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Efficiency of subspace-based DOA estimators

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Abstract

This paper addresses subspace-based direction of arrival (DOA) estimation and its purpose is to complement previously available theoretical results generally obtained for specific algorithms. We focus on asymptotically (in the number of measurements) minimum variance (AMV) estimators based on estimates of orthogonal projectors obtained from singular value decompositions of sample covariance matrices in the general context of noncircular complex signals. After extending the standard AMV bound to statistics whose first covariance matrix of its asymptotic distribution is singular and deriving explicit expressions of this first covariance matrix associated with several projection-based statistics, we give closed-form expressions of AMV bounds based on estimates of different orthogonal projectors. This enable us to prove that these AMV bounds attain the stochastic Cramer–Rao bound (CRB) in the case of circular or noncircular Gaussian signals.

Keywords: Cramer–Rao bound; Efficiency; Subspace-based DOA estimation; Noncircular signals

1. Introduction

Direction of arrival (DOA) subspace-based estimates, i.e., estimates obtained by exploiting the orthogonality between a sample subspace and a parameter-dependant subspace, have proved useful in many algorithms. There is considerable literature about the performance of such algorithms in the context of circular Gaussian signals. The performance of such algorithms are often evaluated using the stochastic and deterministic Cramer–Rao bound (CRB) (see e.g., [1,2]). In particular Porat and Friedlander [3] proved that the MUSIC algorithm is asymptotically efficient for a single source and for uncorrelated sources when the signal-to-noise ratio (SNR) of all the sources tend to infinity, then Stoica and Nehorai [1] extended this result when the number of sensors tend to infinity. Furthermore, they proved that the MUSIC algorithm is not efficient if the sources are correlated and that the difference between the asymptotic covariance given by the MUSIC algorithm and the CRB may be quite large if the sources are nearly coherent. These results have been recently extended to noncircular Gaussian signals where it has been proved [4] that different subspace-based estimates used in the context of noncircular digital modulations are asymptotically efficient for a single source, but for several sources, the efficiency decreases dramatically for uncorrelated sources with low SNR, DOA and noncircularity phase separations.

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This paper offers generic asymptotic results about subspace-based estimates with emphasis on efficiency, based on the notion of asymptotically minimum variance (AMV) and asymptotically best consistent (ABC) estimator introduced by Porat and Friedlander [5] and Stoica et al. [6], respectively, and then applied to Gaussian noncircular signals [7]. But in all these papers, the first[1] covariance matrix of the asymptotic distribution of the involved statistics was nonsingular. In this paper, this notion of AMV estimators is extended to the case of a singular first covariance matrix. This allows us to prove the existence of a lower bound for the covariance of the asymptotic distribution of DOA estimates given by an arbitrary consistent subspace-based algorithm. This bound can be used as a benchmark against which potential subspace-based algorithms are tested. But this AMV bound is generally lower bounded by the CRB because this later bound concerns arbitrary functions of the data. We will prove that this AMV bound associated with different estimated projectors which is function of the second-order statistics of the involved processes only attains the stochastic CRB in the case of circular or noncircular Gaussian signals.

The paper is organized as follows. The array signal model and a motivating example in the context of noncircular signals are given in Section 2. Section 3 extends the standard AMV results to arbitrary statistics whose first covariance matrix of their asymptotic distribution is singular, applies these results to different projection-based statistics, gives closed-form expressions of AMV bounds based on estimates of different orthogonal projectors and finally proves that these AMV bounds attain the stochastic CRB in the case of circular or noncircular Gaussian signals, which is the main contribution of this paper.

The following notations are used throughout the paper. Matrices and vectors are represented by bold upper case and bold lower case characters, respectively. Vectors are by default in column orientation, while T, H, $*$, # and $\perp$ stand for transpose, conjugate transpose, conjugate, Moore–Penrose inverse and ortho-complement of range space, respectively. $E(.)$, Tr(.) and $\Re(.)$ are the expectation, trace and real part operators. $\mathbf{I}$ is the identity matrix. vec($\cdot$) is the "vectorization" operator that turns a matrix into a vector by stacking the columns of the matrix one below another which is used in conjunction with the Kronecker product $\mathbf{A}\otimes\mathbf{B}$ as the block matrix whose (i,j) block element is $a_{i,j}\mathbf{B}$ and with the vec-permutation matrix $\mathbf{K}$ which transforms vec($\mathbf{C}$) to vec($\mathbf{C}^{\mathrm{T}}$).

[1]For noncircular random variables $\mathbf{x}$, the matrices $E[(\mathbf{x}-E(\mathbf{x}))(\mathbf{x}-E(\mathbf{x}))^{\mathrm{H}}]$ and $E[(\mathbf{x}-E(\mathbf{x}))(\mathbf{x}-E(\mathbf{x}))^{\mathrm{T}}]$ are denoted first and second covariance matrices, respectively.

2. Array signal model and motivating example

Let an array of M sensors receive the signals emitted by K narrowband sources with $K<M$. The observations are modeled as

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{n}_t, \quad t = 1,\ldots,T,$$

where $(\mathbf{y}_t)_{t=1,\ldots,T}$ are independent and identically distributed. $\mathbf{A}\overset{\mathrm{def}}{=}[\mathbf{a}_1,\ldots,\mathbf{a}_K]$ is the array response matrix where $\mathbf{a}_k$ is parameterized by the parameter θ_k. In a more general setting, θ_k can contain more parameters per source, e.g., azimuth, elevation, distance, etc. Applications of the presented results to the multiple parameter per source case is straightforward (see Appendix D), but for notational simplicity we assume that θ_k is a real scalar, referred to as the kth DOA. $\mathbf{A}$ is supposed to have full rank for distinct DOAs θ_k. $\mathbf{x}_t = (x_{t,1},\ldots,x_{t,K})^{\mathrm{T}}$ and $\mathbf{n}_t$ model signals transmitted by sources and additive measurement noise, respectively. $\mathbf{x}_t$ and $\mathbf{n}_t$ are independent, zero-mean, $\mathbf{n}_t$ is assumed to be Gaussian complex circular, spatially uncorrelated with $E(\mathbf{n}_t\mathbf{n}_t^{\mathrm{H}}) = \sigma_n^2\mathbf{I}_M$, while $\mathbf{x}_t$ is complex noncircular, not necessarily Gaussian and possibly spatially correlated with nonsingular covariance matrices $\mathbf{R}_x \overset{\mathrm{def}}{=} E(\mathbf{x}_t\mathbf{x}_t^{\mathrm{H}})$ and $\mathbf{R}'_x \overset{\mathrm{def}}{=} E(\mathbf{x}_t\mathbf{x}_t^{\mathrm{T}})$. Consequently, this leads to two covariance matrices of $\mathbf{y}_t$ that convey information about $\Theta \overset{\mathrm{def}}{=} (\theta_1,\ldots,\theta_K)^{\mathrm{T}}$:

$$\mathbf{R}_y = \mathbf{A}\mathbf{R}_x\mathbf{A}^{\mathrm{H}} + \sigma_n^2\mathbf{I}_M \overset{\mathrm{def}}{=} \mathbf{R}_s + \sigma_n^2\mathbf{I}_M \quad \text{and}$$

$$\mathbf{R}'_y = \mathbf{A}\mathbf{R}'_x\mathbf{A}^{\mathrm{T}} \overset{\mathrm{def}}{=} \mathbf{R}_{s'} \neq \mathbf{O}.$$

The noncircularity of the signals $\mathbf{x}_t$ allows us to exploit this second covariance matrix $\mathbf{R}'_y$ to improve the performances of the conventional algorithms based on $\mathbf{R}'_y$ only. Examples of such algorithms are given in the literature (see e.g., [8,4]). We suppose that Θ is uniquely determined by the range space of $\mathbf{A}$ and consequently Θ is uniquely determined by the common orthogonal projector $\mathbf{\Pi}_y$ onto the noise subspace associated with $\mathbf{R}_y$ and $\mathbf{R}'_y$ as well. Using the extended observation $\tilde{\mathbf{y}}_t \stackrel{\text{def}}{=} (\mathbf{y}_t^{\mathrm{T}}, \mathbf{y}_t^{\mathrm{H}})^{\mathrm{T}}$,

$$\mathbf{R}_{\tilde{y}} \stackrel{\text{def}}{=} E(\tilde{\mathbf{y}}_t \tilde{\mathbf{y}}_t^{\mathrm{H}}) = \tilde{\mathbf{A}} \mathbf{R}_{\tilde{x}} \tilde{\mathbf{A}}^{\mathrm{H}} + \sigma_n^2 \mathbf{I}_{2M} \quad \text{with } \tilde{\mathbf{A}} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^* \end{pmatrix}$$

$$\text{and} \quad \mathbf{R}_{\tilde{x}} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{R}_x & \mathbf{R}'_x \\ \mathbf{R}'^*_x & \mathbf{R}^*_x \end{pmatrix},$$

where we suppose here that $\mathbf{R}_{\tilde{x}}$ is nonsingular.[2] Consequently Θ is determined by the orthogonal projector $\mathbf{\Pi}_{\tilde{y}}$ onto the $2(M-K)$-dimensional noise subspace of $\mathbf{R}_{\tilde{y}}$ also.

These covariance matrices are traditionally estimated by $\mathbf{R}_{y,T} = (1/T)\sum_{t=1}^{\mathrm{T}} \mathbf{y}_t \mathbf{y}_t^{\mathrm{H}}$, $\mathbf{R}'_{y,T} = (1/T) \sum_{t=1}^{\mathrm{T}} \mathbf{y}_t \mathbf{y}_t^{\mathrm{T}}$ and $\mathbf{R}_{\tilde{y},T} = (1/T)\sum_{t=1}^{\mathrm{T}} \mathbf{y} \tilde{\mathbf{y}}_t^{\mathrm{H}}$, respectively. There are different alternatives to combine the information conveyed by $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$. The first ones are based directly on the matrices $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$ using the AMV [7] or maximum likelihood approaches [9] and the second ones are based on the orthogonal projectors $(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ and $\mathbf{\Pi}_{\tilde{y},T}$ onto the noise subspace of the sample covariance matrices $\mathbf{R}_{y,T}$, $\mathbf{R}'_{y,T}$ and $\mathbf{R}_{\tilde{y},T}$, respectively. We note that there is not a one-to-one mapping between $(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ and $\mathbf{\Pi}_{\tilde{y},T}$, contrary to the-one-to one mapping $(\mathbf{R}_{y,T}, \mathbf{R}'_{y,T}) \longleftrightarrow \mathbf{R}_{\tilde{y},T}$. It is the reason why we consider in the sequel these two statistics separately.

