
Exercice 1: For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

1. $f(x) = e^x - 1$ on $\mathbb{R}$.
2. $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.
3. $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.
4. $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$.
5. $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R}^2_{++}$.
6. $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, on $\mathbb{R}^2_{++}$, where $0 \leq \alpha \leq 1$.

Exercice 2: For a differentiable objective function f_0, using only the optimality condition : $\forall x$ feasible, $\nabla f_0(x^\star)^\top (x - x^\star) \geq 0$, derive optimality conditions in the following cases :

1. Unconstrained minimization : min. $f_0(x)$.
2. Equality constrained minimization : min. $f_0(x)$ s.t. $Ax = b$.
3. Minimization over nonnegative orthant : min. $f_0(x)$ s.t. $x \succeq 0$.

Exercice 3: Consider the optimization problem

$$\begin{aligned} \text{min. } & f_0(x_1, x_2) \\ \text{s.t. } & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{aligned}$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value.

1. $f_0(x_1, x_2) = x_1 + x_2$.
2. $f_0(x_1, x_2) = -x_1 - x_2$.
3. $f_0(x_1, x_2) = x_1$.
4. $f_0(x_1, x_2) = \max\{x_1, x_2\}$.
5. $f_0(x_1, x_2) = x_1^2 + 9x_2^2$.

Exercice 4: Prove that $x^\star = (1, 1/2, -1)$ is optimal for the optimization problem :

$$\begin{aligned} \text{min. } & \frac{1}{2} x^\top P x + q^\top x + r \\ \text{s.t. } & -1 \leq x_i \leq 1, \quad i = 1, 2, 3, \end{aligned}$$

where :

$$P = \begin{pmatrix} 13 & 12 & -2 \\ 12 & 17 & 6 \\ -2 & 6 & 12 \end{pmatrix}, \quad q = \begin{pmatrix} -22.0 \\ -14.5 \\ 13.0 \end{pmatrix}, \quad r = 1.$$

Exercice 5: Consider the convex problem

$$\begin{aligned} \text{min. } & f_0(x) \\ \text{s.t. } & f_i(x) \leq 0, \quad i = 1, \ldots, m. \end{aligned}$$

Assume $x^\star$ and $\lambda^\star$ satisfy the KKT conditions :

$$\begin{aligned} f_i(x^\star) &\leq 0, & i &= 1, \ldots, m \\ \lambda_i^\star &\geq 0, & i &= 1, \ldots, m \\ \lambda_i^\star f_i(x^\star) &= 0, & i &= 1, \ldots, m \\ \nabla f_0(x^\star) + \textstyle\sum_{i=1}^m \lambda_i^\star \nabla f_i(x^\star) &= 0. \end{aligned}$$

Show that $\nabla f_0(x^\star)^\top (x - x^\star) \geq 0$ for all feasible x.

Exercice 6: Solve by hand the following optimization problem :

$$\begin{aligned} \text{min. } & x_1^2 + x_2^2 \\ \text{s.t. } & -2x_1 - x_2 + 10 \leq 0 \\ & -x_1 \leq 0 \end{aligned}$$

Exercice 7: Solve by hand the following optimization problem :

$$\begin{aligned} \text{min. } & 5x_1^2 + 6x_2^2 \\ \text{s.t. } & x_1 - 4 \leq 0 \\ & 25 - x_1^2 - x_2^2 \leq 0 \end{aligned}$$

Exercice 8: Use the Lagrangian conditions to solve the following problem on the domain $\{(x_1, x_2) \in \mathbb{R}^2 \,|\, x_2 > 0\}$:

$$\begin{aligned} \text{min. } & x_1 + 2/x_2 \\ \text{s.t. } & -x_2 + 1/2 \leq 0 \\ & -x_1 + x_2^2 \leq 0 \end{aligned}$$

