (i/\sqrt{2})[e^{i\phi}, -e^{-i\phi}]^T$, the minimum eigenvalue and the associated unit eigenvector of $\mathbf{R}_w$ are $\frac{1}{2}\lambda$ and $\mathbf{T}^H\mathbf{u} = [-\sin\phi, \cos\phi]^T \perp [\cos\phi, \sin\phi]^T$.   □

Proof of Result 3. When z_t is Gaussian distributed, the log-likelihood function associated with $(z_t)_{t=1,\dots,T}$ can be classically written after dropping the constants as

$$L(\rho, \phi, \sigma_z^2) = -\frac{T}{2}(\ln[\text{Det}(\mathbf{R}_{\tilde{z}})] + \text{Tr}(\mathbf{R}_{\tilde{z}}^{-1}\mathbf{R}_{\tilde{z},T})) \tag{6}$$

with $\mathbf{R}_{\tilde{z},T} \overset{\text{def}}{=} (1/T)\sum_{t=1}^T \tilde{\mathbf{z}}_t\tilde{\mathbf{z}}_t^H$, where the parameter (ρ, ϕ, σ_z^2) is embedded in the covariance matrix $\mathbf{R}_{\tilde{z}}$. Due to the structure $\begin{bmatrix} (\times) & (\diamond) \\ (\diamond)^* & (\times)^* \end{bmatrix}$ of $\mathbf{R}_{\tilde{z}}$, the ML estimation of $\mathbf{R}_{\tilde{z}}$ becomes a constrained optimization problem which is not standard. But maximizing the log-likelihood (6) without any constraint on the Hermitian matrix $\mathbf{R}_{\tilde{z}}$ reduces to a standard maximization problem, whose solution is $\mathbf{R}_{\tilde{z},T}$. Because

$$\mathbf{R}_{\tilde{z},T} = \begin{bmatrix} \frac{1}{T}\sum_{t=1}^T |z_t|^2 & \frac{1}{T}\sum_{t=1}^T z_t^2 \\ \frac{1}{T}\sum_{t=1}^T z_t^{*2} & \frac{1}{T}\sum_{t=1}^T |z_t|^2 \end{bmatrix}$$

is also structured as $\begin{bmatrix} (\times) & (\diamond) \\ (\diamond)^* & (\times)^* \end{bmatrix}$, $\mathbf{R}_{\tilde{z},T}$ is the ML estimate of $\mathbf{R}_{\tilde{z}}$. Using the invariance property of the ML estimate implies that the ML estimate of (ρ, ϕ) is given by $(|\sum_{t=1}^T z_t^2|/\sum_{t=1}^T |z_t|^2, \frac{1}{2}\text{Arg}(\sum_{t=1}^T z_t^2/\sum_{t=1}^T |z_t|^2))$.

Deriving the asymptotic distribution of the empirical estimate $\widehat{\rho}_T$ when z_t is arbitrarily distributed, relies on the standard central limit theorem[6] applied to the independent identically distributed bidimensional complex RVs $\begin{pmatrix} r_{z,T} \\ r'_{z,T} \end{pmatrix}$ with $r_{z,T} = (1/T)\sum_{t=1}^T |z_t|^2$ and $r'_{z,T} = (1/T)\sum_{t=1}^T z_t^2$:

$$\sqrt{T}\begin{pmatrix} r_{z,T} - r_z \\ r'_{z,T} - r'_z \end{pmatrix} \overset{\mathscr{L}}{\to} \mathscr{N}_C\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c_r & c_{r,r'} \\ c_{r',r} & c_{r'} \end{pmatrix}, \begin{pmatrix} c'_r & c'_{r,r'} \\ c'_{r',r} & c'_{r'} \end{pmatrix}\right), \tag{7}$$

where $r_z = \text{E}|z_t^2| = \sigma_z^2$, $r'_z = \text{E}(z_t^2) = \rho\sigma_z^2 e^{i2\phi}$ and where $\begin{pmatrix} c_r & c_{r,r'} \\ c_{r',r} & c_{r'} \end{pmatrix}$ and $\begin{pmatrix} c'_r & c'_{r,r'} \\ c'_{r',r} & c'_{r'} \end{pmatrix}$ denote the covariance and the complementary covariance matrices of the bidimensional RV $(|z_t^2|, z_t^2)^T$. Using the identity

$$\begin{aligned} \text{E}(z_1 z_2 z_3 z_4) = {}& \text{E}(z_1 z_2)\text{E}(z_3 z_4) + \text{E}(z_1 z_3)\text{E}(z_2 z_4) \\ & + \text{E}(z_1 z_4)\text{E}(z_2 z_3) + \text{Cum}(z_1, z_2, z_3, z_4), \end{aligned} \tag{8}$$