The first idea to estimate Θ from $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$ is to use similar subspace-based algorithms derived from the projection matrices $\mathbf{\Pi}_{y,T}$ and $\mathbf{\Pi}'_{y,T}$. For example, the asymptotic performance of the estimates given by the standard MUSIC algorithm and a MUSIC-like algorithm based on $\mathbf{\Pi}_{y,T}$ and $\mathbf{\Pi}'_{y,T}$, respectively, are similar. In particular for only one source, the associated asymptotic variances are, respectively, given by [4]

$$C_{\theta_1} = \frac{1}{\alpha_1}\left[\frac{\sigma_n^2}{\sigma_1^2} + \frac{1}{M}\frac{\sigma_n^4}{\sigma_1^4}\right] \quad \text{and} \quad C_{\theta_1} = \frac{1}{\alpha_1 \rho_1^2}\left[\frac{\sigma_n^2}{\sigma_1^2} + \frac{1}{M}\frac{\sigma_n^4}{\sigma_1^4}\right],$$

with α_1 is a purely geometric factor and where ρ_1 $(0 \leqslant \rho_1 \leqslant 1)$ is the noncircularity rate of $x_{t,1}$ defined by $E(x_{t,1}^2) = \rho_1 \mathrm{e}^{\mathrm{i}\phi_1} E|x_{t,1}^2| = \rho_1 \mathrm{e}^{\mathrm{i}\phi_1} \sigma_1^2$ where ϕ_1 is the phase of noncircularity. Examples of such noncircular signals are given by the rectilinear signals (e.g., unfiltered ASK modulations) for which $x_{t,1} = |x_{t,1}| \mathrm{e}^{\mathrm{i}\phi_1/2}$ and $\rho_1 = 1$.

Consequently a problem crops up: how does one combine the statistics $\mathbf{\Pi}_{y,T}$ and $\mathbf{\Pi}'_{y,T}$ to improve the estimate of Θ?

Another idea to estimate Θ from $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$ is to use subspace-based algorithms derived from the projection matrix $\mathbf{\Pi}_{\tilde{y},T}$. Efficient subspace-based algorithms based on $\mathbf{\Pi}_{\tilde{y},T}$ have been proposed and analyzed in [4] in the particular case of uncorrelated sources with maximum noncircularity rates. However, in the general case of arbitrary extended spatial covariance $\mathbf{R}_{\tilde{x}}$ of the sources, only weighted MUSIC-like algorithms seem to take benefit of the second covariance matrix $\mathbf{R}'_{y,T}$. But the asymptotic performances of these estimates are largely outperformed by those of the AMV estimator based on $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$ [4]. Therefore, a question arises as well: does there exist an algorithm based on the projector $\mathbf{\Pi}_{\tilde{y},T}$ whose performance approaches that of the AMV estimator based on $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$?

A solution of the two aforementioned problems is to use the notion of AMV estimators based, respectively, on the matrix-valued statistics $(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ and $\mathbf{\Pi}_{\tilde{y},T}$. But to apply the standard results [10] on AMV estimators to these projectors, two conditions must be satisfied. First, the involved subspace-based algorithm considered as a mapping must be complex differentiable w.r.t. $(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ [resp., $\mathbf{\Pi}_{\tilde{y},T}$] at the point $(\mathbf{\Pi}_y, \mathbf{\Pi}'_y)$ [resp., $\mathbf{\Pi}_{\tilde{y}}$]. Second, the first covariance matrix $\mathbf{C}_s$ of the asymptotic distribution of $\mathbf{s}_T \stackrel{\text{def}}{=} \mathrm{vec}(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ [resp., $\mathbf{s}_T \stackrel{\text{def}}{=} \mathrm{vec}(\mathbf{\Pi}_{\tilde{y},T})$] must be nonsingular. While the first condition is satisfied because the projection matrices are Hermitian, it will be specified in Section 3.3, that the second is not satisfied. So we have to elaborate a little bit by considering the case of arbitrary sequences of statistics.

3. Asymptotic efficiency of subspace-based AMV estimators

3.1. Asymptotically minimum variance estimator

Consider a general N-multidimensional mixture of real- and complex-valued sequence of statistics $\mathbf{s}_T$ which is a consistent estimate of $\mathbf{s}(\Theta)$ for which the real-valued parameter $\Theta \in \mathbb{R}^K$ is identifiable from $\mathbf{s}(\Theta)$. We suppose that $\mathbf{s}_T$ is asymptotically

[2] The particular case $\mathbf{R}_{\tilde{x}}$ singular is beyond the scope of this paper. This later case occurs for example for uncorrelated rectilinear signals $x_{k,t}$ for which the dimension of the noise subspace becomes $2M-K$.

zero-mean Gaussian distributed where the first covariance matrix $\mathbf{C}_s$ is possibly singular:

$$\sqrt{T}(\mathbf{s}_T - \mathbf{s}(\Theta)) \xrightarrow{\mathscr{L}} \mathscr{N}(\mathbf{0}; \mathbf{C}_s, \mathbf{C}'_s).$$

To consider the asymptotic performance of an algorithm based on $\mathbf{s}_T$, we adopt a functional analysis approach which consists in recognizing that the whole process of constructing an estimate Θ_T of Θ is equivalent to defining a functional relation linking this estimate Θ_T to the statistics $\mathbf{s}_T$ from which it is inferred. This functional dependence is denoted $\mathbf{s}_T \longmapsto \Theta_T = \mathrm{Alg}(\mathbf{s}_T)$. Considering a mapping Alg(.) differentiable w.r.t. $(\Re(\mathbf{s}), \Im(\mathbf{s}))$, the following theorem is proved in [11].

Theorem 1. *The covariance matrix $\mathbf{C}_\Theta$ of the asymptotic distribution of a consistent estimator of Θ given by an arbitrary algorithm based on $\mathbf{s}_T$ is bounded below by the real symmetric matrix $\mathbf{C}_\Theta^{\mathrm{AMV}(s)} = (\mathscr{S}^{\mathrm{H}}\mathbf{C}_s^{\#}\mathscr{S})^{-1}$*

$$\mathbf{C}_\Theta \geqslant (\mathscr{S}^{\mathrm{H}}\mathbf{C}_s^{\#}\mathscr{S})^{-1} \tag{3.1}$$

if the following two conditions hold:

$$\mathrm{Span}(\mathscr{S}) \subset \mathrm{Span}(\mathbf{C}_s) \quad \text{and} \quad \mathbf{s}_T^* = \mathbf{P}\mathbf{s}_T, \tag{3.2}$$

where $\mathbf{P}$ is a permutation matrix[3] *and $\mathscr{S} \stackrel{\mathrm{def}}{=} \mathrm{d}\mathbf{s}(\Theta)/\mathrm{d}\Theta$.*

Remark 1. The second condition (3.2) holds for Hermitian matrix-valued statistics with $\mathbf{P} = \mathbf{K}$. For complex symmetric matrix-valued statistics, the complex conjugate associated terms must be added.

Remark 2. In the trivial case where there are $N - r$ linear relations between the components of $\mathbf{s}_T$ with r components statistically uncorrelated, there exists an $N \times (N - r)$ matrix $\mathbf{B}$ such that $\mathbf{s}_T = \mathbf{B}\mathbf{s}'_T$ with $\mathrm{Cov}(\mathbf{s}'_T)$ nonsingular. Consequently $\mathrm{Span}(\mathscr{S}) \subset \mathrm{Span}(\mathbf{B})$ and $\mathrm{Span}(\mathrm{Cov}(\mathbf{s}_T)) = \mathrm{Span}(\mathbf{B})$. Therefore, the first condition (3.2) holds.

Remark 3. In their discussions about the generalization of the optimal weighted subspace fitting approach, Cardoso and Moulines [12] have introduced a range space condition different from condition (3.2), and they have derived (3.1) as a lower bound to the covariance of the asymptotic distribution of weighted subspace fitting estimates.

Remark 4. Under the assumptions of Theorem 1, it has been proved in [11], that the following nonlinear least square estimate achieves the lower bound (3.1).

$$\Theta_T = \arg\min_{\alpha \in \mathbb{R}^K} [\mathbf{s}_T - \mathbf{s}(\alpha)]^{\mathrm{H}} \mathbf{C}_s^{\#} [\mathbf{s}_T - \mathbf{s}(\alpha)]. \tag{3.3}$$

3.2. Asymptotic distribution of projector estimator

To apply Theorem 1 to the statistics $\mathrm{vec}(\mathbf{\Pi}_{y,T})$, $\mathrm{vec}(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ and $\mathrm{vec}(\mathbf{\Pi}_{\tilde{y},T})$, we need the expression of the first covariance matrice of their asymptotic distribution. They are given by the following lemma proved in [11].