Exercice 9: Consider an optimization problem with feasible set defined by inequalities only :

$$X = \{x \mid f_i(x) \leq 0, i = 1, \ldots, p\}.$$

For any point $\overline{x} \in X$, define the active set

$$I(\overline{x}) = \{i \mid f_i(\overline{x}) = 0\}.$$

Let us remind Slater's constraint qualification (SCQ), linear independence constraint qualification (LICQ), Mangasarian-Fromovitz constraint qualification (MFCQ) :

$$\begin{aligned} \text{(SCQ)} : \quad & \exists x_0 \quad f_i(x_0) < 0, \quad i = 1, \ldots, p. \\ \text{(LICQ) at a point } \overline{x} \in X : \quad & \{\nabla f_i(\overline{x}) : i \in I(\overline{x})\} \text{ linearly independent} \\ \text{(MFCQ) at a point } \overline{x} \in X : \quad & \exists d \quad \langle \nabla f_i(\overline{x}), d \rangle < 0 \quad \text{ for all } i \in I(\overline{x}) \end{aligned}$$

1. Prove that if the f_i are all convex, then (SCQ) implies (MFCQ).
2. Prove that (LICQ) implies (MFCQ).

Exercice 10: Let $A \in \mathbb{R}^{p\times n}$.

1. For $p \geq n$ and $\operatorname{rank} A = n$, solve the problem :

$$\min.\ \|Ax - b\|_2^2$$

2. For $p \leq n$ and $\operatorname{rank} A = p$, consider the problem :

$$\begin{aligned} \min.\ & \frac{1}{2}\|x\|_2^2 \\ \text{s.t. } & Ax = b \end{aligned}$$

   (a) Write the Lagrangian $L(x, \nu)$, derive the dual function $g(\nu)$ and the dual problem.
   (b) Give Slater's sufficient condition for strong duality and solve the dual problem.
   (c) Write the KKT conditions.
   (d) Solve the primal and find $x^\star$.

Exercice 11: constraint qualification Consider the optimization problem :

$$\left[\begin{aligned} \min_{x\in\mathbb{R}^2}.\ & x_1^2 + x_2^2 \\ \text{s.t. } & (x_1-1)^2 + (x_2-1)^2 \leq 1 \\ & (x_1-1)^2 + (x_2+1)^2 \leq 1 \end{aligned}\right.$$

1. Make a sketch of the problem. What is the optimal point $x^\star$? Are Slater's qualification constraints satisfied ?
2. Write the Lagrangian and the KKT optimality conditions. Are there Lagrange parameters proving optimality of $x^\star$?
3. Write and solve the dual. Does strong duality hold ? What is the optimum value for the dual ?

Exercice 12: failure of KKT Consider the problem on $\mathbb{R}^2$:

$$\left[\begin{aligned} \min.\ & (x_1+1)^2 + x_2^2 \\ \text{s.t. } & -x_1^3 + x_2^2 \leq 0 \end{aligned}\right.$$

1. Sketch the feasible set and solve the problem.
2. Find multipliers λ_0, λ_1 satisfying the Fritz-John conditions.
3. Prove there exist no Lagrange multiplier vector for the optimal solution. Explain why not.

Exercice 13: subdifferential Prove the following functions $f : \mathbb{R} \to \mathbb{R}$ are convex and calculate ∂f :

1. $f(x) = |x|$,
2. $f(x) = \imath_{\mathbb{R}_+}(x)$
3. $f(x) = \begin{cases} -\sqrt{x} & \text{if } x \geq 0, \\ +\infty & \text{otherwise.} \end{cases}$
4. $f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x = 0, \\ +\infty & \text{otherwise.} \end{cases}$

Exercice 14: Derive the Fenchel conjugates of the following functions.

1. affine function : $f(x) = ax + b$.
2. exponential : $f(x) = e^x$.
3. negative logarithm : $f(x) = -\log x$.
4. quadratic function : $f(x) = \frac{1}{2}x^\top Qx$ with $Q \in \mathbb{S}^n_{++}$.
5. square norm : $f(x) = \frac{1}{2}\|x\|_2^2$
6. norm : $f(x) = \|x\|$.
7. $f(x) = x \log x$ for $x \geq 0$ and $+\infty$ otherwise.
8. $f(x) = 1/x$ for $x > 0$ and $+\infty$ otherwise.