we straightforwardly obtain

$$\begin{pmatrix} c_r & c_{r,r'} \\ c_{r',r} & c_{r'} \end{pmatrix} = \sigma_z^4\begin{pmatrix} 1 + \rho^2 + \kappa & \rho e^{-i2\phi}(2 + \kappa''^*) \\ \rho e^{i2\phi}(2 + \kappa'') & 2 + \kappa \end{pmatrix}, \tag{9}$$

$$\begin{pmatrix} c'_r & c'_{r,r'} \\ c'_{r',r} & c'_{r'} \end{pmatrix} = \sigma_z^4\begin{pmatrix} 1 + \rho^2 + \kappa & \rho e^{i2\phi}(2 + \kappa'') \\ \rho e^{i2\phi}(2 + \kappa'') & \rho^2 e^{i4\phi}(2 + \kappa') \end{pmatrix}. \tag{10}$$

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[6] $\overset{\mathscr{L}}{\to}$ means the convergence in distribution when $T \to \infty$, while $\mathscr{N}_R(\mathbf{m}, \mathbf{C})$ and $\mathscr{N}_C(\mathbf{m}, \mathbf{C}, \mathbf{C}')$ denote Gaussian real and complex distribution with mean, covariance and complementary covariance are $\mathbf{m}$, $\mathbf{C}$ and $\mathbf{C}'$, respectively.

Then, considering the following mappings:

$$(r_{z,T}, r'_{z,T}) \mapsto m_T = \frac{r'_{z,T}}{r_{z,T}} \mapsto \widehat{\rho}_T = \sqrt{m_T m_T^*},$$

with their associated differentials

$$dm = -\frac{r'}{r^2}\,dr + \frac{1}{r}\,dr' \quad \text{and} \quad d\rho = \frac{1}{2\rho}(m^* dm + m dm^*),$$

the standard theorem of continuity (see e.g. [19, Th. A, p. 122]) on regular functions of asymptotically Gaussian statistics applies. Consequently, we obtain the following convergence in distribution with $m = r'_z/r_z = \rho e^{i2\phi}$:

$$\sqrt{T}(m_T - m) \overset{\mathscr{L}}{\to} \mathscr{N}_C(0, c_m, c'_m),$$

where

$$c_m = \begin{pmatrix} -\dfrac{r'_z}{r_z^2} & \dfrac{1}{r_z} \end{pmatrix}\begin{pmatrix} c_r & c_{r,r'} \\ c_{r',r} & c_{r'} \end{pmatrix}\begin{pmatrix} -\dfrac{r'^*_z}{r_z^2} \\ \dfrac{1}{r_z} \end{pmatrix},$$

$$c'_m = \begin{pmatrix} -\dfrac{r'_z}{r_z^2} & \dfrac{1}{r_z} \end{pmatrix}\begin{pmatrix} c'_r & c'_{r,r'} \\ c'_{r',r} & c'_{r'} \end{pmatrix}\begin{pmatrix} -\dfrac{r'_z}{r_z^2} \\ \dfrac{1}{r_z} \end{pmatrix} \tag{11}$$

and

$$\sqrt{T}(\widehat{\rho}_T - \rho) \overset{\mathscr{L}}{\to} \mathscr{N}_R(0, c_\rho),$$

where

$$c_\rho = \frac{1}{4\rho^2}\begin{pmatrix} m^* & m \end{pmatrix}\begin{pmatrix} c_m & c'_m \\ c'^*_m & c^*_m \end{pmatrix}\begin{pmatrix} m \\ m^* \end{pmatrix} = \frac{1}{2}(c_m + \Re(c'_m e^{-4i\phi})).$$

Inserting (9) and (10) into expressions (11) of c_m and c'_m, (3) follows thanks to simple algebraic manipulations.

When z is replaced with $ze^{i\alpha}$ with α fixed, using the multilinearity of $\text{Cum}(z_1, z_2, z_3, z_4)$, the normalized-like cumulants κ, κ' and κ'' are invariant, so c_p as well, whereas the non-circularity phase ϕ is replaced by $\phi + \alpha$.   □

Proof of Result 4. The proof follows the same steps that for Result 3. When $\mathbf{z}_t$ is Gaussian distributed, the log-likelihood function $L(\rho, \phi, \mathbf{R}_z)$ has form (6) and consequently $\mathbf{R}_{\tilde{z},T}$ is the ML estimate of $\mathbf{R}_{\tilde{z}}$. Using the SVD of the coherence matrix [20, 4.4.4] $\mathbf{M} = \mathbf{U}\Delta\mathbf{U}^T = \mathbf{R}_z^{-1/2}\mathbf{R}'_z\mathbf{R}_z^{-T/2}$, the invariance property of the ML implies that the ML estimate of $\boldsymbol{\rho}$ is given by the Kth vector diag(Δ_T) containing the K singular values of the empirical coherence matrix $\mathbf{M}_T = \mathbf{R}_{z,T}^{-1/2}\mathbf{R}'_{z,T}\mathbf{R}_{z,T}^{-T/2}$.