Lemma 1. *The first covariance matrices $\mathbf{C}_\Pi$, $\mathbf{C}_{\substack{\Pi \\ \Pi'}}$ and $\mathbf{C}_{\tilde{\Pi}}$ of the asymptotic distribution of $\mathrm{vec}(\mathbf{\Pi}_{y,T})$, $\mathrm{vec}(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ and $\mathrm{vec}(\mathbf{\Pi}_{\tilde{y},T})$ are given by*

$$\mathbf{C}_\Pi = (\mathbf{\Pi}_y^* \otimes \mathbf{U}) + (\mathbf{U}^* \otimes \mathbf{\Pi}_y), \tag{3.4}$$

$$\mathbf{C}_{\substack{\Pi \\ \Pi'}} = \begin{pmatrix} \mathbf{\Pi}_y^* \otimes \mathbf{U} & \mathbf{\Pi}_y^* \otimes \mathbf{U}'' \\ \mathbf{\Pi}_y^* \otimes \mathbf{U}''^{\mathrm{H}} & \mathbf{\Pi}_y^* \otimes \mathbf{U}' \end{pmatrix} + \begin{pmatrix} \mathbf{U}^* \otimes \mathbf{\Pi}_y & \mathbf{U}''^* \otimes \mathbf{\Pi}_y \\ \mathbf{U}''^{\mathrm{T}} \otimes \mathbf{\Pi}_y & \mathbf{U}'^* \otimes \mathbf{\Pi}_y \end{pmatrix}, \tag{3.5}$$

$$\mathbf{C}_{\tilde{\Pi}} = (\mathbf{I} + \mathbf{K}(\mathbf{J} \otimes \mathbf{J}))((\mathbf{\Pi}_{\tilde{y}}^* \otimes \tilde{\mathbf{U}}) + (\tilde{\mathbf{U}}^* \otimes \mathbf{\Pi}_{\tilde{y}})), \tag{3.6}$$

with $\mathbf{U} \stackrel{\mathrm{def}}{=} \sigma_n^2 \mathbf{R}_s^{\#} \mathbf{R}_y \mathbf{R}_s^{\#}$, $\mathbf{U}' \stackrel{\mathrm{def}}{=} \sigma_n^2 \mathbf{R}_s^{\prime *\#} \mathbf{R}_y^* \mathbf{R}_s^{\prime\#}$, $\mathbf{U}'' \stackrel{\mathrm{def}}{=} \sigma_n^2 \mathbf{R}_s^{\#} \mathbf{R}'_y \mathbf{R}_s^{\prime\#}$ *and* $\tilde{\mathbf{U}} \stackrel{\mathrm{def}}{=} \sigma_n^2 \mathbf{R}_{\tilde{s}}^{\#} \mathbf{R}_{\tilde{y}} \mathbf{R}_{\tilde{s}}^{\#}$, *and where* $\mathbf{K}$ *is the vec-permutation matrix of appropriate dimension which transforms* vec(.) *to* $\mathrm{vec}(.^{\mathrm{T}})$ *for any square matrix and*

$$\mathbf{J} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}.$$

We note that the previous expressions of $\mathbf{C}_\Pi$, $\mathbf{C}_{\substack{\Pi \\ \Pi'}}$ and $\mathbf{C}_{\tilde{\Pi}}$, do not depend on the fourth-order moments of the sources. Furthermore, $\mathbf{C}_\Pi$ does not depend on $\mathbf{R}'_y$. Consequently, we have proved the following:

Theorem 2. *The asymptotic performance given by an arbitrary subspace-based algorithm built from $\mathbf{R}_{y,T}$, $(\mathbf{R}_{y,T}, \mathbf{R}'_{y,T})$ or $\mathbf{R}_{\tilde{y},T}$ depends on the distribution of $\mathbf{x}_t$ through its second-order moments only. Furthermore, for subspace-based algorithms built from $\mathbf{R}_{y,T}$, this asymptotic performance depends only on the first covariance matrix $\mathbf{R}_x$.*

3.3. Asymptotically minimum variance subspace-based estimator

We can now consider the two conditions (3.2) of Theorem 1 to prove that this theorem applies to the

[3] We note that in this case $\mathbf{C}_s = \mathbf{C}'_s\mathbf{P}$, and the second covariance matrix $\mathbf{C}'_s$ of the asymptotic distribution of $\mathbf{s}_T$ is deduced from the first covariance matrix $\mathbf{C}_s$.

statistics $\text{vec}(\mathbf{\Pi}_{y,T})$, $\text{vec}(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ and $\text{vec}(\mathbf{\Pi}_{\tilde{y},T})$. It is proved in Appendix A that

$$\text{Null space}(\mathbf{C}_{\Pi}) = \text{Span}\{\mathbf{u}^*_{l'} \otimes \mathbf{u}_{l''} | 1 \leqslant l', l'' \leqslant K \text{ or } K < l', l'' \leqslant M\}, \tag{3.7}$$

$$\text{Null space}(\mathbf{C}_{\substack{\Pi \\ \Pi'}}) = \text{Span}\left\{ \begin{matrix} \mathbf{u}^*_{l'} \otimes \mathbf{u}_{l''} \\ \mathbf{0} \end{matrix}, \begin{matrix} \mathbf{0} \\ \mathbf{u}^*_{l'} \otimes \mathbf{u}_{l''} \end{matrix} | 1 \leqslant l', l'' \leqslant K \text{ or } K < l', l'' \leqslant M \right\}, \tag{3.8}$$

$$\text{Null space}(\mathbf{C}_{\tilde{\Pi}}) = \text{Span}\{\tilde{\mathbf{u}}^*_{l'} \otimes \tilde{\mathbf{u}}_{l''} | 1 \leqslant l', l'' \leqslant 2K \text{ or } 2K < l', l'' \leqslant 2M\}. \tag{3.9}$$

This allows us to prove in Appendix B that

$$\frac{\partial \text{vec}(\mathbf{\Pi}_y)}{\partial \theta_k} \perp \text{Null space}(\mathbf{C}_{\Pi}), \quad k = 1, \ldots, K, \tag{3.10}$$

and

$$\begin{pmatrix} \frac{\partial \text{vec}(\mathbf{\Pi}_y)}{\partial \theta_k} \\ \frac{\partial \text{vec}(\mathbf{\Pi}_y)}{\partial \theta_k} \end{pmatrix} \perp \text{Null space}(\mathbf{C}_{\substack{\Pi \\ \Pi'}}), \quad k = 1, \ldots, K. \tag{3.11}$$

Consequently, because the nullspaces of the Hermitian matrices $\mathbf{C}_{\Pi}$ and $\mathbf{C}_{\substack{\Pi \\ \Pi'}}$ are the complementary orthogonal of $\text{span}(\mathbf{C}_{\Pi})$ and $\text{span}(\mathbf{C}_{\substack{\Pi \\ \Pi'}})$, respectively, the first condition (3.2) is satisfied for the statistics $\text{vec}(\mathbf{\Pi}_{y,T})$ and $\text{vec}(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$. This condition is proved in the same way for $\text{vec}(\Pi_{\tilde{y},T})$. Furthermore, because these matrix-valued statistics are Hermitian, the second condition of (3.2) is satisfied. Consequently, Theorem 1 applies to the statistics $\text{vec}(\mathbf{\Pi}_{y,T})$, $\text{vec}(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ and $\text{vec}(\mathbf{\Pi}_{\tilde{y},T})$.

Remark 5. We note that the asymptotic covariance of the nonlinear least square estimate (3.3) is preserved if the weighting matrix is replaced by any consistent estimate $\mathbf{W}_T$ of $\mathbf{C}^{\#}_s$ satisfying $\mathbf{W}_T = \mathbf{C}^{\#}_s + \text{o}(\mathbf{s}_T - \mathbf{s}(\Theta))$ by checking that the Jacobian $\mathbf{D}^{\text{Alg}}_s = (\mathscr{S}^{\text{H}} \mathbf{C}^{\#}_s \mathscr{S})^{-1} \mathscr{S}^{\text{H}} \mathbf{C}^{\#}_s$ of the mapping Alg(.) involved by (3.3) is preserved by following a perturbation analysis similar to that of the proof of Remark 4 given in [11]. Moreover, consistent estimates of σ^2, $\mathbf{\Pi}_y$, $\mathbf{\Pi}_{\tilde{y}}$, $\mathbf{R}_s$, $\mathbf{R}'_s$, $\mathbf{R}_{\tilde{s}}$ are available from the singular value decompositions of $\mathbf{R}_{y,T}$, $\mathbf{R}'_{y,T}$ and $\mathbf{R}_{\tilde{y},T}$ and consequently, consistent estimates of $\mathbf{C}^{\#}_{\Pi}$, $\mathbf{C}^{\#}_{\substack{\Pi \\ \Pi'}}$ and $\mathbf{C}^{\#}_{\tilde{\Pi}}$ can be derived as well from Lemma 1.

3.4. Relation to the Cramer– Rao bound in the Gaussian case

To evaluate the efficiency of the subspace-based AMV estimators previously introduced, we consider the particular case where the sources $\mathbf{x}_t$ are Gaussian distributed. The following main contribution of this paper is proved in Appendix C.

Theorem 3. *When the sources are Gaussian distributed, the AMV bound* (3.1) *associated with the statistics* $\text{vec}(\mathbf{\Pi}_{y,T})$ [*resp.* $\text{vec}(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ *and* $\text{vec}(\mathbf{\Pi}_{\tilde{y},T})$] *are equal to the statistical CRB associated with the circular* [*resp. noncircular*] *Gaussian distribution.*

$$\mathbf{C}^{\text{AMV}(\Pi)}_{\Theta} = \mathbf{CRB}^{\text{CG}}_{\Theta} = \frac{\sigma^2_n}{2} \{\Re[\mathbf{D}^{\text{H}} \mathbf{\Pi}_y \mathbf{D} \odot (\mathbf{R}_x \mathbf{A}^{\text{H}} \mathbf{R}^{-1}_y \mathbf{A} \mathbf{R}_x)^{\text{T}}]\}^{-1}, \tag{3.12}$$

$$\mathbf{C}^{\text{AMV}(\Pi,\Pi')}_{\Theta} = \mathbf{CRB}^{\text{NCG}}_{\Theta} = \frac{\sigma^2_n}{2} \left\{ \Re \left[\mathbf{D}^{\text{H}} \mathbf{\Pi}_y \mathbf{D} \odot \left([\mathbf{R}_x \mathbf{A}^{\text{H}}, \mathbf{R}'_x \mathbf{A}^{\text{T}}] \mathbf{R}^{-1}_{\tilde{y}} \times \begin{bmatrix} \mathbf{A} \mathbf{R}_x \\ \mathbf{A}^* \mathbf{R}'^*_x \end{bmatrix} \right)^{\text{T}} \right] \right\}^{-1}, \tag{3.13}$$

$$\mathbf{C}^{\text{AMV}(\tilde{\Pi})}_{\Theta} = \mathbf{CRB}^{\text{NCG}}_{\Theta}, \tag{3.14}$$

with $\mathbf{D} \stackrel{\text{def}}{=} \text{d}\mathbf{A}(\Theta)/\text{d}\Theta$.