Exercice 15: Derive the Fenchel conjugate and biconjugate of the following functions ($a > 0$ and all function are $\mathbb{R} \to \mathbb{R}$).

1. $f(x) = a|x|$.
2. $f(x) = \imath_{[-a,a]}(x)$.
3. $f(x) = +\infty$ for $x > a$ and $f(x) = 0$ for $x \leq a$.
4. $f(x) = \imath_{\{0\}}(x)$.

Exercice 16: Farkas' lemma Consider the following linear program (LP) with variable $x \in \mathbb{R}^n$ ($c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^n$ are given) :

$$p^* : \quad \begin{cases} \text{min. } c^\top x \\ \text{s.t. } Ax = b \\ \qquad x \succeq 0 \end{cases}$$

1. Write the Lagrangian and give the Lagrange dual function.
2. Write the dual problem.
3. Consider the case $c = 0$ and prove that exactly one of the assertions below holds, but not both :

$$\text{Either :} \quad \exists x \in \mathbb{R}^n, Ax = b, x \succeq 0 \qquad \text{or :} \quad \exists y \in \mathbb{R}^m, A^\top y \succeq 0, b^\top y < 0$$

Exercice 17: Lagrange and Fenchel duality for LPs

1. Let $A \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Consider the following (primal) LP with variable $x \in \mathbb{R}^n$:

$$\begin{aligned} \text{min. } & c^\top x \\ \text{s.t. } & Ax = b \\ & x \succeq 0 \end{aligned}$$

   Find the dual problem using Lagrange duality. We suggest to denote y (resp. s) the vector of dual variables associated to equality (resp. inequality) constraints ; note that s can be eliminated.

2. Write the KKT optimality conditions associated to the above LP optimization problem.

3. Define

$$f(x) = \begin{cases} c^\top x & \text{if } x \succeq 0, \\ +\infty & \text{otherwise.} \end{cases} \qquad \imath_{\{b\}}(x) = \begin{cases} 0 & \text{if } x = b, \\ +\infty & \text{otherwise.} \end{cases}$$

   Derive the Fenchel conjugates of both functions.
4. Derive the Fenchel dual of the problem $\min f(x) + \imath_{\{b\}}(Ax)$ and show that the Lagrange and Fenchel dual of the primal LP are the same.
5. Consider now the problem

$$\left[\begin{array}{l} \text{min. } c^\top x - \mu \sum_{i=1}^{n} \log x_i \\ \quad \text{s.t. } Ax = b \end{array}\right.$$

   Write the Lagrangian and the KKT optimality conditions. Show that the above problem can be considered as a perturbation of the previous problem.

Exercice 18: Consider the problem

$$\text{min. } \|x\|_1 \quad \text{s.t. } Ax = b\,.$$

We write $A = [a_1, \dots, a_n]$ where $(a_i)_{i=1,\dots,n}$ are the columns of A.

1. Defining $x^+, x^- \succeq 0$ and decomposing $x = x^+ - x^-$, write an equivalent problem.
2. Write the Lagrangian, derive the Lagrange dual function and the dual problem.
3. Write the KKT optimality conditions. Show that the optimal x has a nonzero ith component x_i only if $|{a_i}^\top u| = 1$.
4. Determine the Fenchel conjugates of $f(x) = \|x\|_1$ and $g(y) = \imath_b(y)$.
5. Write the above optimization problem using the previous two functions and determine its Fenchel dual problem.

Exercice 19:

1. Let $f(x) = \frac{1}{2}\|x\|_2^2$. Derive the Fenchel conjugate of f.
2. Let f be a function such that $f^* = f$, where f^* is the Fenchel conjugate of f.
   (a) Write the Fenchel-Young inequality and show that necessarily $f(y) \geq \frac{1}{2}\|y\|_2^2$ for all y.
   (b) Deduce that $f^*(y) \leq \frac{1}{2}\|y\|_2^2$.
3. Conclude about sufficient and necessary conditions for a function to be equal to its Fenchel conjugate.