Deriving the asymptotic distribution of the empirical estimate $\widehat{\boldsymbol{\rho}}_T$ when $\mathbf{z}_t$ is arbitrarily distributed is based on the following mapping:

$$(\mathbf{R}_{z,T}, \mathbf{R}'_{z,T}) \mapsto \mathbf{M}_T \overset{\text{def}}{=} \mathbf{R}_{z,T}^{-1/2}\mathbf{R}'_{z,T}\mathbf{R}_{z,T}^{-T/2} = \mathbf{U}_T\Delta_T\mathbf{U}_T^T \mapsto \widehat{\boldsymbol{\rho}}_T \overset{\text{def}}{=} \text{diag}(\Delta_T).$$

Thus, the first step consists in deriving the asymptotic distribution of $(\mathbf{R}_{z,T}, \mathbf{R}'_{z,T})$ given by the standard central

limit theorem:

$$\sqrt{T}(\text{vec}(\mathbf{R}_{z,T}, \mathbf{R}'_{z,T}) - \text{vec}(\mathbf{R}_z, \mathbf{R}'_z))$$
$$\xrightarrow{\mathscr{L}} \mathscr{N}_C\left(\mathbf{0}, \begin{pmatrix} \mathbf{C}_R & \mathbf{C}_{R,R'} \\ \mathbf{C}_{R',R} & \mathbf{C}_{R'} \end{pmatrix}, \begin{pmatrix} \mathbf{C}'_R & \mathbf{C}'_{R,R'} \\ \mathbf{C}'_{R',R} & \mathbf{C}'_{R'} \end{pmatrix}\right),$$

where the expressions of the different covariance matrices are deduced from identity (8). For example

$$(\mathbf{C}_R)_{i+(j-1)K,k+(l-1)K} = \text{E}(z_{t,i}z_{t,j}^*z_{t,k}^*z_{t,l}) - \text{E}(z_{t,i}z_{t,j}^*)\text{E}(z_{t,k}^*z_{t,l})$$
$$= \text{E}(z_{t,i}z_{t,k}^*)\text{E}(z_{t,j}^*z_{t,l}) + \text{E}(z_{t,i}z_{t,l})\text{E}(z_{t,j}^*z_{t,k}^*)$$
$$+ \text{Cum}(z_{t,i}, z_{t,j}^*, z_{t,k}^*, z_{t,l})$$

which gives $\mathbf{C}_R = \mathbf{R}_z^* \otimes \mathbf{R}_z + \mathbf{K}(\mathbf{R}'_z \otimes \mathbf{R}'^*_z) + \mathbf{Q}_z$ where $(\mathbf{Q}_z)_{i+(j-1)K,k+(l-1)K} = \text{Cum}(z_{t,i}, z_{t,j}^*, z_{t,k}^*, z_{t,l})$ and where $\otimes$ denotes the standard Kronecker product of matrices and $\mathbf{K}$ the vec-permutation matrix which transforms $\text{vec}(\mathbf{C})$ to $\text{vec}(\mathbf{C}^T)$ for any square matrix $\mathbf{C}$. The other covariance matrices are deduced in the same way (see e.g. [3]) and are given by

$$\mathbf{C}_{R'} = \mathbf{R}_z \otimes \mathbf{R}_z + \mathbf{K}(\mathbf{R}_z \otimes \mathbf{R}_z) + \mathbf{Q}'_z,$$
$$\mathbf{C}'_R = \mathbf{C}_R \mathbf{K},$$
$$\mathbf{C}'_{R'} = \mathbf{R}'_z \otimes \mathbf{R}'_z + \mathbf{K}(\mathbf{R}'_z \otimes \mathbf{R}'_z) + \mathbf{Q}''_z,$$
$$\mathbf{C}_{R,R'} = \mathbf{C}_{R',R}^H = \mathbf{R}'^*_z \otimes \mathbf{R}_z + \mathbf{K}(\mathbf{R}_z \otimes \mathbf{R}'^*_z) + \mathbf{Q}'''_z,$$
$$\mathbf{C}'_{R,R'} = \mathbf{C}'^T_{R',R} = \mathbf{R}'_z \otimes \mathbf{R}^*_z + \mathbf{K}(\mathbf{R}'_z \otimes \mathbf{R}^*_z) + \mathbf{Q}''''_z,$$

where $(\mathbf{Q}'_z)_{i+(j-1)K,k+(l-1)K} = \text{Cum}(z_{t,i}, z_{t,j}, z_{t,k}^*, z_{t,l}^*)$, $(\mathbf{Q}''_z)_{i+(j-1)K,k+(l-1)K} = \text{Cum}(z_{t,i}, z_{t,j}, z_{t,k}, z_{t,l})$, $(\mathbf{Q}'''_z)_{i+(j-1)K,k+(l-1)K} = \text{Cum}(z_{t,i}, z_{t,j}^*, z_{t,k}^*, z_{t,l}^*)$ and $(\mathbf{Q}''''_z)_{i+(j-1)K,k+(l-1)K} = \text{Cum}(z_{t,i}, z_{t,j}^*, z_{t,k}, z_{t,l})$.