Consequently the nonlinear least square DOA estimators described at the end of Section 3.3 are asymptotically efficient in the Gaussian context.

Remark 6. Because the statistic $\mathbf{\Pi}_{y,T}$ is a function of $(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$, we have $\mathbf{C}^{\text{AMV}(\Pi,\Pi')}_{\Theta} \leqslant \mathbf{C}^{\text{AMV}(\Pi)}_{\Theta}$ and consequently $\mathbf{CRB}^{\text{NCG}}_{\Theta} \leqslant \mathbf{CRB}^{\text{CG}}_{\Theta}$ for Gaussian sources of same first spatial covariance matrices $\mathbf{R}_x$.

4. Conclusion

This paper provides generic asymptotic results about DOA subspace-based estimates with emphasis on efficiency. The standard AMV bound has been extended to statistics whose first covariance matrices of their asymptotic distributions are singular. This bound has been applied to several projector estimators using the first covariance matrices of their asymptotic distributions that have been derived. This enables us to prove that these AMV bounds attain the stochastic CRB in the case of circular or noncircular Gaussian signals. Consequently, there always exists asymptotically efficient subspace-based DOA algorithms in the Gaussian context.

Appendix A. Proof of rels. (3.7), (3.8) and (3.9)

(3.7) is straightforwardly proved, thanks to the eigenvalue decomposition $\sum_{l=1}^{M}\lambda_l\mathbf{u}_l\mathbf{u}_l^{\mathrm{H}}$ of $\mathbf{R}_y$ which implies $\mathbf{\Pi}_y=\sum_{l=K+1}^{M}\mathbf{u}_l\mathbf{u}_l^{\mathrm{H}}$ and $\mathbf{U}=\sum_{l=1}^{M}(\lambda_l/(\lambda_l-\sigma_n^2)^2)\mathbf{u}_l\mathbf{u}_l^{\mathrm{H}}$. Consequently $\mathbf{C}_{\Pi}$ becomes from (3.4)

$$\mathbf{C}_{\Pi}=\sum_{l',l''\in\mathscr{L}}\lambda_{l',l''}(\mathbf{u}_{l'}^{*}\otimes\mathbf{u}_{l''})(\mathbf{u}_{l'}^{\mathrm{T}}\otimes\mathbf{u}_{l''}^{\mathrm{H}}),$$

where $\mathscr{L}$ is the set $\{(l',l'')|1\leqslant l'\leqslant K<l''\leqslant M$ or $1\leqslant l''\leqslant K<l'\leqslant M\}$ and the values of $\lambda_{l',l''}\neq 0$ are irrelevant.

(3.8) is more involved to prove, but using the singular value decomposition of $\mathbf{U}'$ and $\mathbf{U}''$, we can write from the following expressions proved in [11]:

$$\mathbf{C}_{\Pi'}=(\mathbf{\Pi}_y^{*}\otimes\mathbf{U}')+(\mathbf{U}^{'*}\otimes\mathbf{\Pi}_y)\quad\text{and}$$

$$\mathbf{C}_{\Pi,\Pi'}=(\mathbf{\Pi}_y^{*}\otimes\mathbf{U}'')+(\mathbf{U}''^{*}\otimes\mathbf{\Pi}_y),$$

$$\mathbf{C}_{\Pi'}=\sum_{l',l''\in\mathscr{L}}\lambda'_{l',l''}(\mathbf{u}_{l'}^{'*}\otimes\mathbf{u}'_{l''})(\mathbf{u}_{l'}^{'\mathrm{T}}\otimes\mathbf{u}_{l''}^{'\mathrm{H}}),$$

$$\mathbf{C}_{\Pi,\Pi'}=\sum_{l',l''\in\mathscr{L}_1}\lambda''_{l',l''}(\mathbf{u}_{l'}^{*}\otimes\mathbf{u}''_{l''})(\mathbf{u}_{l'}^{\mathrm{T}}\otimes\mathbf{u}_{l''}^{'''\mathrm{H}})+\sum_{l',l''\in\mathscr{L}_2}\lambda''_{l',l''}(\mathbf{u}_{l'}^{''*}\otimes\mathbf{u}_{l''})(\mathbf{u}_{l'}^{'''\mathrm{T}}\otimes\mathbf{u}_{l''}^{\mathrm{H}}),$$

where $(\mathbf{u}'_l)_{l=1,\ldots,K}$, $(\mathbf{u}''_l)_{l=1,\ldots,K}$ and $(\mathbf{u}'''_l)_{l=1,\ldots,K}$ are orthogonal basis of Span($\mathbf{A}$), $(\mathbf{u}'_l)_{l=K+1,\ldots,M}$ is an orthogonal bases of Span$(\mathbf{A})^{\perp}$ and where $\mathscr{L}_1$ and $\mathscr{L}_2$ are the sets $\{(l',l'')|1\leqslant l'\leqslant K<l''\leqslant M\}$ and $\{(l',l'')|1\leqslant l''\leqslant K<l'\leqslant M\}$, respectively, and the values of $\lambda'_{l',l''}\neq 0$ and $\lambda''_{l',l''}\neq 0$ are irrelevant.Considering the partitioned matrix $\mathbf{C}_{\Pi \atop \Pi'}$ constituted by $\mathbf{C}_{\Pi}$, $\mathbf{C}_{\Pi'}$ and $\mathbf{C}_{\Pi,\Pi'}$, the proof of (3.8) follows.

(3.9) is proved similarly by considering the eigenvalue decomposition $\sum_{l=1}^{2M}\tilde{\lambda}_l\tilde{\mathbf{u}}_l\tilde{\mathbf{u}}_l^{\mathrm{H}}$ of $\mathbf{R}_{\tilde{y}}$ which implies $\mathbf{\Pi}_{\tilde{y}}=\sum_{l=2K+1}^{2M}\tilde{\mathbf{u}}_l\tilde{\mathbf{u}}_l^{\mathrm{H}}$and $\tilde{\mathbf{U}}=\sum_{l=1}^{2K}(\tilde{\lambda}_l/(\tilde{\lambda}_l-\sigma_n^2)^2)\ \tilde{\mathbf{u}}_l\tilde{\mathbf{u}}_l^{\mathrm{H}}$. Consequently

$$(\mathbf{\Pi}_{\tilde{y}}^{*}\otimes\tilde{\mathbf{U}})+(\tilde{\mathbf{U}}^{*}\otimes\mathbf{\Pi}_{\tilde{y}})=\sum_{l',l''\in\mathscr{L}}\tilde{\lambda}_{l',l''}(\tilde{\mathbf{u}}_{l'}^{*}\otimes\tilde{\mathbf{u}}_{l''})(\tilde{\mathbf{u}}_{l'}^{\mathrm{T}}\otimes\tilde{\mathbf{u}}_{l''}^{\mathrm{H}}),$$

where $\mathscr{L}$ is the set $\{(l',l'')|1\leqslant l'\leqslant 2K<l''\leqslant 2M$ or $1\leqslant l''\leqslant 2K<l'\leqslant 2M\}$ and the values of $\tilde{\lambda}_{l',l''}\neq 0$ are irrelevant. Then from (3.6) and the property [14, Theorem 9(b), p. 47] of $\mathbf{K}$, we have

$$\mathbf{C}_{\tilde{\Pi}}=\sum_{l',l''\in\mathscr{L}}\tilde{\lambda}_{l',l''}(\tilde{\mathbf{u}}_{l'}^{*}\otimes\tilde{\mathbf{u}}_{l''}+\mathbf{J}\tilde{\mathbf{u}}_{l''}\otimes\mathbf{J}\tilde{\mathbf{u}}_{l'}^{*})(\tilde{\mathbf{u}}_{l'}^{\mathrm{T}}\otimes\tilde{\mathbf{u}}_{l''}^{\mathrm{H}})$$

and the proof is complete because $\tilde{\mathbf{u}}_{l'}^{*}\otimes\tilde{\mathbf{u}}_{l''}+\mathbf{J}\tilde{\mathbf{u}}_{l''}\otimes\mathbf{J}\tilde{\mathbf{u}}_{l'}^{*}\neq 0$ for all $(l',l'')\in\mathscr{L}$.

Appendix B. Proof of (3.10) and (3.11)

Since $\{\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_M\}$ is an orthonormal basis of C^M, we have

$$\begin{aligned}
&\left[\frac{\partial\mathrm{vec}(\mathbf{\Pi}_y)}{\partial\theta_k}\right]^{\mathrm{H}}(\mathbf{u}_{l'}^{*}\otimes\mathbf{u}_{l''})\\
&=-\sum_{k'=1}^{K}\left(\mathbf{u}_{k'}^{*}\otimes\frac{\partial\mathbf{u}_{k'}}{\partial\theta_k}+\frac{\partial\mathbf{u}_{k'}^{*}}{\partial\theta_k}\otimes\mathbf{u}_{k'}\right)^{\mathrm{H}}(\mathbf{u}_{l'}^{*}\otimes\mathbf{u}_{l''})\\
&=-\sum_{k'=1}^{K}\left((\mathbf{u}_{k'}^{\mathrm{T}}\mathbf{u}_{l'}^{*})\left(\frac{\partial\mathbf{u}_{k'}^{\mathrm{H}}}{\partial\theta_k}\mathbf{u}_{l''}\right)+\left(\frac{\partial\mathbf{u}_{k'}^{\mathrm{T}}}{\partial\theta_k}\mathbf{u}_{l'}^{*}\right)(\mathbf{u}_{k'}^{\mathrm{H}}\mathbf{u}_{l''})\right)\\
&=0\quad\text{for}\quad K<l',\ l''\leqslant M\\
&=(\mathbf{u}_{l'}^{\mathrm{T}}\mathbf{u}_{l'}^{*})\left(\frac{\partial\mathbf{u}_{l'}^{\mathrm{H}}}{\partial\theta_k}\mathbf{u}_{l''}\right)+\left(\frac{\partial\mathbf{u}_{l''}^{\mathrm{T}}}{\partial\theta_{l'}}\mathbf{u}_{l'}^{*}\right)(\mathbf{u}_{l''}^{\mathrm{H}}\mathbf{u}_{l''})\\
&=\frac{\partial(\mathbf{u}_{l'}^{\mathrm{H}}\mathbf{u}_{l''})}{\partial\theta_k}=0\quad\text{for }1\leqslant l'<l''\leqslant K\\
&=(\mathbf{u}_{l}^{\mathrm{T}}\mathbf{u}_{l}^{*})\left(\frac{\partial\mathbf{u}_{l}^{\mathrm{H}}}{\partial\theta_k}\mathbf{u}_{l}\right)+\left(\frac{\partial\mathbf{u}_{l}^{\mathrm{T}}}{\partial\theta_{l}}\mathbf{u}_{l}^{*}\right)(\mathbf{u}_{l}^{\mathrm{H}}\mathbf{u}_{l})\\
&=\frac{\partial\|\mathbf{u}_l\|^2}{\partial\theta_k}=0\quad\text{for }1\leqslant l'=l''\overset{\mathrm{def}}{=}l\leqslant K,
\end{aligned}$$

which proves (3.10) using (3.7). From the range space of $\mathbf{C}_{\Pi \atop \Pi'}$ given in (3.8), (3.11) is proved in the same way.