Exercice 20: Fenchel weak/strong duality Let $f : \mathbb{R}^n \to]-\infty, +\infty]$ and $g : \mathbb{R}^m \to]-\infty, +\infty]$ be l.s.c. convex proper functions. Let $A \in \mathbb{R}^{m\times n}$ be a matrix. Consider the following primal optimization problem with value $p^\star$ and its associated dual problem with value $d^\star$:

$$(p^\star): \quad \min_{x\in\mathbb{R}^n} f(x) + g(Ax) \qquad\qquad (d^\star): \quad \max_{v\in\mathbb{R}^m} -f^*(-A^\top v) - g^*(v)$$

1. Show that $d^\star \leq p^\star$ (weak duality).
2. Suppose $d^\star = p^\star$ (strong duality). Show that $x^\star, v^\star$ are primal, dual optimal if and only if :

$$-A^\top v^\star \in \partial f(x^\star) \quad \text{ and } \quad v^\star \in \partial g(Ax^\star)$$

Exercice 21: penalized linear regression Consider the following optimization problem, where $\lambda > 0$, $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ are fixed :

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax - b\|_2^2 + \frac{\lambda}{2}\|x\|_2^2$$

1. Find the analytic optimal solution $x^\star$.
2. Note that the above problem can be written :

$$\begin{aligned} \min_{x\in\mathbb{R}^n,\xi\in\mathbb{R}^m} \quad & \frac{1}{2}\|\xi\|_2^2 + \frac{\lambda}{2}\|x\|_2^2 \\ \text{s.t.} \quad & \xi = Ax - b \end{aligned}$$

   (a) Write the Lagrangian of the above constrained problem (where α denotes the vector of dual variables).
   (b) Write the KKT optimality conditions of the above constrained problem.
   (c) Solve the KKT conditions. Give first the optimal dual variable $\alpha^\star$, then derive the optimal primal variable $x^\star$.
   (d) Compare with the original analytic solution.
3. We consider now the Fenchel dual problem of the original problem.
   (a) Write the Fenchel dual. We suggest to write the problem in the form $\min_{x\in\mathbb{R}^n} f(x) + g(Ax)$ and recall properties of the Fenchel conjugate (scaled/translated function, case of $\frac{1}{2}\|.\|_2^2$).
   (b) Write the subdifferential conditions for $(x^\star, \alpha^\star)$ to be primal/dual optimal. Show that they correspond to the KKT conditions.

Exercice 22: For the following functions, derive the proximal operator $\text{prox}_{\lambda f}$ (where $\lambda > 0$) :

1. square loss : $f(x) = \frac{1}{2}x^2$.
2. absolute value : $f(x) = |x|$.
3. support function : $f(x) = \sigma_{[a,b]}(x) = \begin{cases} ax & \text{if } x \leq 0 \\ bx & \text{if } x \geq 0 \end{cases}$ with $a \leq b$.

   Note that $\sigma_{[a,b]} = \imath^*_{[a,b]}$ is known as the support function of the set $[a, b]$.
4. ReLU function : $f(x) = \sigma_{[0,1]}(x) = \max(x, 0)$.
5. dead-zone linear/ ω-insensitive loss : $f(x) = \max(0, |x| - \omega)$.
6. Huber : $f(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } |x| \leq \omega \\ \omega|x| - \frac{\omega^2}{2} & \text{otherwise.} \end{cases}$
7. Log : $f(x) = -\log x$ if $x > 0$ and $+\infty$ otherwise.
8. Quadratic : $f(x) = \frac{1}{2}x^\top Ax + b^\top x + c$ with $A \succeq 0$ and symmetric.

Exercice 23: Let $m \leq n$ be two integers, $A \in \mathbb{R}^{m\times n}$ a rank m matrix and $b \in \mathbb{R}^m$. Let $v \in \mathbb{R}^n$.

1. Consider the problem :

$$\begin{aligned} &\min_x \frac{1}{2}\|x - v\|^2 \\ &\text{s.t. } Ax = b. \end{aligned}$$

   (a) Write the KKT conditions.
   (b) Solve the KKT equations and find the optimal point.
2. Define $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ by $f(x) = 0$ if $Ax = b$ and $f(x) = +\infty$ if $Ax \neq b$. Determine $\text{prox}_f(v)$.