With the differential of the mapping $(\mathbf{R}_{z,T}, \mathbf{R}'_{z,T}) \mapsto \mathbf{M}_T = \mathbf{R}_{z,T}^{-1/2} \mathbf{R}'_{z,T} \mathbf{R}_{z,T}^{-T/2}$

$$\text{vec}(d\mathbf{M}) = -((\mathbf{R}_z^{-1/2}\mathbf{R}'^T_z \otimes \mathbf{I}) + (\mathbf{I} \otimes \mathbf{R}_z^{-1/2}\mathbf{R}'_z))$$
$$\times ((\mathbf{R}_z^{-T/2} \otimes \mathbf{I}) + (\mathbf{I} \otimes \mathbf{R}_z^{-1/2}))^{-1}(\mathbf{R}_z^{-T} \otimes \mathbf{R}_z^{-1})\text{vec}(d\mathbf{R}_z)$$
$$+ (\mathbf{R}_z^{-T/2} \otimes \mathbf{R}_z^{-1/2})\text{vec}(d\mathbf{R}'_z)$$
$$\overset{\text{def}}{=} \mathbf{A}_r\text{vec}(d(\mathbf{R}_z)) + \mathbf{A}_{r'}\text{vec}(d\mathbf{R}'_z)),$$

we obtain the following convergence in distribution by the standard theorem of continuity (see e.g., [19, Th.A, p. 122]):

$$\sqrt{T}(\text{vec}(\mathbf{M}_T) - \text{vec}(\mathbf{M})) \xrightarrow{\mathscr{L}} \mathscr{N}_C(\mathbf{0}, \mathbf{C}_M, \mathbf{C}'_M), \tag{12}$$

where $\mathbf{C}_M = \mathbf{A}_r\mathbf{C}_R\mathbf{A}_r^H + \mathbf{A}_r\mathbf{C}_{R,R'}\mathbf{A}_{r'}^H + \mathbf{A}_{r'}\mathbf{C}_{R'R}\mathbf{A}_r^H + \mathbf{A}_{r'}\mathbf{C}_{R'}\mathbf{A}_{r'}^H$ and $\mathbf{C}'_M = \mathbf{A}_r\mathbf{C}'_R\mathbf{A}_r^T + \mathbf{A}_r\mathbf{C}'_{R,R'}\mathbf{A}_{r'}^T + \mathbf{A}_{r'}\mathbf{C}'_{R'R}\mathbf{A}_r^T + \mathbf{A}_{r'}\mathbf{C}'_{R'}\mathbf{A}_{r'}^T$.

Finally, consider the mapping $\mathbf{M}_T \mapsto \hat{\boldsymbol{\rho}}_T = \text{diag}(\boldsymbol{\Delta}_T)$. Noting that the eigenvalues of $\mathbf{M}_T\mathbf{M}_T^H$ are the squares of the singular values of $\mathbf{M}_T$, the differential of the simple eigenvalues of $\mathbf{M}_T\mathbf{M}_T^H$ gives (see e.g., [21, th. 8, ch. 9])

$$d(\rho_k^2) = \mathbf{u}_k^H d(\mathbf{M}\mathbf{M}^H)\mathbf{u}_k, \tag{13}$$

where $\mathbf{u}_k$ denotes the left singular vector of $\mathbf{M}$ associated with the singular value ρ_k. Using $\mathbf{M}^H\mathbf{u}_k = \rho_k\mathbf{u}_k^*$, (13) gives

$d\rho_k = \frac{1}{2}(\mathbf{u}_k^H d\mathbf{M}\mathbf{u}_k^* + \mathbf{u}_k^T d\mathbf{M}^H\mathbf{u}_k^*)$, and more compactly

$$d\boldsymbol{\rho} = \frac{1}{2}((\mathbf{U} \circ \mathbf{U})^H\text{vec}(d\mathbf{M}) + (\mathbf{U} \circ \mathbf{U})^T\text{vec}(d\mathbf{M}^*))$$
$$\overset{\text{def}}{=} \mathbf{A}_\rho\text{vec}(d\mathbf{M}) + \mathbf{A}_\rho^*\text{vec}(d\mathbf{M}^*).$$

Consequently, the covariance matrix of the asymptotic Gaussian distribution of $\hat{\boldsymbol{\rho}}_T$ is given by (4). $\square$

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