Appendix C. Proof of Theorem 3

We separately consider the three statistics where we will make relatively frequent use of the following identities (see e.g., [13, Theorems 7.16 and 7.17]):

$$\mathrm{vec}(\mathbf{ABC}) = (\mathbf{C}^{\mathrm{T}} \otimes \mathbf{A})\,\mathrm{vec}(\mathbf{B}), \tag{C.1}$$

$$\mathrm{Tr}(\mathbf{ABCD}) = \mathrm{vec}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}})(\mathbf{D}^{\mathrm{T}} \otimes \mathbf{B})\,\mathrm{vec}(\mathbf{C}). \tag{C.2}$$

Projector $\mathrm{vec}(\mathbf{\Pi}_{y,T})$:

Because Null space $(\mathbf{R}_s^{\#}) = \mathrm{Span}(\mathbf{\Pi}_y)$, we have $\mathbf{U}\mathbf{\Pi}_y = \mathbf{O}$. This implies the two relations

$$(\mathbf{\Pi}_y^* \otimes \mathbf{U})(\mathbf{U}^* \otimes \mathbf{\Pi}_y)^{\mathrm{H}} = \mathbf{\Pi}_y^*\mathbf{U}^{\mathrm{T}} \otimes \mathbf{U}\mathbf{\Pi}_y = \mathbf{O},$$
$$(\mathbf{\Pi}_y^* \otimes \mathbf{U})^{\mathrm{H}}(\mathbf{U}^* \otimes \mathbf{\Pi}_y) = \mathbf{\Pi}_y^*\mathbf{U}^* \otimes \mathbf{U}\mathbf{\Pi}_y = \mathbf{O},$$

which, thanks to [13, Theorem 5.17], enables one to write, the Moore–Penrose inverse of $\mathbf{C}_{\Pi}$ given by (3.4) in the form:

$$\begin{aligned}\mathbf{C}_{\Pi}^{\#} &= (\mathbf{\Pi}_y^* \otimes \mathbf{U})^{\#} + (\mathbf{U}^* \otimes \mathbf{\Pi}_y)^{\#} = (\mathbf{\Pi}_y^{\#^*} \otimes \mathbf{U}^{\#}) \\ &\quad + (\mathbf{U}^{\#^*} \otimes \mathbf{\Pi}_y^{\#}) = (\mathbf{\Pi}_y^* \otimes \mathbf{U}^{\#}) + (\mathbf{U}^{\#^*} \otimes \mathbf{\Pi}_y) \\ &= \frac{1}{\sigma_n^2}((\mathbf{\Pi}_y^* \otimes \mathbf{AHA}^{\mathrm{H}}) + (\mathbf{A}^*\mathbf{H}^*\mathbf{A}^{\mathrm{T}} \otimes \mathbf{\Pi}_y)),\end{aligned}$$

where the second equality is by [14, Theorem 5 (xvii), p. 33] and the last equality is deduced from $\mathbf{U}^{\#} = (1/\sigma_n^2)\,\mathbf{R}_s\mathbf{R}_y^{-1}\mathbf{R}_s = (1/\sigma_n^2)\mathbf{AR}_x\mathbf{A}^{\mathrm{H}}\mathbf{R}_y^{-1}\mathbf{AR}_x\mathbf{A}^{\mathrm{H}}$ $= (1/\sigma_n^2)\mathbf{AHA}^{\mathrm{H}}$ with $\mathbf{H} \stackrel{\mathrm{def}}{=} \mathbf{R}_x\mathbf{A}^{\mathrm{H}}\mathbf{R}_y^{-1}\mathbf{AR}_x$, thanks to [13, Theorems 5.6 and 5.7] because the Hermitian matrices $\mathbf{R}_s$ and $\mathbf{R}_y$ have a common basis of orthonormal eigenvectors. So, from Theorem 1

$$\begin{aligned}[(\mathbf{C}_{\Theta}^{\mathrm{AMV}(\Pi)})^{-1}]_{k,l} &= \frac{1}{\sigma_n^2}\frac{\partial \mathrm{vec}^{\mathrm{T}}(\mathbf{\Pi}_y^{\mathrm{T}})}{\partial\theta_k}((\mathbf{\Pi}_y^{\mathrm{T}} \otimes \mathbf{AHA}^{\mathrm{H}}) \\ &\quad + ((\mathbf{AHA}^{\mathrm{H}})^{\mathrm{T}} \otimes \mathbf{\Pi}_y))\frac{\partial \mathrm{vec}(\mathbf{\Pi}_y)}{\partial\theta_l} \\ &= \frac{1}{\sigma_n^2}\,\mathrm{Tr}\left(\frac{\partial\mathbf{\Pi}_y}{\partial\theta_k}\mathbf{AHA}^{\mathrm{H}}\frac{\partial\mathbf{\Pi}_y}{\partial\theta_l}\mathbf{\Pi}_y \right. \\ &\quad \left. + \frac{\partial\mathbf{\Pi}_y}{\partial\theta_k}\mathbf{\Pi}_y\frac{\partial\mathbf{\Pi}_y}{\partial\theta_l}\mathbf{AHA}^{\mathrm{H}}\right) \\ &= \frac{2}{\sigma_n^2}\Re\left[\mathrm{Tr}\left(\mathbf{A}^{\mathrm{H}}\frac{\partial\mathbf{\Pi}_y}{\partial\theta_k}\mathbf{\Pi}_y\frac{\partial\mathbf{\Pi}_y}{\partial\theta_l}\mathbf{AH}\right)\right],\end{aligned}$$

where we have used identity (C.2) in the second equality.

Then $\mathbf{\Pi}_y\mathbf{A} = \mathbf{O}$ implying

$$\frac{\partial\mathbf{\Pi}_y}{\partial\theta_i}\mathbf{A} + \mathbf{\Pi}_y\frac{\partial\mathbf{A}}{\partial\theta_i} = \mathbf{O}, \quad i = k, l, \tag{C.3}$$

we have

$$\begin{aligned}[(\mathbf{C}_{\Theta}^{\mathrm{AMV}(\Pi)})^{-1}]_{k,l} &= \frac{2}{\sigma_n^2}\Re\left[\mathrm{Tr}\left(\frac{\partial\mathbf{A}^{\mathrm{H}}}{\partial\theta_k}\mathbf{\Pi}_y\frac{\partial\mathbf{A}}{\partial\theta_l}\mathbf{H}\right)\right] \\ &= \frac{2}{\sigma_n^2}\Re\left[\frac{\mathrm{d}\mathbf{a}_k^{\mathrm{H}}}{\mathrm{d}\theta_k}\mathbf{\Pi}_y\frac{d\mathbf{a}_l}{d\theta_l}(\mathbf{H})_{l,k}\right].\end{aligned} \tag{C.4}$$

This proves (3.12), thanks to the expression of the circular Gaussian CRB (see e.g., [2]).

Projector $\mathrm{vec}(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$:

As for the statistic $\mathrm{vec}(\mathbf{\Pi}_{y,T})$, we have $\mathbf{U}\mathbf{\Pi}_y = \mathbf{U}'\mathbf{\Pi}_y = \mathbf{U}''\mathbf{\Pi}_y = \mathbf{O}$, which implies after straightforward algebraic manipulations, the two relations

$$\begin{pmatrix}\mathbf{\Pi}_y^* \otimes \mathbf{U} & \mathbf{\Pi}_y^* \otimes \mathbf{U}'' \\ \mathbf{\Pi}_y^* \otimes \mathbf{U}''^{\mathrm{H}} & \mathbf{\Pi}_y^* \otimes \mathbf{U}'\end{pmatrix} \times \begin{pmatrix}\mathbf{U}^* \otimes \mathbf{\Pi}_y & \mathbf{U}''^* \otimes \mathbf{\Pi}_y \\ \mathbf{U}''^{\mathrm{T}} \otimes \mathbf{\Pi}_y & \mathbf{U}'^* \otimes \mathbf{\Pi}_y\end{pmatrix}^{\mathrm{H}} = \mathbf{O},$$

$$\begin{pmatrix}\mathbf{\Pi}_y^* \otimes \mathbf{U} & \mathbf{\Pi}_y^* \otimes \mathbf{U}'' \\ \mathbf{\Pi}_y^* \otimes \mathbf{U}''^{\mathrm{H}} & \mathbf{\Pi}_y^* \otimes \mathbf{U}'\end{pmatrix}^{\mathrm{H}} \times \begin{pmatrix}\mathbf{U}^* \otimes \mathbf{\Pi}_y & \mathbf{U}''^* \otimes \mathbf{\Pi}_y \\ \mathbf{U}''^{\mathrm{T}} \otimes \mathbf{\Pi}_y & \mathbf{U}'^* \otimes \mathbf{\Pi}_y\end{pmatrix} = \mathbf{O}. \tag{C.5}$$

This enables one to write, thanks to [13, Theorem 5.17], the Moore–Penrose inverse of $\mathbf{C}_{\substack{\Pi \\ \Pi'}}$ given by (3.5) in the form:

$$\begin{aligned}\mathbf{C}_{\substack{\Pi \\ \Pi'}}^{\#} &= \begin{pmatrix}\mathbf{\Pi}_y^* \otimes \mathbf{U} & \mathbf{\Pi}_y^* \otimes \mathbf{U}'' \\ \mathbf{\Pi}_y^* \otimes \mathbf{U}''^{\mathrm{H}} & \mathbf{\Pi}_y^* \otimes \mathbf{U}'\end{pmatrix}^{\#} \\ &\quad + \begin{pmatrix}\mathbf{U}^* \otimes \mathbf{\Pi}_y & \mathbf{U}''^* \otimes \mathbf{\Pi}_y \\ \mathbf{U}''^{\mathrm{T}} \otimes \mathbf{\Pi}_y & \mathbf{U}'^* \otimes \mathbf{\Pi}_y\end{pmatrix}^{\#} \\ &= \left(\begin{pmatrix}\mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{K}\end{pmatrix}\left(\begin{pmatrix}\mathbf{U} & \mathbf{U}'' \\ \mathbf{U}''^{\mathrm{H}} & \mathbf{U}'\end{pmatrix} \otimes \mathbf{\Pi}_y^*\right)\begin{pmatrix}\mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{K}\end{pmatrix}\right)^{\#} \\ &\quad + \left(\begin{pmatrix}\mathbf{U}^* & \mathbf{U}''^* \\ \mathbf{U}''^{\mathrm{T}} & \mathbf{U}'^*\end{pmatrix} \otimes \mathbf{\Pi}_y\right)^{\#} \\ &= \left(\begin{pmatrix}\mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{K}\end{pmatrix}\left(\begin{pmatrix}\mathbf{U} & \mathbf{U}'' \\ \mathbf{U}''^{\mathrm{H}} & \mathbf{U}'\end{pmatrix}^{\#} \otimes \mathbf{\Pi}_y^*\right)\begin{pmatrix}\mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{K}\end{pmatrix}\right) \\ &\quad + \left(\begin{pmatrix}\mathbf{U}^* & \mathbf{U}''^* \\ \mathbf{U}''^{\mathrm{T}} & \mathbf{U}'^*\end{pmatrix}^{\#} \otimes \mathbf{\Pi}_y\right),\end{aligned} \tag{C.6}$$

where we have used the identity $\mathbf{A} \otimes \mathbf{B} = \mathbf{K}(\mathbf{B} \otimes \mathbf{A})\mathbf{K}$ [14, Theorem 4, p. 47] in the second equality, and [13, Theorem 5.8] and [14, Theorem 5 (xvii), p. 33] in the third equality. Noting that

$$\begin{pmatrix} \mathbf{K} & \mathbf{O} \\ \mathbf{O} & \mathbf{K} \end{pmatrix} \frac{\partial \mathrm{vec}(\boldsymbol{\Pi}_y, \boldsymbol{\Pi}_y)}{\partial \theta_i} = \begin{pmatrix} \mathbf{K}\frac{\partial \mathrm{vec}(\boldsymbol{\Pi}_y)}{\partial \theta_i} \\ \mathbf{K}\frac{\partial \mathrm{vec}(\boldsymbol{\Pi}_y)}{\partial \theta_i} \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathrm{vec}(\boldsymbol{\Pi}_y^*)}{\partial \theta_i} \\ \frac{\partial \mathrm{vec}(\boldsymbol{\Pi}_y^*)}{\partial \theta_i} \end{pmatrix}, \quad i = k, l,$$

we have from (C.6)

$$\left[\left(\mathbf{C}_{\Theta}^{\mathrm{AMV}(\Pi,\Pi')}\right)^{-1}\right]_{k,l} = 2\Re\left[\frac{\partial \mathrm{vec}^{\mathrm{T}}}{\partial \theta_k}\begin{pmatrix} \boldsymbol{\Pi}_y \\ \boldsymbol{\Pi}_y \end{pmatrix}^{\mathrm{T}} \left(\left(\begin{pmatrix} \mathbf{U} & \mathbf{U}'' \\ \mathbf{U}''^{\mathrm{H}} & \mathbf{U}' \end{pmatrix}^{\#}\right)^{\mathrm{T}} \otimes \boldsymbol{\Pi}_y\right) \times \frac{\partial \mathrm{vec}(\boldsymbol{\Pi}_y, \boldsymbol{\Pi}_y)}{\partial \theta_l}\right]$$
$$= 2\Re\left[\mathrm{Tr}\left(\begin{pmatrix} \frac{\partial \boldsymbol{\Pi}_y}{\partial \theta_k} \\ \frac{\partial \boldsymbol{\Pi}_y}{\partial \theta_k} \end{pmatrix} \boldsymbol{\Pi}_y \begin{pmatrix} \frac{\partial \boldsymbol{\Pi}_y}{\partial \theta_l} & \frac{\partial \boldsymbol{\Pi}_y}{\partial \theta_l} \end{pmatrix} \begin{pmatrix} \mathbf{U} & \mathbf{U}'' \\ \mathbf{U}''^{\mathrm{H}} & \mathbf{U}' \end{pmatrix}^{\#}\right)\right],$$

where identity (C.2) is used in the second equality. Then from the definition of the matrices $\mathbf{U}$, $\mathbf{U}'$ and $\mathbf{U}''$ given in Lemma 1, we have

$$\begin{pmatrix} \mathbf{U} & \mathbf{U}'' \\ \mathbf{U}''^{\mathrm{H}} & \mathbf{U}' \end{pmatrix} = \sigma_n^2 \begin{pmatrix} \mathbf{R}_s^{\#} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{*\#} \end{pmatrix} \mathbf{R}_{\tilde{y}} \begin{pmatrix} \mathbf{R}_s^{\#} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{\#} \end{pmatrix}.$$

Since

$$\mathrm{rank}\begin{pmatrix} \mathbf{R}_s^{\#} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{*\#} \end{pmatrix} = \mathrm{rank}\left(\mathbf{R}_{\tilde{y}}\begin{pmatrix} \mathbf{R}_s^{\#} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{\#} \end{pmatrix}\right),$$

theorem [13, Theorem 5.9] applies and using [13, Theorem 5.14], we get

$$\begin{pmatrix} \mathbf{U} & \mathbf{U}'' \\ \mathbf{U}''^{H} & \mathbf{U}' \end{pmatrix}^{\#} = \frac{1}{\sigma_n^2}\left(\mathbf{R}_{\tilde{y}}\begin{pmatrix} \mathbf{R}_s^{\#} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{\#} \end{pmatrix}\right)^{\#} \times \begin{pmatrix} \mathbf{R}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{*} \end{pmatrix}. \tag{C.7}$$

Now, we must prove that

$$\left(\mathbf{R}_{\tilde{y}}\begin{pmatrix} \mathbf{R}_s^{\#} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{\#} \end{pmatrix}\right)^{\#} = \begin{pmatrix} \mathbf{R}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s' \end{pmatrix}\mathbf{R}_{\tilde{y}}^{-1}. \tag{C.8}$$

With $\mathscr{A} \stackrel{\mathrm{def}}{=} \mathbf{R}_{\tilde{y}}\left(\begin{smallmatrix} \mathbf{R}_s^{\#} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{\#} \end{smallmatrix}\right)$ and $\mathscr{X} \stackrel{\mathrm{def}}{=} \left(\begin{smallmatrix} \mathbf{R}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s' \end{smallmatrix}\right)\mathbf{R}_{\tilde{y}}^{-1}$, let us prove that $\mathscr{X}$ is the Moore–Penrose inverse of $\mathscr{A}$, by proving that it satisfies the four axioms [13, Definition 5.1] defining this Moore–Penrose inverse. Since $\mathbf{R}_s^{\#}$ and $\mathbf{R}_s'^{\#}$satisfy these four axioms, we get after some algebraic manipulations:

$$\mathscr{A}\mathscr{X}\mathscr{A} = \mathbf{R}_{\tilde{y}}\begin{pmatrix} \mathbf{R}_s^{\#}\mathbf{R}_s\mathbf{R}_s^{\#} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{\#}\mathbf{R}_s'\mathbf{R}_s'^{\#} \end{pmatrix} = \mathbf{R}_{\tilde{y}}\begin{pmatrix} \mathbf{R}_s^{\#} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{\#} \end{pmatrix} = \mathscr{A},$$

$$\mathscr{X}\mathscr{A}\mathscr{X} = \begin{pmatrix} \mathbf{R}_s\mathbf{R}_s^{\#}\mathbf{R}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'\mathbf{R}_s'^{\#}\mathbf{R}_s' \end{pmatrix}\mathbf{R}_{\tilde{y}}^{-1} = \begin{pmatrix} \mathbf{R}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s' \end{pmatrix}\mathbf{R}_{\tilde{y}}^{-1} = \mathscr{X},$$

$$(\mathscr{X}\mathscr{A})^{\mathrm{H}} = \begin{pmatrix} \mathbf{R}_s\mathbf{R}_s^{\#} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'\mathbf{R}_s'^{\#} \end{pmatrix} = \mathscr{X}\mathscr{A}.$$

It remains to prove $(\mathscr{A}\mathscr{X})^{\mathrm{H}} = \mathscr{A}\mathscr{X}$. Since $\mathrm{Span}(\mathbf{R}_s') = \mathrm{Span}(\mathbf{R}_s)$ implies

$$\mathbf{R}_s'^{*}\mathbf{R}_s^{\#}\mathbf{R}_s = \mathbf{R}_s'^{*},$$
$$\mathbf{R}_s^{*}\mathbf{R}_s'^{\#}\mathbf{R}_s' = \mathbf{R}_s^{*}, \tag{C.9}$$

this give with the decomposition $\mathbf{R}_{\tilde{y}} = \left(\begin{smallmatrix} \mathbf{R}_s & \mathbf{R}_s'^{*} \\ \mathbf{R}_s' & \mathbf{R}_s^{*} \end{smallmatrix}\right) + \sigma_n^2\mathbf{I}_{2M} \stackrel{\mathrm{def}}{=} \mathbf{R}_{\tilde{s}} + \sigma_n^2\mathbf{I}_{2M}$:

$$\mathbf{R}_{\tilde{s}}\begin{pmatrix} \mathbf{R}_s^{\#}\mathbf{R}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{\#}\mathbf{R}_s' \end{pmatrix} = \begin{pmatrix} \mathbf{R}_s\mathbf{R}_s^{\#}\mathbf{R}_s & \mathbf{R}_s'\mathbf{R}_s'^{\#}\mathbf{R}_s' \\ \mathbf{R}_s'^{*}\mathbf{R}_s^{\#}\mathbf{R}_s & \mathbf{R}_s^{*}\mathbf{R}_s'^{\#}\mathbf{R}_s' \end{pmatrix} = \mathbf{R}_{\tilde{s}}. \tag{C.10}$$

After straightforward algebraic manipulations using $\mathbf{R}_{\tilde{y}}^{-1} = \sigma_n^{-2}\mathbf{I}_{2M} - \sigma_n^{-2}\mathbf{R}_{\tilde{s}}\mathbf{R}_{\tilde{y}}^{-1}$ and (C.10), we get

$$\mathbf{R}_{\tilde{y}}\begin{pmatrix} \mathbf{R}_s^{\#}\mathbf{R}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{\#}\mathbf{R}_s' \end{pmatrix}\mathbf{R}_{\tilde{y}}^{-1} = \begin{pmatrix} \mathbf{R}_s^{\#}\mathbf{R}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{\#}\mathbf{R}_s' \end{pmatrix} + \left(\mathbf{R}_{\tilde{s}} - \begin{pmatrix} \mathbf{R}_s^{\#}\mathbf{R}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{\#}\mathbf{R}_s' \end{pmatrix}\mathbf{R}_{\tilde{s}}\right)\mathbf{R}_{\tilde{y}}^{-1}$$

and using

$$\begin{pmatrix} \mathbf{R}_s^{\#}\mathbf{R}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{\#}\mathbf{R}_s' \end{pmatrix}\mathbf{R}_{\tilde{s}} = \begin{pmatrix} \mathbf{R}_s^{\#}\mathbf{R}_s\mathbf{R}_s & \mathbf{R}_s^{\#}\mathbf{R}_s\mathbf{R}_s' \\ \mathbf{R}_s'^{\#}\mathbf{R}_s'\mathbf{R}_s'^{*} & \mathbf{R}_s'^{\#}\mathbf{R}_s'\mathbf{R}_s^{*} \end{pmatrix} = \mathbf{R}_{\tilde{s}}$$

obtained from [14, Theorem 5, rel.(vii), p. 33] and (C.9), we get

$$\mathscr{A}\mathscr{X} = \mathbf{R}_{\tilde{y}} \begin{pmatrix} \mathbf{R}_s^{\#}\mathbf{R}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{\#}\mathbf{R}_s' \end{pmatrix} \mathbf{R}_{\tilde{y}}^{-1} = \begin{pmatrix} \mathbf{R}_s^{\#}\mathbf{R}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{\#}\mathbf{R}_s' \end{pmatrix},$$

and therefore $(\mathscr{A}\mathscr{X})^{\mathrm{H}} = \mathscr{A}\mathscr{X}$ is proved. Consequently from (C.7) and (C.8), we get

$$\begin{aligned}
\begin{pmatrix} \mathbf{U} & \mathbf{U}'' \\ \mathbf{U}''^{\mathrm{H}} & \mathbf{U}' \end{pmatrix}^{\#} &= \frac{1}{\sigma_n^2} \begin{pmatrix} \mathbf{R}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s' \end{pmatrix} \mathbf{R}_{\tilde{y}}^{-1} \begin{pmatrix} \mathbf{R}_s & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_s'^{*} \end{pmatrix} \\
&= \frac{1}{\sigma_n^2} \begin{pmatrix} \mathbf{A}\mathbf{R}_x\mathbf{A}^{\mathrm{H}} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}\mathbf{R}_x'\mathbf{A}^{\mathrm{T}} \end{pmatrix} \mathbf{R}_{\tilde{y}}^{-1} \\
&\quad \times \begin{pmatrix} \mathbf{A}\mathbf{R}_x\mathbf{A}^{\mathrm{H}} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^{*}\mathbf{R}_x'^{*}\mathbf{A}^{\mathrm{H}} \end{pmatrix} \\
&= \frac{1}{\sigma_n^2} \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} \end{pmatrix} \begin{pmatrix} \mathbf{R}_x\mathbf{A}^{\mathrm{H}} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_x'\mathbf{A}^{\mathrm{T}} \end{pmatrix} \mathbf{R}_{\tilde{y}}^{-1} \\
&\quad \times \begin{pmatrix} \mathbf{A}\mathbf{R}_x & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^{*}\mathbf{R}_x'^{*} \end{pmatrix} \begin{pmatrix} \mathbf{A}^{\mathrm{H}} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^{\mathrm{H}} \end{pmatrix} \\
&= \frac{1}{\sigma_n^2} \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} \end{pmatrix} \tilde{\mathscr{H}} \begin{pmatrix} \mathbf{A}^{\mathrm{H}} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^{\mathrm{H}} \end{pmatrix},
\end{aligned}$$

with $\tilde{\mathscr{H}} \stackrel{\mathrm{def}}{=} \left(\begin{smallmatrix} \mathbf{R}_x\mathbf{A}^{\mathrm{H}} & \mathbf{O} \\ \mathbf{O} & \mathbf{R}_x'\mathbf{A}^{\mathrm{T}} \end{smallmatrix}\right) \mathbf{R}_{\tilde{y}}^{-1} \left(\begin{smallmatrix} \mathbf{A}\mathbf{R}_x & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^{*}\mathbf{R}_x'^{*} \end{smallmatrix}\right)$. Consequently,

$$\begin{aligned}
&\left[\left(\mathbf{C}_{\Theta}^{\mathrm{AMV}(\Pi,\Pi')}\right)^{-1}\right]_{k,l} \\
&= \frac{2}{\sigma_n^2} \Re \left[\mathrm{Tr} \left(\begin{pmatrix} \frac{\partial \mathbf{\Pi}_y}{\partial \theta_k} \\ \frac{\partial \mathbf{\Pi}_y}{\partial \theta_k} \end{pmatrix} \mathbf{\Pi}_y \begin{pmatrix} \frac{\partial \mathbf{\Pi}_y}{\partial \theta_l} & \frac{\partial \mathbf{\Pi}_y}{\partial \theta_l} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{O} & \mathbf{A} \end{pmatrix} \right.\right. \\
&\quad \left.\left. \times \tilde{\mathscr{H}} \begin{pmatrix} \mathbf{A}^{\mathrm{H}} & \mathbf{O} \\ \mathbf{O} & \mathbf{A}^{\mathrm{H}} \end{pmatrix} \right) \right] \\
&= \frac{2}{\sigma_n^2} \Re \left[\mathrm{Tr} \left(\begin{pmatrix} \mathbf{A}^{\mathrm{H}} \frac{\partial \mathbf{\Pi}_y}{\partial \theta_k} \\ \mathbf{A}^{\mathrm{H}} \frac{\partial \mathbf{\Pi}_y}{\partial \theta_k} \end{pmatrix} \mathbf{\Pi}_y \begin{pmatrix} \frac{\partial \mathbf{\Pi}_y}{\partial \theta_l}\mathbf{A} & \frac{\partial \mathbf{\Pi}_y}{\partial \theta_l}\mathbf{A} \end{pmatrix} \tilde{\mathscr{H}} \right) \right].
\end{aligned}$$

Applying identity (C.3), we obtain

$$\begin{aligned}
&[(\mathbf{C}_{\Theta}^{\mathrm{AMV}(\Pi,\Pi')})^{-1}]_{k,l} \\
&= \frac{2}{\sigma_n^2} \Re \left[\mathrm{Tr} \left(\begin{pmatrix} \frac{\partial \mathbf{A}^{\mathrm{H}}}{\partial \theta_k} \mathbf{\Pi}_y \frac{\partial \mathbf{A}}{\partial \theta_l} & \frac{\partial \mathbf{A}^{\mathrm{H}}}{\partial \theta_k} \mathbf{\Pi}_y \frac{\partial \mathbf{A}}{\partial \theta_l} \\ \frac{\partial \mathbf{A}^{\mathrm{H}}}{\partial \theta_k} \mathbf{\Pi}_y \frac{\partial \mathbf{A}}{\partial \theta_l} & \frac{\partial \mathbf{A}^{\mathrm{H}}}{\partial \theta_k} \mathbf{\Pi}_y \frac{\partial \mathbf{A}}{\partial \theta_l} \end{pmatrix} \tilde{\mathscr{H}} \right) \right]
\end{aligned}$$

which gives after straightforward algebraic manipulations

$$\begin{aligned}
&[(\mathbf{C}_{\Theta}^{\mathrm{AMV}(\tilde{\Pi})})^{-1}]_{k,l} \\
&= \frac{2}{\sigma_n^2} \Re \left[\mathrm{Tr} \left(\frac{\partial \mathbf{A}^{\mathrm{H}}}{\partial \theta_k} \mathbf{\Pi}_y \frac{\partial \mathbf{A}}{\partial \theta_l} [\mathbf{R}_x\mathbf{A}^{\mathrm{H}}, \mathbf{R}_x'\mathbf{A}^{\mathrm{T}}] \mathbf{R}_{\tilde{y}}^{-1} \right.\right. \\
&\quad \left.\left. \times \begin{bmatrix} \mathbf{A}\mathbf{R}_x \\ \mathbf{A}^{*}\mathbf{R}_x'^{*} \end{bmatrix} \right) \right],
\end{aligned}$$

which proves (3.13), thanks to the expression of the noncircular Gaussian CRB [9].

Projector $\mathrm{vec}(\mathbf{\Pi}_{\tilde{y},T})$:

To prove Theorem 3 for this statistic, we first must simplify the expression of $\mathbf{C}_{\Theta}^{\mathrm{AMV}(\tilde{\Pi})}$. Because $\mathbf{L} \stackrel{\mathrm{def}}{=} \mathbf{I} + \mathbf{K}(\mathbf{J} \otimes \mathbf{J})$ of (3.6) satisfies $\mathbf{L}^2 = 2\mathbf{L}$, the Hermitian matrix $\mathbf{C}_{\tilde{\Pi}}$ becomes $\mathbf{C}_{\tilde{\Pi}} = \frac{1}{2}\mathbf{L}\mathbf{C}\mathbf{L}$ with $\mathbf{C} \stackrel{\mathrm{def}}{=} (\mathbf{\Pi}_{\tilde{y}}^{*} \otimes \tilde{\mathbf{U}}) + (\tilde{\mathbf{U}}^{*} \otimes \mathbf{\Pi}_{\tilde{y}})$ and a simpler expression of the AMV bound can be obtained from the following minimization problem:

$$\mathbf{C}_{\Theta}^{\mathrm{AMV}(\tilde{\Pi})} = \min_{\mathbf{D}\mathscr{S}=\mathbf{I}_K} \mathbf{D}\mathbf{C}_{\tilde{\Pi}}\mathbf{D}^{\mathrm{H}} = \tfrac{1}{2} \min_{\mathbf{D}\mathscr{S}=\mathbf{I}_K} \mathbf{D}\mathbf{L}\mathbf{C}\mathbf{L}\mathbf{D}^{\mathrm{H}}.$$

Checking that $\mathbf{L}\mathscr{S} = (\mathbf{I} + \mathbf{K}(\mathbf{J} \otimes \mathbf{J}))\frac{\mathrm{dvec}(\mathbf{\Pi}_{\tilde{y}})}{\mathrm{d}\Theta} = \mathscr{S} + \mathbf{K}\mathrm{vec}(\mathbf{J}(\mathrm{d}\mathbf{\Pi}_{\tilde{y}}/\mathrm{d}\Theta)\mathbf{J}) = \mathscr{S} + \mathbf{K}\,\mathrm{vec}(\mathrm{d}\mathbf{\Pi}_{\tilde{y}}/\mathrm{d}\Theta^{\mathrm{T}}) = 2\mathscr{S}$, thanks to identity (C.1) for the second equality and the property $\mathbf{J}\mathbf{\Pi}_{\tilde{y}}\mathbf{J} = \mathbf{\Pi}_{\tilde{y}}^{\mathrm{T}}$ [4] for the third equality; the constraints $\mathbf{D}\mathscr{S} = \mathbf{I}$ and $\mathbf{D}\mathbf{L}\mathscr{S} = 2\mathbf{I}$ are equivalent.Consequently, the previous minimization is tantamount to

$$\mathbf{C}_{\Theta}^{\mathrm{AMV}(\tilde{\Pi})} = 2 \min_{(\mathbf{D}\mathbf{L}/2)\mathscr{S}=\mathbf{I}_K} \left(\frac{\mathbf{D}\mathbf{L}}{2}\right) \mathbf{C} \left(\frac{\mathbf{D}\mathbf{L}}{2}\right)^{\mathrm{H}}.$$

Because $\mathbf{C}$ is structured similarly as $\mathbf{C}_{\Pi}$ (see (3.4)), $\mathrm{Span}(\mathscr{S}) \subset \mathrm{Span}(\mathbf{C})$. Consequently, the proof of Theorem 1 given in [11] applies and $\mathbf{C}_{\Theta}^{\mathrm{AMV}(\tilde{\Pi})} = 2\left(\mathscr{S}^{\mathrm{H}}\mathbf{C}^{\#}\mathscr{S}\right)^{-1}$.

Noting that $\mathbf{C} = (\mathbf{\Pi}_{\tilde{y}}^{*} \otimes \tilde{\mathbf{U}}) + (\tilde{\mathbf{U}}^{*} \otimes \mathbf{\Pi}_{\tilde{y}})$ is structured similarly to $\mathbf{C}_{\Pi}$, all the steps of the proof given for the statistic $\mathrm{vec}(\mathbf{\Pi}_{y,T})$ extend up to equality (C.4) by replacing $\mathbf{A}$, $\mathbf{\Pi}_y$ and $\mathbf{H} = \mathbf{R}_x\mathbf{A}^{\mathrm{H}}\mathbf{R}_y^{-1}\mathbf{A}\mathbf{R}_x$, by $\tilde{\mathbf{A}},\mathbf{\Pi}_{\tilde{y}} = \left(\begin{smallmatrix} \mathbf{\Pi}_y & \mathbf{O} \\ \mathbf{O} & \mathbf{\Pi}_y^{*} \end{smallmatrix}\right)$ (from [4]) and $\tilde{\mathbf{H}} \stackrel{\mathrm{def}}{=} \mathbf{R}_{\tilde{x}}\tilde{\mathbf{A}}^{\mathrm{H}}\mathbf{R}_{\tilde{y}}^{-1}\tilde{\mathbf{A}}\mathbf{R}_{\tilde{x}}$,respectively, and consequently

$$[(\mathbf{C}_{\Theta}^{\mathrm{AMV}(\tilde{\Pi})})^{-1}]_{k,l} = \frac{1}{2}\frac{2}{\sigma_n^2} \Re \left[\mathrm{Tr} \left(\frac{\partial \tilde{\mathbf{A}}^{\mathrm{H}}}{\partial \theta_k} \mathbf{\Pi}_{\tilde{y}} \frac{\partial \tilde{\mathbf{A}}}{\partial \theta_l} \tilde{\mathbf{H}} \right) \right].$$

Because all the matrices involved in $\tilde{\mathbf{H}}$ are structured in the form $\begin{pmatrix} (\square) & (\times) \\ (\times)^* & (\square)^* \end{pmatrix}$, $\tilde{\mathbf{H}}$ is structured in the same form as well, i.e., $\tilde{\mathbf{H}} = \begin{pmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ \mathbf{H}_2^* & \mathbf{H}_1^* \end{pmatrix}$ with $\mathbf{H}_1 = [\mathbf{R}_x\mathbf{A}^{\mathrm{H}}, \mathbf{R}'_x\mathbf{A}^{\mathrm{T}}]\mathbf{R}_{\tilde{y}}^{-1}\begin{bmatrix} \mathbf{A}\mathbf{R}_x \\ \mathbf{A}^*\mathbf{R}'^*_x \end{bmatrix}$. Then

$$\frac{\partial\tilde{\mathbf{A}}^{\mathrm{H}}}{\partial\theta_k}\mathbf{\Pi}_{\tilde{y}}\frac{\partial\tilde{\mathbf{A}}}{\partial\theta_l}\tilde{\mathbf{H}} = \begin{pmatrix} \frac{\partial\mathbf{A}^{\mathrm{H}}}{\partial\theta_k}\mathbf{\Pi}_y\frac{\partial\mathbf{A}}{\partial\theta_l}\mathbf{H}_1 & (\times) \\ (\times)^* & \frac{\partial\mathbf{A}^{\mathrm{T}}}{\partial\theta_k}\mathbf{\Pi}_y^*\frac{\partial\mathbf{A}^*}{\partial\theta_l}\mathbf{H}_1^* \end{pmatrix}$$

and

$$[(\mathbf{C}_{\Theta}^{\mathrm{AMV}(\tilde{\Pi})})^{-1}]_{k,l} = \frac{2}{\sigma_n^2}\Re\left[\frac{\mathrm{d}\mathbf{a}_k^{\mathrm{H}}}{\mathrm{d}\theta_k}\mathbf{\Pi}_y\frac{\mathrm{d}\mathbf{a}_l}{\mathrm{d}\theta_l}(\mathbf{H}_1)_{l,k}\right],$$

which proves (3.14) thanks to the expression of the noncircular Gaussian CRB [9].

Appendix D. The case of multiple parameters per source

It is straightforward to extend Theorem 3 to the case of multiple parameters per source. One the one hand, the circular and noncircular Gaussian CRB are derived from slight modifications of the end of the proofs given in [2] and of the proof given [9, Appendix C], respectively. They are given by

$$\mathbf{CRB}_{\Theta}^{\mathrm{CG}} = \frac{\sigma_n^2}{2}\{\Re[\mathbf{D}^{\mathrm{H}}\mathbf{\Pi}_y\mathbf{D} \odot ((\mathbf{R}_x\mathbf{A}^{\mathrm{H}}\mathbf{R}_y^{-1}\mathbf{A}\mathbf{R}_x)^{\mathrm{T}} \otimes \mathbf{1})]\}^{-1},$$

$$\mathbf{CRB}_{\Theta}^{\mathrm{NCG}} = \frac{\sigma_n^2}{2}\left\{\Re\left[\mathbf{D}^{\mathrm{H}}\mathbf{\Pi}_y\mathbf{D} \odot \left(\left([\mathbf{R}_x\mathbf{A}^{\mathrm{H}}, \mathbf{R}'_x\mathbf{A}^{\mathrm{T}}]\mathbf{R}_{\tilde{y}}^{-1} \times \begin{bmatrix} \mathbf{A}\mathbf{R}_x \\ \mathbf{A}^*\mathbf{R}'^*_x \end{bmatrix}\right)^{\mathrm{T}} \otimes \mathbf{1}\right)\right]\right\}^{-1},$$

where $\mathbf{1}$ is a $L \times L$ matrix of 1 if there are L parameters per source. The parameter Θ and the matrix of derivative $\mathbf{D}$ are organized as $(\theta_1, \ldots, \phi_1, \ldots, \theta_K, \ldots, \phi_K)^{\mathrm{T}}$ and

$$\mathbf{D} \stackrel{\mathrm{def}}{=} \left[\frac{\mathrm{d}\mathbf{a}_1(\theta_1, \ldots, \phi_1)}{\mathrm{d}\theta_1}, \ldots, \frac{\mathrm{d}\mathbf{a}_1(\theta_1, \ldots, \phi_1)}{\mathrm{d}\phi_1}, \ldots, \frac{\mathrm{d}\mathbf{a}_K(\theta_K, \ldots, \phi_K)}{\mathrm{d}\theta_K}, \ldots, \frac{\mathrm{d}\mathbf{a}_K(\theta_K, \ldots, \phi_K)}{\mathrm{d}\phi_K}\right].$$

On the other hand, the derivation of the AMV bound follows the same lines as for a single parameter per source except the last step when the matrix $\mathbf{A}$ is decomposed in the different steering vectors $\mathbf{a}_k$.

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